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Some experiences in the equilibrium path determination

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The paper presents numerical techniques useful in nonlinear analysis of structures. The main attention is focused on procedures for the determination of equilibrium paths and examination of critical points. Most of them are connected with the arc length method and can be treated as additional tools, that improve the effectiveness and reliability of computations. It was confirmed in a number of test examples, presented in authors' earlier papers [1–4]. In this paper only one, but the most representative example including different types of critical points has been chosen. The conclusions are that even highly nonlinear problems can effectively be solved using relatively simple algorithms.

1. Introduction

One of the primary tasks in nonlinear analysis of structures is the determination of the equilibrium path and examination of critical points on it. At these points the loss of stability can occur, and this is the main reason that we look for them. Unfortunately, the numerical analysis of critical points is difficult, especially in the case of bifurcation points. The standard incremental-iterative approach based on the steady load increments fails near the critical points, because the tangent stiffness matrix is singular or ill-conditioned. As the result, slow convergence or divergence of iteration procedures are noticed. So in order to proceed effectively around the critical points, special computing techniques should be used. In the last two decades a number of different path-following procedures have been proposed, among which the length method is the most popular in practical applications. This method has been proposed by Riks [5] and Wempner [6], who independently and simultaneously introduced the concept of using the path length as the control parameter in the incremental-iterative process. It basically consists of completing the system of equilibrium equations with an additional control equation followed by an iterative calculation of the load increment together with the displacement increments. Such an approach assures good convergence of iteration process and its reliability in the vicinity of the limit points. The detailed discussion of this topic was presented by Riks [7]. The arc length method gained wide acceptance and was followed by many modifications. Batoz and Dhatt [8] developed a technique of solution of the extended system of equations, preserving the symmetry of the stiffness matrix. This technique was used by Crisfield [9] in his spherical arc length method. Many of the later papers (e.g. [10–15]) concern the modifications of the control equation and an automatic governing of the incremental process. A brief review of the literature can be found in [16].

The arc length method can be used with success in a bifurcation point analysis, however some additional, more advanced procedures must be found. It is the main topic of our considerations. In this paper the attention is given to an effective approach to the critical point and to passing onto the secondary, post-bifurcation branch. Two techniques for leaving the bifurcation point are presented. One of them is based on the paper by Riks [7]. The other one was originally introduced by the authors. It eliminates the singularity of Riks' extended matrix by using the temporary load imperfection, consistent with the first eigenvector of the tangent stiffness matrix. The main ideas of applied computational strategy have been presented in [2, 3]. The present paper contains a few improvements and summarises the authors' researches.

2. THE EQUILIBRIUM PATH

The discretized nonlinear equilibrium equations (e.g. in FEM) for a proportional, one-parametric load are given by

$$\mathbf{F}(\mathbf{q}, \mu) = \mathbf{R}(\mathbf{q}) - \mu\mathbf{P} = \mathbf{0}, \tag{1}$$

where $\mathbf{P}$ is the comparative load vector, μ is the load factor, $\mathbf{R}$ is the vector of nodal forces due to the internal stresses in elements and $\mathbf{q}$ is the unknown displacement vector. The above equations define a curve, called the equilibrium path (see Fig. 1), in $N+1$ dimensional space $\{\mathbf{q}, \mu\} : R^N \times R$, and could be treated as the implicit form of the functions

$$\mathbf{q} = \mathbf{q}(\mu). \tag{2}$$

Unfortunately, the load factor is not a correct parameter for the curve under consideration, because in the general case, it is not increasing monotonically (at the limit points it attains the local extremum). The arc length s can be a proper parameter instead. Using it, the equilibrium path can be written as

$$\mathbf{q} = \mathbf{q}(s), \quad \mu = \mu(s). \tag{3}$$

In this section our considerations are limited to smooth parts of the equilibrium path and a few techniques connected with the incremental-iterative process are presented. The analysis of critical points will be presented in the next section.

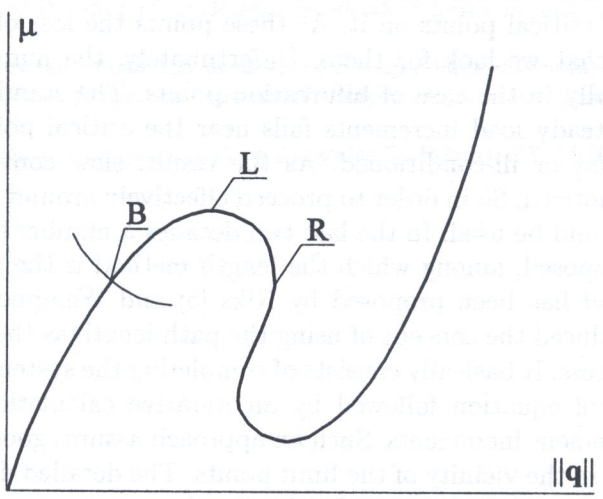

Fig. 1. The equilibrium path with bifurcation (B), limit (L) and turning (R) points

2.1. The arc length method

The nonlinear equilibrium equations (1) can be solved by different numerical techniques. Most of them are based on the incremental-iterative approach, in which the succeeding points of the equilibrium path are determined in subsequent incremental steps. Such an approach has been used in the arc length method. A brief review of this method is presented here for better understanding of the following consideration and proposed improvements.

In the arc length method a special equation (constraint condition)

$$\Delta s = g(\Delta\mathbf{q}, \Delta\mu) \tag{4}$$

is added to the basic system of equations (1) and the arc length s is used as a control parameter in the incremental-iterative process. For each step a new, but fixed increment Δs is assumed. Thanks to this, the standard iteration scheme with the constant load increment $\Delta\mu$ can be replaced by a more general scheme, in which the displacements $\mathbf{q}$ and the load factor μ are the values corrected in every iteration (see Fig. 2). Equation (4) defines the approximate length of the arc. Different approximations can be used [2, 7, 9, 10]. For constant Δs, Eq. (4) determines some hypersurface that intersects the equilibrium path at the point we are looking for. To find this point the incremental-iterative algorithm of the predictor-corrector type is commonly used, in which the task of the corrector is fulfilled by the Newton–Raphson method (full — FNR, modified — MNR) or BFGS method [17].

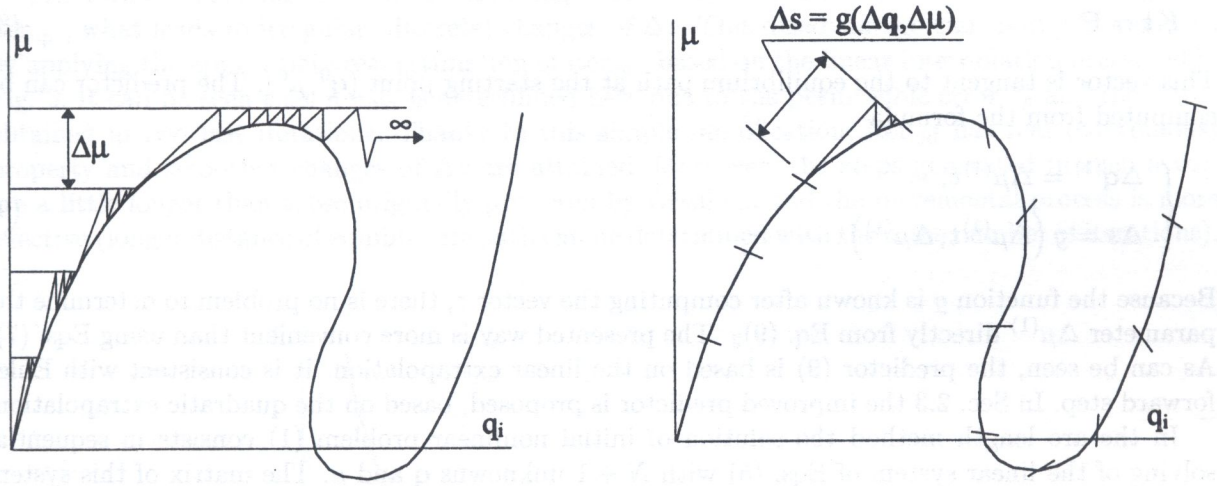

Fig. 2. The incremental-iterative process: a) standard, b) in the arc length method

When a point $(\mathbf{q}^0, \mu^0)$ on the equilibrium path is known, the next point $(\mathbf{q}^n, \mu^n)$ can be computed using the following iteration scheme (for $\Delta s = \text{const}$).

$$\begin{cases} \mathbf{K}^{(i-1}\delta\mathbf{q}^{(i)} - \mathbf{P}\delta\mu^{(i)} = \mu^{(i-1)}\mathbf{P} - \mathbf{R}^{(i-1)}, \\ \dfrac{\partial g}{\partial \mathbf{q}}\delta\mathbf{q}^{(i)} - \dfrac{\partial g}{\partial \mu}\delta\mu^{(i)} = 0, \end{cases} \qquad \text{for } i > 1, \tag{5}$$

in which the quantities $\mathbf{q}, \mu$ and their increments are updated in each iteration, according to

$$\begin{aligned} \mathbf{q}^{(i)} &= \mathbf{q}^{(i-1)} + \delta\mathbf{q}^{(i)}, \\ \mu^{(i)} &= \mu^{(i-1)} + \delta\mu^{(i)}, \\ \Delta\mathbf{q}^{(i)} &= \Delta\mathbf{q}^{(i-1)} + \delta\mathbf{q}^{(i)}, \\ \Delta\mu^{(i)} &= \Delta\mu^{(i-1)} + \delta\mu^{(i)}. \end{aligned} \tag{6}$$

Δ denotes the increments and δ their iterative corrections. In Eq. $(5)_1$ $\mathbf{K}$ is the tangent stiffness matrix, defined by the formula $\mathbf{K} = \partial\mathbf{R}/\partial\mathbf{q}$. In FNR or BFGS methods, matrix $\mathbf{K}$ is updated after each iteration, and in the MNR method — at the beginning of each step only. The right side of Eq. $(5)_1$ is called the vector of unbalanced (residual) forces. It is updated in each iteration (independently of the applied iteration method).

The predictor is the initial approximation used in the first iteration. It can be computed from Eqs. (5) with modified right sides. Assuming: $\mu^{(0)}\mathbf{P} - \mathbf{R}^{(0)} = \mathbf{0}$, $\Delta\mathbf{q}^{(1)} = \delta\mathbf{q}^{(1)}$, $\Delta\mu^{(1)} = \delta\mu^{(1)}$, the

system of Eqs. (5), for $i = 1$, can be rewritten in the form

$$\begin{cases} \mathbf{K}\Delta\mathbf{q}^{(1)} - \Delta\mu^{(1)}\mathbf{P} = \mathbf{0}, \\ \dfrac{\partial g}{\partial \mathbf{q}}\Delta\mathbf{q}^{(1)} - \dfrac{\partial g}{\partial \mu}\Delta\mu^{(1)} = \Delta s. \end{cases} \tag{7}$$

However, in the arc length method the control equation as well as function $g(\Delta\mathbf{q}, \Delta\mu)$ are often fixed just after first iteration (Riks [7]). Because of this, the direct application of Eqs. (7) is not convenient. A simpler algorithm can be proposed instead. Let as assume that the matrix $\mathbf{K}$ is not singular (the case when it is singular will be presented in Sec. 3). At this assumption we can compute the vector $\mathbf{t}$, defined as follows

$$\mathbf{K}\mathbf{t} = \mathbf{P}. \tag{8}$$

This vector is tangent to the equilibrium path at the starting point $(\mathbf{q}^0, \mu^0)$. The predictor can be computed from the formula

$$\begin{cases} \Delta\mathbf{q}^{(1)} = \Delta\mu^{(1)}\mathbf{t}, \\ \Delta s = g\left(\Delta\mu^{(1)}\mathbf{t}, \Delta\mu^{(1)}\right). \end{cases} \tag{9}$$

Because the function g is known after computing the vector $\mathbf{t}$, there is no problem to determine the parameter $\Delta\mu^{(1)}$ directly from Eq. $(9)_2$. The presented way is more convenient than using Eqs. (7). As can be seen, the predictor (9) is based on the linear extrapolation. It is consistent with Euler forward step. In Sec. 2.3 the improved predictor is proposed, based on the quadratic extrapolation.

In the arc length method the solution of initial nonlinear problem (1) consists in sequential solving of the linear system of Eqs. (5) with $N + 1$ unknowns $\mathbf{q}$ and μ. The matrix of this system has the following form (it is called by us the extended Riks' matrix)

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{K} & -\mathbf{P} \\ \hline \partial g/\partial \mathbf{q} & \partial g/\partial \mu \end{array}\right]. \tag{10}$$

In general, matrix $\mathbf{A}$ is non-symmetric. However, using a specific factorization procedure of $\mathbf{A}$, the symmetry and profile form of matrix $\mathbf{K}$ can be preserved [1]. This non-symmetry can be also eliminated by using the Batoz–Dhatt technique [8], in which the solution is split into two components. A comprehensive discussion and comparison of the different procedures for solving the system of Eqs. (5) was presented in [2] and [4]. Also the problem of different form of Eq. (4) was explored there. It should be pointed out, that use of extended Riks' matrix is more reliable than Batoz–Dhatt technique. Matrix $\mathbf{A}$ has an important advantage: it is not singular at limit points. This interesting fact was proved by Riks [7] (an independent and shorter proof is presented in Sec. 3.1). Unfortunately, at the bifurcation points, matrix $\mathbf{A}$ is singular as well as matrix $\mathbf{K}$. Therefore, the incremental-iterative process mentioned in this section cannot be used directly and some additional procedures are needed. This topic is discussed in detail in Sec. 3.

2.2. The automatic control of the step length

The step length is one of the most important parameters which influences the incremental process effectiveness. However, the speed of computations should not be the only criterion of effectiveness. According to the authors, the quality of information gathered on the equilibrium path is more important. A proper spacing of points should give a good view of the shape of the path, avoiding at the same time superfluous information. In most cases, the constant increment of the arc length

is not suitable. The equilibrium path is approximated by straight segments and the increment Δs should depend on the arc curvature (see [2]). The commonly used method of step length changing was proposed by Crisfield [11], and has the form

$$\Delta s_{\text{new}} = \Delta s_{\text{old}} \sqrt{\frac{\text{iter}_{\text{opt}}}{\text{iter}_{\text{old}}}}, \tag{11}$$

where iter_{old} denotes the number of iterations in the last step and iter_{opt} is the optimum (desired) number of iterations. It should be related to the required accuracy level, the chosen iteration scheme and the number of degrees of freedom (DOF). In most cases, iter_{opt} can be assumed equal to 5 or 6 for MNR, and 3 or 4 for FNR.

The formula (11) has a small disadvantage consisting of the integer character of iter_{old} and iter_{opt}, what leads to irregular (discrete) changes of Δs. This disadvantage can simply be removed by applying the appropriate real estimation of iter_{old}, based on the linear interpolation presented in Fig. 3. It can be seen that iter_{old} is determined here due to the permissible error ε_p and the errors obtained in two last iterations. Thanks to this simple modification, iter_{old} has real (continuous) property and smoother changes of Δs are attained. Moreover, the steps generated in such a way are a little longer than those originally proposed by Crisfield, and the incremental process is more effective (longer distance of equilibrium path can be determined with the same number of iterations).

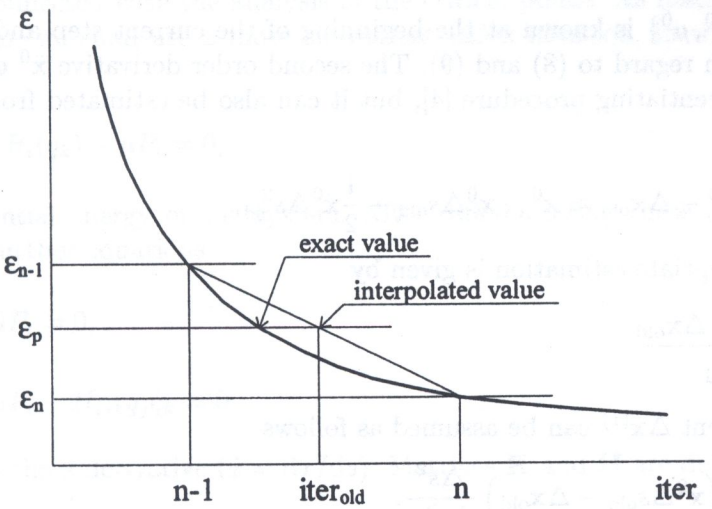

Fig. 3. Estimation of iter_{old}

If the incremental step is too long the problem of convergence occurs. In this case the step reduction technique is commonly used. It consists in halving the Δs and repeating the last step from the beginning. More advanced approach to the step size adjustment has been proposed by Eriksson and Kouhia [18]. They introduced two algorithms for automatic control of step size during the iterations. However, their final conclusions are that the benefit to the total efficiency is questionable. Nevertheless, the idea of using the information on the current iteration state to automatic adjustment of step size is very stimulating, and should be taken into consideration in new applications.

When using Eq. (11) or similar, it is advisable to adopt additional constraints for Δs

$$\Delta s_{\min} \leq \Delta s \leq \Delta s_{\max}, \tag{12}$$

where $\Delta s_{\min}$ and $\Delta s_{\max}$ should be established depending on the structure under consideration. These values can be defined independently for various segments of the path. This allows to hand-control the increments Δs for a chosen set of steps. In other words the automatic control of the

step length can be corrected by the user defined conditions. In authors opinion, it is very valuable in practice, because full automation, based on the formula (11), is not sufficient in some cases (see examples 1 and 2 in [2]).

2.3. The improved predictor

There are many techniques to improve the efficiency of iteration procedures. For example, Aitken's method [17] can be mentioned here. The algorithms accelerating the iteration, however, are fairly complicated. The number of iterations to attain the recommended convergence depends in great part on the precision of the initial approximation (predictor) $\Delta\mathbf{x}^{(1)} = \{\Delta\mathbf{q}^{(1)}, \Delta\mu^{(1)}\}$. In Sec. 2.1 the standard predictor based on the linear extrapolation has been described. Its accuracy can be significantly improved using the higher order extrapolation. A comprehensive review and comparison of different predictions have been presented by Eriksson [19]. It should be pointed out that the better prediction of the solution reduces the number of iterations and improves the reliability of computation, because of smaller risk of divergence. But the overall efficiency can decrease if the computation of the predictor is too expensive. So in practice, simpler (perhaps less exact) predictors should be preferred. One of the possibilities is the use of quadratic extrapolation

$$\mathbf{x}(s^0 + \Delta s) \approx \mathbf{x}^0 + \dot{\mathbf{x}}^0 \Delta s + \frac{1}{2}\ddot{\mathbf{x}}^0 \Delta s^2. \tag{13}$$

The vector $\mathbf{x}^0 = \{\mathbf{q}^0, \mu^0\}$ is known at the beginning of the current step and vector $\dot{\mathbf{x}}^0$ is defined as $\dot{\mathbf{x}}^0 = \Delta\mathbf{x}/\Delta s$ with regard to (8) and (9). The second order derivative $\ddot{\mathbf{x}}^0$ can be determined by an appropriate differentiating procedure [4], but it can also be estimated from the solution of the previous step

$$\mathbf{x}(s - \Delta s_{\text{old}}) = \mathbf{x}^0 - \Delta\mathbf{x}_{\text{old}} \approx \mathbf{x}^0 - \dot{\mathbf{x}}^0 \Delta s_{\text{old}} + \frac{1}{2}\ddot{\mathbf{x}}^0 \Delta s_{\text{old}}^2. \tag{14}$$

Therefore, the appropriate estimation is given by

$$\frac{1}{2}\ddot{\mathbf{x}}^0 \approx \frac{\dot{\mathbf{x}}^0 \Delta s_{\text{old}} - \Delta\mathbf{x}_{\text{old}}}{\Delta s_{\text{old}}^2} \tag{15}$$

and the first increment $\Delta\mathbf{x}^{(1)}$ can be assumed as follows

$$\Delta\mathbf{x}^{(1)} = \dot{\mathbf{x}}^0 \Delta s + \left(\dot{\mathbf{x}}^0 \Delta s_{\text{old}} - \Delta\mathbf{x}_{\text{old}}\right) \frac{\Delta s^2}{\Delta s_{\text{old}}^2}. \tag{16}$$

The above vector should suitably be scaled so its length equals to the length of vector $\dot{\mathbf{x}}^0 \Delta s$. The second addend of the Eq. (16) is a correction resulting from the expected curvature of the path (see Fig. 4) The computation of this correction is simple and does not require additional storage space. Usually, on sufficiently smooth parts of the path, this correction considerably decreases the number of iterations [20]. Of course such a predictor based on the path history cannot be used in branching problems, because the curvature and direction of the post-bifurcation branch is completely different, so the estimation (15) is meaningless. Besides, a similar problem appears at the beginning of computations, when the previous solution is unknown. Prediction of the post-bifurcation branch requires special procedures. They are described in Sec. 3.2.

3. Critical points

At the critical points the tangent stiffness matrix $\mathbf{K}$ is singular and solving of the system of Eqs. (5) is difficult. First of all, the predictor cannot be determined in the conventional manner, described in the previous section, because the tangent vector $\mathbf{t}$ cannot be computed from Eq. (8). In the case

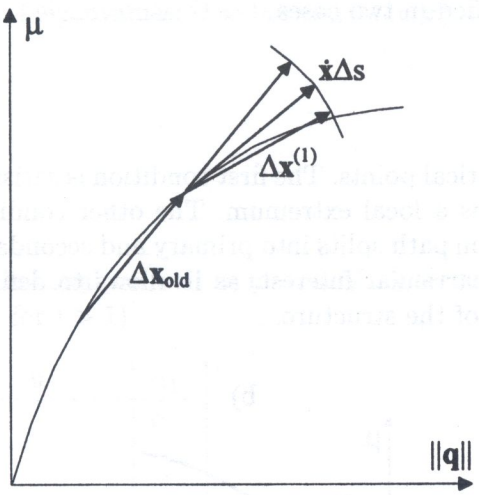

Fig. 4. The improved predictor

of limit point, the suitably chosen approximation of $\mathbf{t}$ can be applied, but at the bifurcation point it is impossible. Moreover, at the bifurcation points the extended Riks' matrix $\mathbf{A}$ is singular and it is necessary to resort to special numerical techniques. Before we discuss them, let us present the essential equations connected with the analysis of the critical points. As matrices of higher orders appear in this section, we shall use a more convenient index notation, in which the equilibrium Eq. (1) take the form

$$\frac{\partial V}{\partial q_i} = F_i(q_k, \mu) = R_i(q_k) - \mu P_i = 0, \tag{17}$$

where V is the potential energy of the system. After differentiating these equations by the arc length s, we obtain further equations

$$\frac{\mathrm{d}F_i}{\mathrm{d}s} = K_{ij}\dot{q}_j - \dot{\mu}P_i = 0, \tag{18}$$

$$\frac{\mathrm{d}^2F_i}{\mathrm{d}s^2} = K_{ij}\ddot{q}_j - \ddot{\mu}P_i + H_{ijk}\dot{q}_j\dot{q}_k = 0, \tag{19}$$

where the dot means the s derivative ($\dot{x} = \mathrm{d}x/\mathrm{d}s$). Matrices $\mathbf{K}$ and $\mathbf{H}$ are defined as follows

$$K_{ij} = \frac{\partial R_i}{\partial q_j} = \frac{\partial^2 V}{\partial q_i \partial q_j}, \tag{20}$$

$$H_{ijk} = \frac{\partial K_{ij}}{\partial q_k} = \frac{\partial^3 V}{\partial q_i \partial q_j \partial q_k}. \tag{21}$$

At the critical points, the determinant and the first eigenvalue of the stiffness matrix are equal to zero: $\det(\mathbf{K}) = 0$, $\omega_1 = 0$. Let us restrict the discussion to single critical points, for which $0 = \omega_1 <| \omega_2 |\leq| \omega_3 |\leq \ldots \leq| \omega_N |$. The first eigenvector $\mathbf{v}$, associated with ω_1 satisfies the following conditions

$$K_{ij}v_j = 0, \quad v_i v_i = 1. \tag{22}$$

Note that $\mathbf{v}$ is the only non-zero vector satisfying the above equations. After multiplication of Eqs. (18) and (19) by v_i and taking (22) into account, we obtain the following equations

$$\dot{\mu}P_i v_i = 0, \tag{23}$$

$$-\ddot{\mu}P_i v_i + v_i H_{ijk}\dot{q}_j\dot{q}_k = 0. \tag{24}$$

Equation (23) can be satisfied in two cases

$$\begin{aligned} &\text{a)} \quad P_i v_i \neq 0 \;\Rightarrow\; \dot{\mu} = 0, \\ &\text{b)} \quad P_i v_i = 0, \end{aligned} \tag{25}$$

relevant to the two types of critical points. The first condition is satisfied at the limit points (Fig. 1), where the parameter μ attains a local extremum. The other condition is satisfied at bifurcation points, at which the equilibrium path splits into primary and secondary branch (see Fig. 5). Passing on the secondary path is of particular interest, as it allows to define the type of the bifurcation point and assess the stability of the structure.

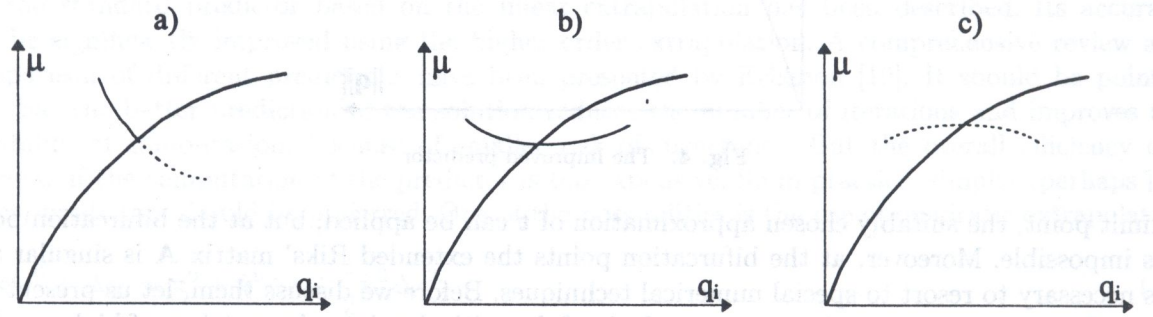

Fig. 5. Classification of bifurcation points: a) nonsymmetrical, b) symmetrical stable, c) symmetrical unstable

3.1. Limit point

Let us now proceed with the determination of the vector $\dot{\mathbf{q}}$, tangent to the equilibrium path at critical points. Further, we are assuming that the s parameter denotes the length of the path in N-dimensional space $\{\mathbf{q}\} : R^N$. Hence, vector $\dot{\mathbf{q}} = \mathrm{d}\mathbf{q}/\mathrm{d}s$ is normalized and satisfies the equation

$$\dot{\mathbf{q}}^{\mathrm{T}}\dot{\mathbf{q}} = \dot{q}_i \dot{q}_i = 1. \tag{26}$$

Analysing Eq. (18) with the condition $(25)_1$ and having in mind the remark mentioned at Eq. (22), it can be noted that the tangent to the equilibrium path at the limit point is defined by the eigenvector $\mathbf{v}$. This fact can be written as follows

$$\dot{\mathbf{q}}(s = s^{lp}) = \pm\mathbf{v} \quad \text{and} \quad \dot{\mu}(s = s^{lp}) = 0, \tag{27}$$

where the upper index lp denotes the limit point. In the above equation a suitable sign should be chosen to make the product $\dot{\mathbf{q}}^{\mathrm{T}}\Delta\mathbf{q}_{\mathrm{old}} > 0$. Moreover, $\ddot{\mu}$ can be computed from Eq. (24). It allows to establish the type of the local extremum (minimum or maximum).

Let us now investigate the Riks' extended matrix $\mathbf{A}$ defined in (10). Assuming the control Eq. (4) has the form: $\Delta s = \dot{\mathbf{q}}^{\mathrm{T}}\Delta\mathbf{q}$ (see [2]), and taking into account Eq. (27), matrix $\mathbf{A}$ is equal to

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{K} & -\mathbf{P} \\ \hline \mathbf{v}^{\mathrm{T}} & 0 \end{array}\right]. \tag{28}$$

This matrix is non-singular, what can be proved by using the eigenvector transformation. Let us assume a matrix $\mathbf{V}_{N\times N}$ containing in its columns succeeding eigenvectors. This matrix satisfies the following equations

$$\mathbf{V}^{\mathrm{T}}\mathbf{K}\mathbf{V} = \Omega \quad \text{and} \quad \mathbf{V}^{\mathrm{T}}\mathbf{V} = \mathbf{I}_{N\times N}, \tag{29}$$

where Ω is the diagonal matrix of eigenvalues: $\Omega = \lceil \omega_1, \omega_2, \ldots, \omega_N \rfloor$. Let us also assume an extended matrix $\mathbf{W}$ defined as

$$\mathbf{W} = \left[\begin{array}{c|c} \mathbf{V} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right]. \tag{30}$$

Note that this matrix is orthogonal, as well as matrix $\mathbf{V}$. After the transformation $\mathbf{W}^{\mathrm{T}}\mathbf{A}\mathbf{W}$ we obtain (for $\omega_1 = 0$, and $\omega_i \neq 0$ for $i > 1$)

$$\tilde{\mathbf{A}} = \mathbf{W}^{\mathrm{T}}\mathbf{A}\mathbf{W} == \left[\begin{array}{c|ccc|c} 0 & & \mathbf{0} & & \alpha_1 \\ \hline & \omega_2 & & & \alpha_2 \\ \mathbf{0} & & \ddots & & \vdots \\ & & & \omega_N & \alpha_N \\ \hline 1 & & \mathbf{0} & & 0 \end{array}\right], \tag{31}$$

where $\alpha_i = -V_{ki}P_k$ (and $\alpha_1 = -V_{k1}P_1 = -P_k v_k$). The determinant of the above matrix can be written as

$$\det \tilde{\mathbf{A}} = -\alpha_1 \prod_{i=2}^{N} \omega_i = P_k v_k \prod_{i=2}^{N} \omega_i. \tag{32}$$

Because $P_k v_k \neq 0$ at the limit point, then $\det \tilde{\mathbf{A}} \neq 0$. The matrix $\mathbf{W}$ is orthogonal, then $\mathbf{W}^{\mathrm{T}}\mathbf{W} = \mathbf{I}$ and $\det \mathbf{W} = 1$. Finally we obtain

$$\det \tilde{\mathbf{A}} = \det \mathbf{W}^{\mathrm{T}} \cdot \det \mathbf{A} \cdot \det \mathbf{W} = \det \mathbf{A} \neq 0, \tag{33}$$

which concludes the proof. Thanks to the eigenvector transformation the above proof is very concise and simpler then the analogous proof presented by Riks [7].

3.2. Bifurcation point

From the above proof it can be perceived that the matrix $\mathbf{A}$ is singular at the bifurcation point independently of the control equation, because $\alpha_1 = -P_k v_k = 0$. It leads to serious numerical difficulties, and continuation methods cannot be used directly (some additional specialized procedures have to be used). One of the possibilities is using the perturbation method introduced by Koiter [21, 22]. However it has a limited application in practice, because of its complication and computational cost. A lot of different methods have been proposed instead. A comprehensive review of the algorithms connected with the bifurcation point analysis can be found in the book by Seydel [23].

In the present paper we propose relatively simple approach based on the tangents to the path. First it should be noted that if a certain vector $\dot{\mathbf{q}} = \mathbf{z}$ satisfies Eqs. (18), then they are also satisfied by the vector $\dot{\mathbf{q}} = \mathbf{z} + \alpha\mathbf{v}$, for any arbitrary parameter α. Therefore, without restraining the general character of the solution it can be assumed that

$$\dot{\mathbf{q}} = \mu\mathbf{n} + \alpha\mathbf{v}, \tag{34}$$

where the vector $\mathbf{n}$ is defined from the equations

$$\mathbf{K}\mathbf{n} = \mathbf{P}, \quad \mathbf{v}^{\mathrm{T}}\mathbf{n} = 0. \tag{35}$$

Equation $(35)_2$ is necessary for the unique definition of the vector $\mathbf{n}$, because matrix $\mathbf{K}$ is singular here. It can be verified that Eqs. (18) are satisfied identically for the vector $\dot{\mathbf{q}}$ defined by

the formula (34). After using Eqs. (24), (26), (34) and $(25)_2$ we obtain the following system of quadratic equations

$$\begin{cases} \dot{\mu}^2(H_{ijk}v_i n_j n_k) + 2\alpha\dot{\mu}(H_{ijk}v_i v_j n_k) + \alpha^2(H_{ijk}v_i v_j v_k) = 0, \\ \dot{\mu}^2 n_i n_i + \alpha^2 = \mathbf{1}. \end{cases} \tag{36}$$

In the general case, this system has two solutions (μ_1, α_1) and (μ_2, α_2), corresponding to the primary and the secondary branches of the equilibrium path. After adopting the auxiliary variable $x = \dot{\mu}/\alpha$ and denoting the expressions in brackets in Eq. $(36)_1$ by a, b, c, the solution can be presented in simple form

$$\begin{aligned} &ax^2 + 2bx + c = 0, \\ &\alpha = \pm\sqrt{\frac{1}{1 + x^2\mathbf{n}^2}}, \\ &\dot{\mu} = \alpha x = \pm\sqrt{\frac{x^2}{1 + x^2\mathbf{n}^2}}. \end{aligned} \tag{37}$$

The case when the c parameter is equal to zero

$$c = H_{ijk}v_i v_j v_k = 0, \tag{38}$$

requires a separate discussion. One of the results is $\dot{\mu} = 0$, $\alpha = \pm 1 \Rightarrow \dot{\mathbf{q}} = \pm\mathbf{v}$. The point fulfilling this condition is called the symmetrical bifurcation point (Fig. 5b, c). In this case, the vector tangent to the secondary branch is equal to

$$\dot{\mathbf{q}}(s = s^{sbp}) = \pm\mathbf{v} \quad \text{and} \quad \dot{\mu}(s = s^{sbp}) = 0, \tag{39}$$

and it is not necessary to use Eqs. (36) for computing $\dot{\mathbf{q}}$. This interesting result is important because the symmetrical bifurcation points are most frequently encountered in practice. The choice of the sign in above equation is arbitrary and depends on the decision of the user. The best way, however, is to check the secondary branch of the path on both sides of the bifurcation point, as this allows the exact determination of its type to be found.

3.3. Passing onto the secondary branch

The computing of the coefficients in Eq. $(36)_1$ is very expensive. It should be pointed out that the explicit determination of the **H** matrix is completely ineffective and unnecessary. The products $H_{ijk}x_i y_j z_k$ can be computed at the level of the finite element. This way reduces the storage capacity needed and the duration of computations because many elements of **H** matrix as well as of stiffness matrix **K** are equal to zero. But even using such a way, the calculation of the above mentioned coefficients is costly. The determination of **H** matrix can also cause difficulties for some types of the finite element. In view of this, the authors propose the following strategy for passing onto the secondary branch:

1) To start, it is necessary to approach the bifurcation point with possibly high accuracy in order to obtain good evaluation of the first eigenvector **v** (see Secs. 3.5 and 3.4).

2) Next, assuming $\dot{\mathbf{q}} = \mathbf{v}$ passing onto the secondary branch should be tried; this topic is presented in the following part of this section.

3) If this attempt fails, then the Eqs. (36) are used for more exact determination of the tangent vector $\dot{\mathbf{q}}$ and step 2) is repeated.

Using this approach, passing onto the secondary branch is possible without resorting to (36), even in the case of asymmetrical bifurcation points. The arbitrary assumption $\dot{\mathbf{q}} = \mathbf{v}$ valid for symmetrical bifurcation points will be satisfied in most of cases.

The MNR iteration procedure cannot be used directly at bifurcation points, because the system of Eqs. (5) is singular. (A similar problem also occurs at the limit points if Batoz–Dhatt technique is used.) The singularity can be eliminated by using FNR method or by leaving the critical point in accordance with the computed predictor and then updating the tangent stiffness matrix (see [2, 4] for details). However, the above techniques of leaving the bifurcation point, can sometimes prove unreliable. This is mainly caused by ill-condition of the tangent stiffness matrix on the secondary branches. To understand this disadvantage the short proof is presented below.

The determinant of the tangent stiffness matrix and its derivative can be written by the eigenvalues as follows

$$D = \det \mathbf{K} = \prod_{i=1}^{N} \omega_i, \tag{40}$$

$$\dot{D} = \sum_{i=1}^{N} \dot{\omega}_i \prod_{j \neq 1}^{N} \omega_j. \tag{41}$$

At the critical points: $\omega_1 = 0 \Rightarrow D = 0$, hence

$$\dot{D} = \dot{\omega}_1 \prod_{i=2}^{N} \omega_i. \tag{42}$$

The first eigenvalue ω_1 and the corresponding eigenvector $\mathbf{v}$ satisfy the equation

$$\mathbf{K}\mathbf{v} = \omega_1 \mathbf{v}. \tag{43}$$

After its differentiation, then multiplication by $\mathbf{v}^{\mathrm{T}}$, and taking into account Eqs. (21), (22) and $\mathbf{v}^{\mathrm{T}}\dot{\mathbf{v}} = 0$, we obtain

$$\dot{\omega}_1 = \mathbf{v}^{\mathrm{T}}\dot{\mathbf{K}}\mathbf{v} = v_i H_{ijk} v_j \dot{q}_k. \tag{44}$$

From Eqs. (38), (39), (42) and (44), it can be found, that on the secondary branch at the symmetrical bifurcation point the derivatives $\dot{D}$ as well as $\dot{\omega}_1$ are equal to zero. So the singularity of the tangent stiffness matrix disappears slowly in this direction.

Therefore, it is difficult to invent a fully reliable algorithm. On one hand the leaving distance from the bifurcation point should be sufficiently large to allow the $\mathbf{K}$ matrix not to be singular, but on the other hand this distance should be sufficiently small to assure the convergence of iterations. The contradiction between these two criteria constitutes here the key problem, which cannot always be solved. When changing the step length Δs, its reduction as well as its increasing should be taken into account.

3.3.1. Temporary load imperfection technique

An alternative way for passing onto the secondary branch can be the technique based on a temporary load imperfection. Let us recall, that the occurrence of the bifurcation point is closely connected with the adopted load vector $\mathbf{P}$. Even when a very small change is introduced in it, the bifurcation point will not appear. After reaching the bifurcation point, a new load vector $\mathbf{P}_{tmp}$ can be adopted for the moment. It is useful to assume

$$\mathbf{P}_{tmp} = \mathbf{P} \pm \varepsilon \mathbf{v}, \tag{45}$$

where ε a is a suitably chosen small number. Note that the condition $(25)_2$ is not satisfied now. Since the matrix $\mathbf{K}$ is singular the condition $(25)_1$ must be satisfied therefore. This means that

instead of a bifurcation point, a limit point appears on some abstract branch, connected with the load vector adopted for the moment. It is no problem to obtain the next point on this branch, because the extended Riks' matrix is non-singular. The proof can be performed in the way similar to that presented in Sec. 3.1. With regard to an insignificant simplification $\mathbf{P}_{tmp} = -\mathbf{v}$, the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{K} & \mathbf{v} \\ \hline \mathbf{v}^{\mathrm{T}} & 0 \end{array}\right]. \tag{46}$$

After eigenvector transformation we obtain (for $0 = \omega_1$, $\omega_i \neq 0$ for $i \neq 0$) the matrix

$$\tilde{\mathbf{A}} = \mathbf{W}^{\mathrm{T}}\mathbf{A}\mathbf{W} == \left[\begin{array}{c|c|c} 0 & \mathbf{0} & 1 \\ \hline & \omega_2 & \\ \mathbf{0} & \ddots & \mathbf{0} \\ & & \omega_N \\ \hline 1 & \mathbf{0} & 0 \end{array}\right]. \tag{47}$$

Because matrix $\mathbf{W}$ is orthogonal ($\mathbf{W}^{\mathrm{T}}\mathbf{W} = \mathbf{I}$, $\det \mathbf{W} = 1$), then

$$\det \mathbf{A} = \det \tilde{\mathbf{A}} = -\prod_{i=2}^{N} \omega_i \neq 0, \tag{48}$$

which concludes the proof.

It should be noticed that the point obtained in the presented way is not situated on the path we are looking for, but on an abstract branch connected with the perturbed load vector. After return to the original loads, the unbalanced forces will appear and the proper branch can be attained by using the conventional iteration procedure.

3.4. Computation of the first eigenvector

As it can be seen from Eqs. (22)–(39), the first eigenvector $\mathbf{v}$ is of great weight in critical point analysis. For its correct computation the critical point must be reached with sufficient accuracy. This problem is presented in Sec. 3.5. The computation of the mentioned eigenvector can be performed without resorting to general procedures of eigenproblems. It is sufficient to use directly the homogeneous system of Eq. $(22)_1$. Using the Gaussian elimination with full choice of the pivot element (see Ralston [24]), the $\mathbf{K}$ matrix can be presented in the form

$$\mathbf{K} = \mathbf{P}_r\mathbf{L}\mathbf{D}\mathbf{U}\mathbf{P}_c, \tag{49}$$

where $\mathbf{P}_r$ and $\mathbf{P}_c$ are the permutation matrices, $\mathbf{L}$ is the lower and $\mathbf{U}$ is the upper triangular matrix and $\mathbf{D}$ is the diagonal matrix. At the single critical point the rank of matrix $\mathbf{K}$ is equal to $N-1$. Using appropriate permutation matrices, it can be satisfied, that the last element of the $\mathbf{D}$ matrix is equal to zero, while the remaining are not zero. Thanks to this, a non-zero solution of $(22)_1$ exists and can be calculated from the following equation

$$\mathbf{U}\mathbf{P}_c\mathbf{v} = \mathbf{x}, \quad \text{where} \quad \mathbf{x} = \{0, 0, \ldots, 1\}_{N\times 1}. \tag{50}$$

The vector $\mathbf{x}$ satisfies the equality $\mathbf{D}\mathbf{x} = \mathbf{0}$. It should be noticed, that only the last element of this vector is not equal to zero and can be freely assumed (e.g. $x_N = 1$). Vector $\mathbf{v}$ is computed by the so called backward elimination [24]. Finally it should be normalized in accordance with $(22)_2$.

As can be seen, the problem of computation of the first eigenvector has been reduced to the solution of a linear system of equations. The knowledge that $\omega_1 = 0$ was here taken to good account. The presented algorithm is simpler and faster than the general procedures for solving of the eigenproblem: $(\mathbf{K} - \omega\mathbf{I})\mathbf{v} = \mathbf{0}$.

3.5. Localization of critical points

Formula (11) can be applied to smooth branches of the equilibrium path. When looking for critical points, the formula based on interpolation (extrapolation) of such parameters as the determinant or the first eigenvalue of the stiffness matrix is more suitable. This approach was presented in [7]. The calculation of the first eigenvalue is relatively costly, but the determinant can be evaluated at little cost during the decomposition of the matrix. It is sufficient to take very large or very small values of determinant into account. The best way is to store separately its mantissa and exponent. This approach protects the calculations against the overflow or underflow of real numbers. The determinant can be normalized relatively to the initial (undeformed) configuration

$$\det_{\text{norm}} = \frac{\det \mathbf{K}\,|_{t=t}}{\det \mathbf{K}\,|_{t=0}}. \tag{51}$$

Such a normalized parameter is more convenient in calculations and in interpreting of the results. It is equal to one at the beginning and close to zero near the critical points. The ordinary determinant can be a large number, even in close vicinity of the critical point. Only its order of magnitude decreases distinctly. It is well known, that a sign of the determinant cannot be a sure parameter of system stability [23] (e.g. if ω_1 and ω_2 are less then zero) as well as of positive definition of the stiffness matrix. The most reliable method for the detection of critical points is observation of a few first eigenvalues, however, it is very expensive in practice. A reliable indicator of passing through the critical point is the number of negative diagonal elements (NDE) of the tangent stiffness matrix after decomposition. This parameter is very simple and its computation is fast and cheap. If NDE_1 and NDE_2 are different in two subsequent points then the critical point exists between them (see Fig. 6). For an accurate estimation of this point a special searching procedure should be used. We suggest to apply the locally defined function $f(s)$, given by

$$f(s) = \begin{cases} +\,|\det_{\text{norm}}| & \text{if NDE} = \text{NDE}_1, \\ -\,|\det_{\text{norm}}| & \text{if NDE} \neq \text{NDE}_1. \end{cases} \tag{52}$$

This function has a single root s^* ($f(s^*) = 0$), which can be determined by one of the conventional searching algorithms. In our own computer program we implemented a mixed method, derived from the regula falsi and secant methods (Ralston [24]). This method has proved to be sufficiently effective because of an initially small interval of searching. In all investigated tests, the critical points were attained in 3 to 5 iterations, for relative s error equal to $1 \cdot 10^{-7}$.

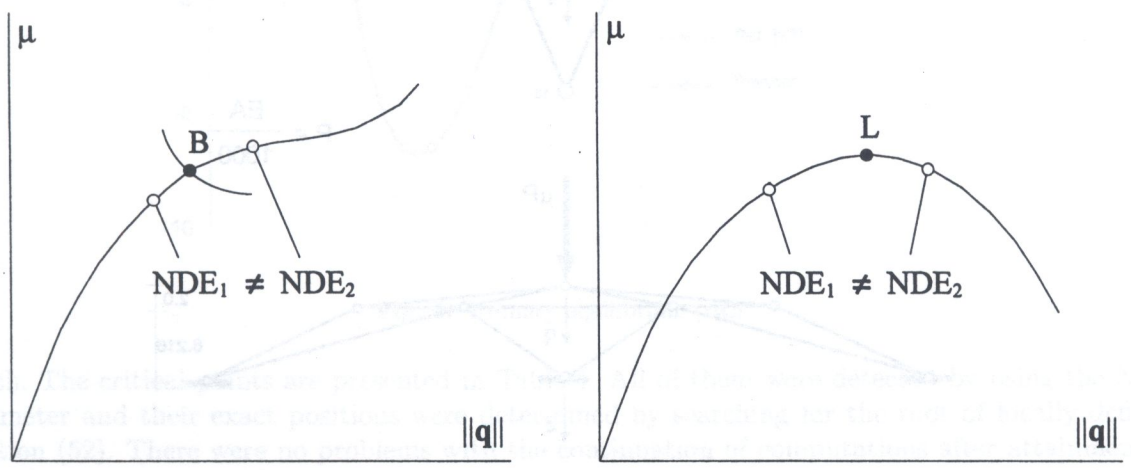

Fig. 6. Detection of critical points by NDE parameter

The only condition for the effective localization the critical point is observation of $\det_{norm}$ and NDE parameters during the computation. However, it is also recommended to observe other parameters like number of iterations (iter) or the current stiffness parameter (C_{SP}), introduced by Bergan et all [25]. A simultaneous observation of $\det_{norm}$, NDE, iter and C_{SP} parameters gives a good evaluation of the nonlinear behaviour of the system and makes the interpretation of the results easier.

4. EXAMPLE — SHALLOW TRUSS DOME

The analysis of the shallow truss dome shown in Fig. 7 is a complicated numerical problem. This structure has been investigated in a number of earlier papers (e.g. [26–28]), and can be treated as a particular benchmark test, because of its highly nonlinear behaviour. Many different critical points

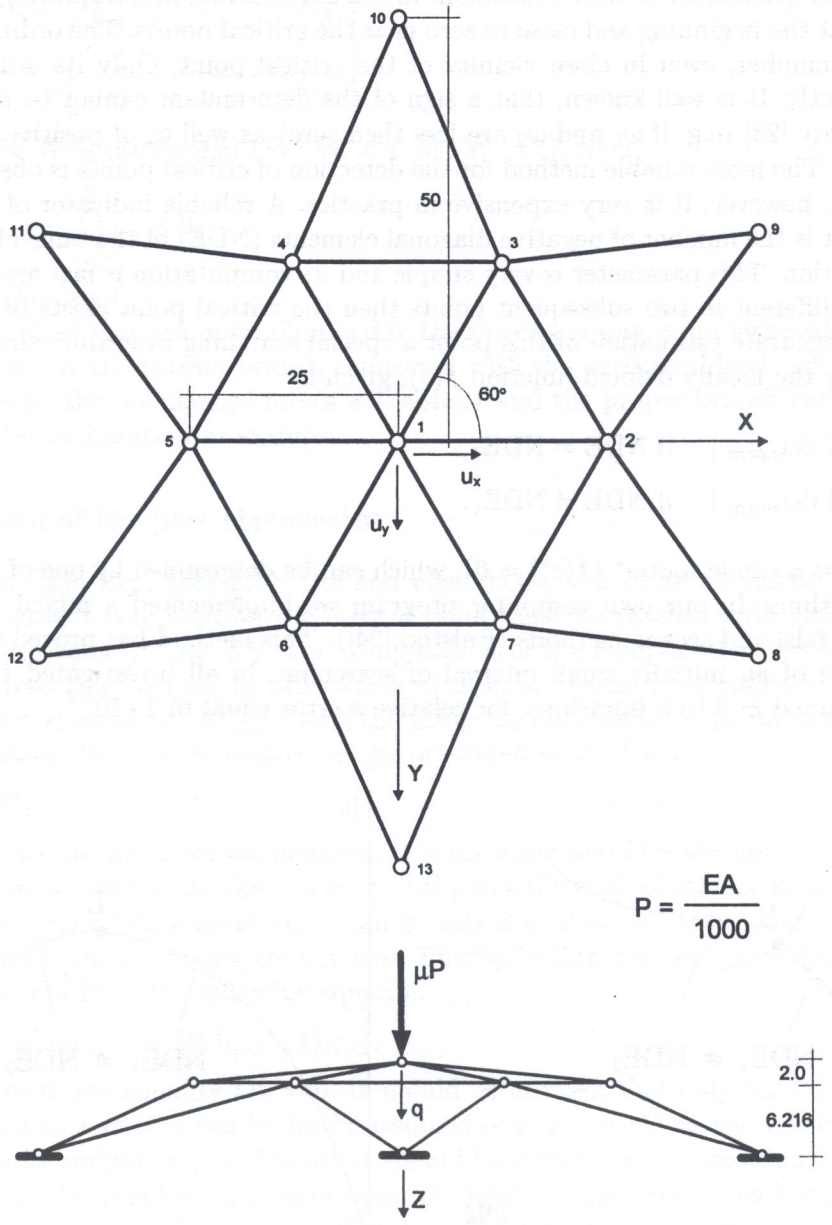

Fig. 7. Shallow truss dome

occur on its equilibrium path. Our investigation is limited here to elastic, geometrically nonlinear analysis. Local buckling of elements is neglected (compare with [28]). The simple two-node element has been used, based on the stationary Lagrange formulation and linear relationship between the Green strain and the second Piola–Kirchhoff stress. Full detailed description of this element can be found in [4].

In the first stage of computations the primary equilibrium path, shown in Fig. 8, has been determined. The obtained results are very close to those presented in the paper by Woźniak and Kleiber [27]. The formula (11) with iter_{opt} equal to 5, was used for the automatic control of step

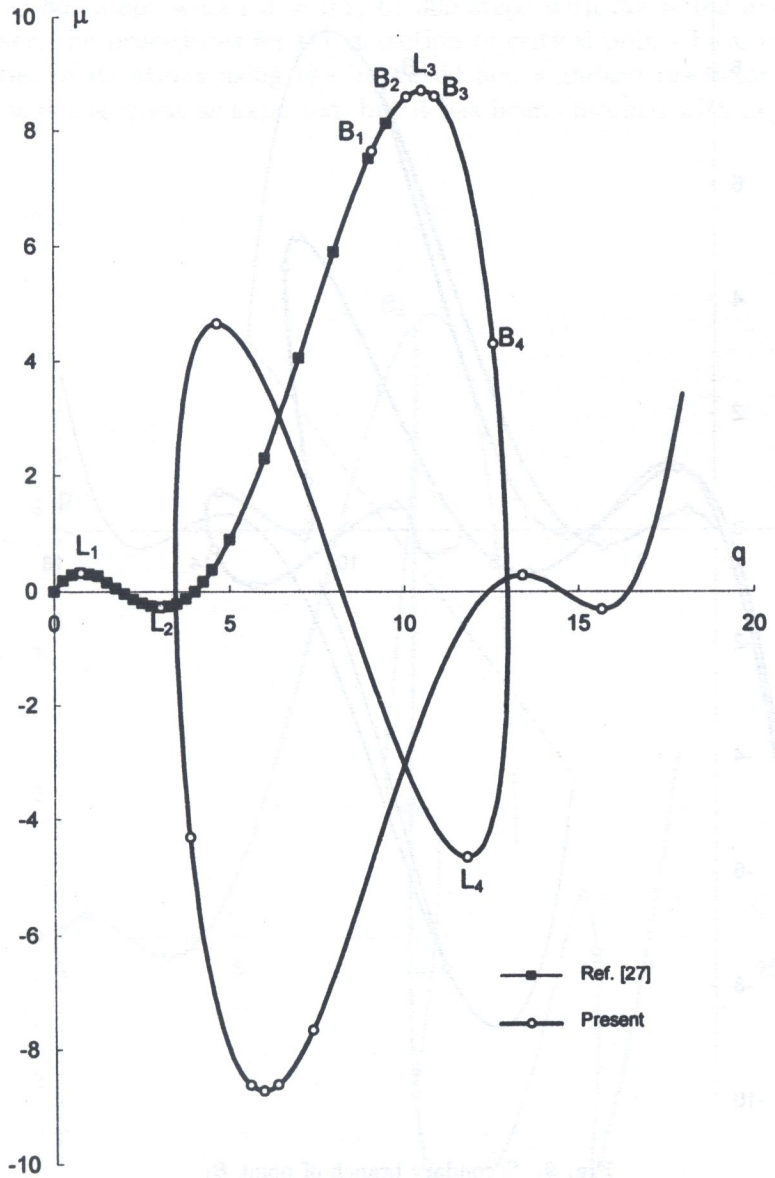

Fig. 8. Primary equilibrium path

length. The critical points are presented in Table 1. All of them were detected by using the NDE parameter and their exact positions were determined by searching for the root of locally defined function (52). There were no problems with the continuation of computations after attainment of critical points. It should be noticed here, that the first and third bifurcation points proved to be multiple ($\omega_i = 0$ for $i = 1$ and 2). It was found that the difference between NDE_1 and NDE_2 is

equal to 2 there. In such cases, the sign of the determinant does not change. Therefore, it is not a sufficient indicator of stability. The NDE parameter should be used instead. It is equal to zero for positive defined matrices and greater than zero otherwise.

The succeeding computations have been made to determine the secondary branches of some bifurcation points. Analysing the first of them (point B_1 in Fig. 9), the double singularity was removed by an additional boundary condition, imposed on the horizontal displacement of the central node. After such an intervention, the computations could be continued on the secondary branch

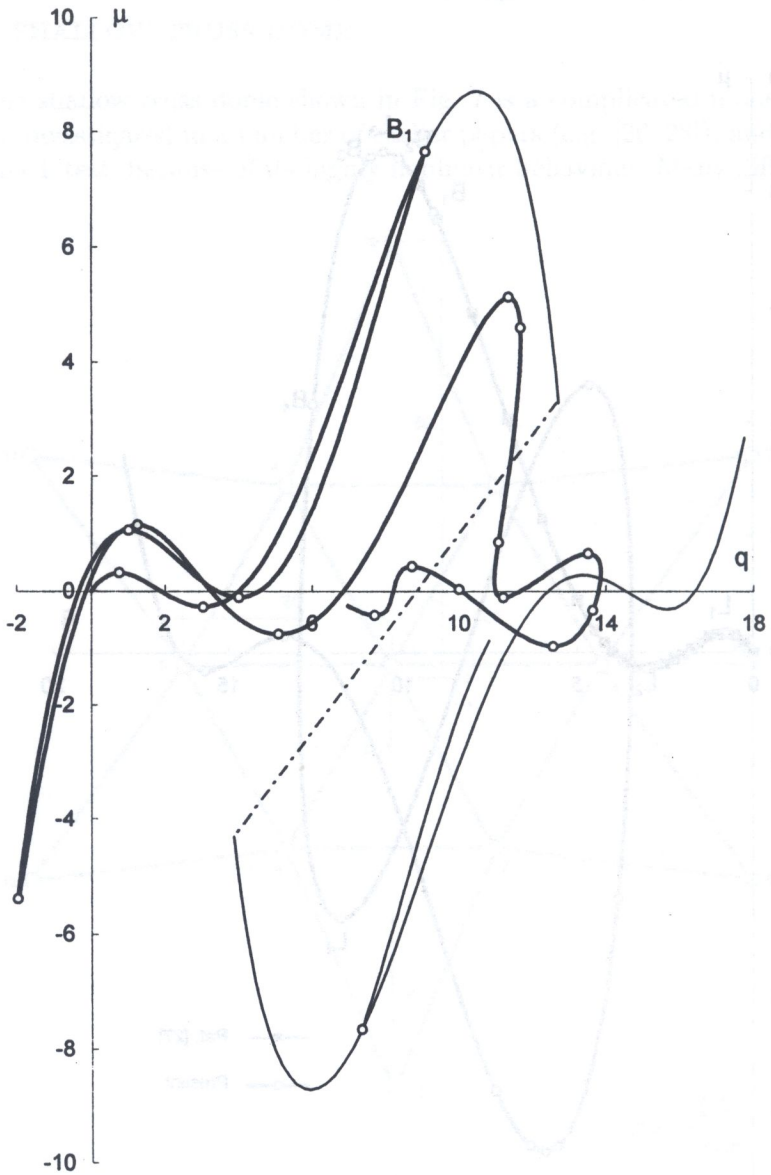

Fig. 9. Secondary branch of point B_1

without great difficulties. At the beginning, the first eigenvector $\mathbf{v}$ was computed (see Sec. 3.4) to determine the predictor. There was no need to use complicated Eqs. (36), because the investigated point proved to be symmetrical. However, it was checked in this case that indeed $H_{ijk}v_iv_jv_k$ is a negligibly small number. To pass onto the secondary branch, the temporary load imperfection technique, described in Sec. 3.3.1, was used. The secondary branches were investigated in both directions, i.e. for $\dot{\mathbf{q}} = +\mathbf{v}$ and $\dot{\mathbf{q}} = -\mathbf{v}$. In the presented subspace (Fig. 9), the projections of these branches superpose.

Similar computations have been made for the other two bifurcation points (B_2 in Fig. 10, and B4 in Fig. 11). As can be seen, many critical points appear also on the secondary branches (these are marked by circles). There are no obstacles to investigate these points in the same way as in the case of those on the primary branch: it is presented in Figs. 10 and 11. Of course, in practice, such a detailed analysis is unnecessary. The first critical point is the most important. Nevertheless, the above examples were performed here to show the efficiency and reliability of the presented algorithms.

On the occasion of this example, the effectiveness of the improved predictor (Sec. 2.3) has been checked. In comparative tests, the primary equilibrium path was determined by using arbitrarily fixed increments a) 600 steps with $\Delta s = 0.1$, b) 300 steps with $\Delta s = 0.2$ and c) 200 steps with $\Delta s = 0.3$. Moreover, the procedures for the detection of critical points have been turned off. The ratio of the number of iterations using the improved and standard predictor was within 0.66 to 0.78. The benefit is not as great as expected, but it has been obtained with negligibly small effort.

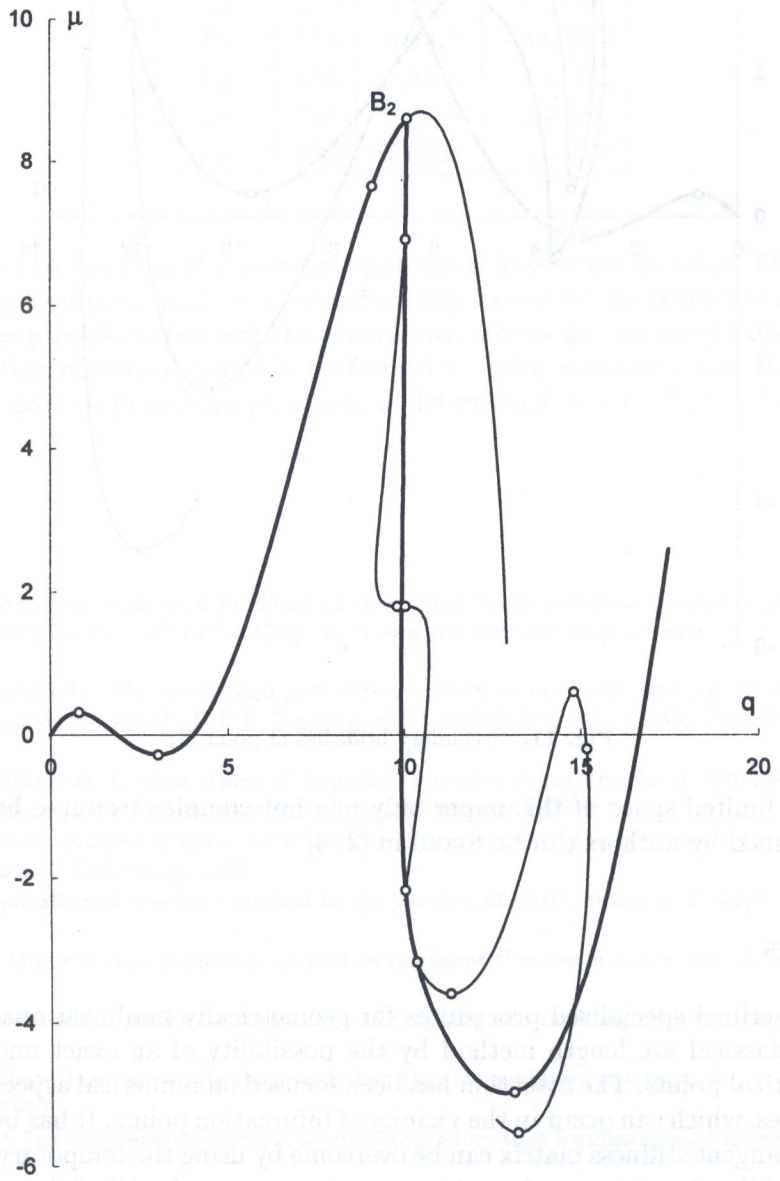

Fig. 10. Secondary branches of point B_2

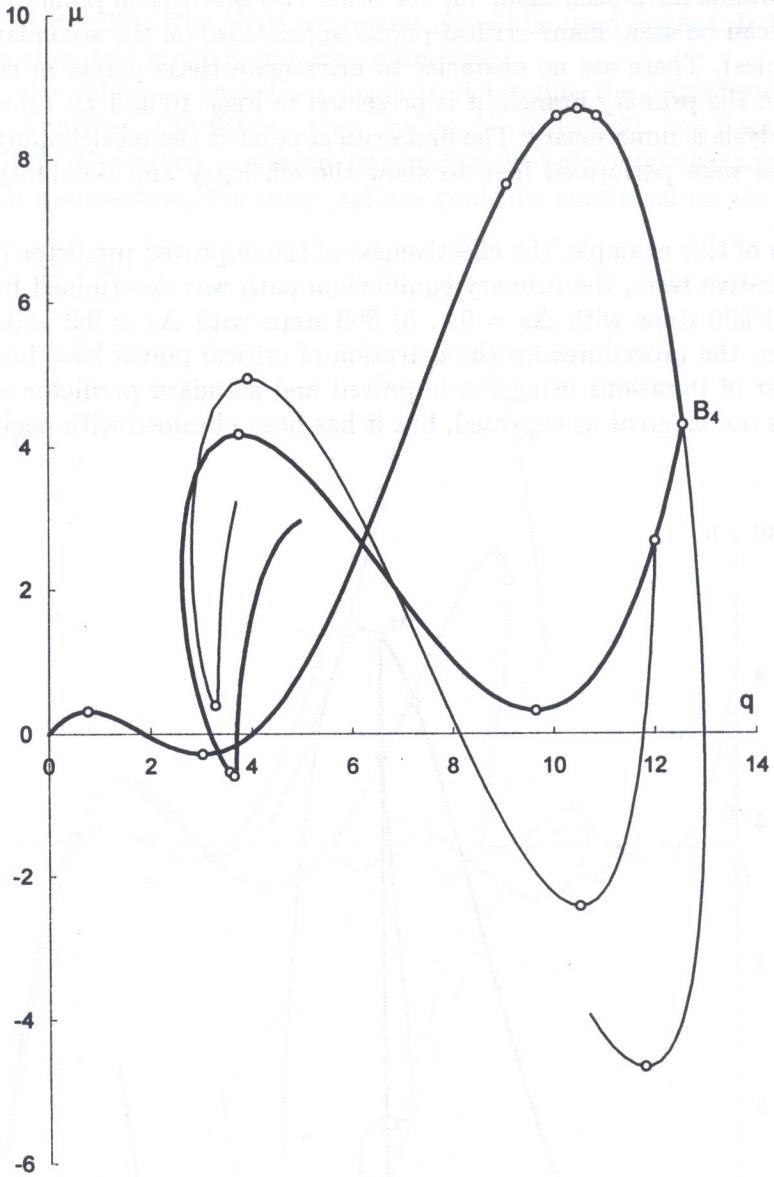

Fig. 11. Secondary branches of point B_4

Because of the limited space of this paper only one but complex example has been presented. Other tests performed by authors can be found in [2, 4].

5. Conclusions

The paper has described specialized procedures for geometrically nonlinear analysis. They are an extension of the classical arc length method by the possibility of an exact and relatively simple examination of critical points. The attention has been focused on numerical aspects of computations and some difficulties, which can occur in the vicinity of bifurcation points. It has been shown that the singularity of the tangent stiffness matrix can be overcome by using the temporary load imperfection technique. Its reliability has been confirmed in several test examples [4]. Other methods presented in this paper proved to be very effective in spite of their simplicity. They are the result of authors' researches connected with developing of own computer program, however most of the described

Table 1. Critical points on the primary branch

crit. p.	step	q	μ
L_1	5	0.7686	0.31558
L_2	16	3.0279	−0.27605
B_1	47	9.0965	7.65387
B_2	51	10.0992	8.60963
L_3	53	10.5128	8.71532
B_3	55	10.8872	8.61690
B_4	66	12.5741	4.30916
L_4	88	11.7873	−4.65750
L_5	114	4.6447	4.65750
B_5	140	3.8579	−4.30916
B_6	151	5.5448	−8.61689
L_6	153	5.9192	−8.71532
B_7	155	6.3328	−8.60963
B_8	159	7.3355	−7.65387
L_7	181	13.4041	0.27605
L_8	193	15.6634	−0.31558

procedures can also be found in the literature published in two last decades. The main conclusion is that even highly nonlinear problems can effectively be solved by relatively simple algorithms. For example, in the proposed approach the general procedures for eigenproblems are not necessary, because the detection of critical points is performed by NDE parameter and the first eigenvector, which plays a key part in branching problems, is determined directly from a homogenous system of equations.

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