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H-adaptive Element-Free Trefftz Method

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The Element-Free Trefftz method can solve the problem only by taking the collocation points on the boundary when the domain under consideration is governed by the linear and homogeneous differential equation. Only the coordinates and the boundary-specified values on the boundary collocation points are required as the input data and therefore, input data generation is much simpler than the other solution procedures. However, the computational accuracy is strongly dependent on the positions of the collocation points. For determining the positions with the desired accuracy, this paper presents h-adaptive scheme for the placement of the collocation points. Global and local error estimators are defined by the residuals of the boundary conditions. The refinement of the positions is performed so that new collocation points are placed in the center of the boundary segments with larger local error estimators than the global estimator. The present scheme is applied to the two-dimensional potential problem in order to confirm its validity.

1. Introduction

The Trefftz Method was first presented by E. Trefftz [1] in 1926 and since then, many researchers have been studying it, e.g., [2–7]. In this method, the solution of a problem is approximated by the superposition of the regular T-complete functions satisfying the governing equation. The unknown parameters are determined so that the approximate solution satisfies the boundary condition by means of the collocation, the least square or the Galerkin method. This paper employs the collocation method. The collocation Trefftz method requires as input data only the coordinates and the boundary-specified values in the boundary collocation points. Therefore, the input data generation is very easy. Besides, since the integrals on the boundary elements are not performed to construct the coefficient matrices, the algorithm is simple and the computational efficiency is very high. So we shall call it the "Element-Free Trefftz Method". However, the computational accuracy is strongly dependent on the positions of the collocation points. For determining the positions with a desired accuracy, this paper presents the adaptive placement scheme of the collocation points. Similar schemes were already presented by Xiaoping *et al.* [8] for the plate bending problem, and by Kita *et al.* [9] for the two-dimensional potential problem with the Dirichlet condition. In this paper, a two-dimensional potential problem with the complex condition is considered as an example. Since, in this case, the potential or the flux error yields on different part of the boundary, different kinds of the errors, the potential and flux errors must be compared in the same manner. For this purpose, this paper presents the error estimation scheme using a weight parameter, which is simple but very efffective.

In the present method, the global and local error estimators are defined by the residuals of the boundary conditions. If the global error etimator is small enough, the process is terminated. If it is not so, the positions of the collocation points are refined. The refinement is performed so that the new collocation points are placed in the center of the elements on which local error

estimators are greater than the global error estimator. Finally, the present method is applied to the two-dimensional potential problem in order to confirm the validity.

2. Trefftz formulation

2.1. T-complete function

We shall consider the two-dimensional potential problem. The governing equation and the boundary conditions are presented in the form

$$\nabla^2 u = 0 \quad \text{in } \Omega \tag{1}$$

and

$$\begin{aligned} u &= \bar{u} && \text{on } \Gamma_u\,, \\ q &\equiv \frac{\partial u}{\partial n} = \bar{q} && \text{on } \Gamma_q\,, \end{aligned} \tag{2}$$

where Ω, Γ_u and Γ_q ($\Gamma_u + \Gamma_q = \Gamma$) are the domain under consideration, the potential-specified and the flux-specified boundaries, respectively. Symbol $(\bar{})$ denotes the value prescribed at the boundary and n — the unit outward normal vector to the boundary.

The Trefftz method is formulated by introducing the regular T-complete functions. The T-complete functions of the two-dimensional potential problem in a bounded domain are represented by [10–12]

$$\begin{aligned} u_n^* &= \{r^n \mathrm{e}^{jn\theta}\} \\ &= \{r^n \cos n\theta,\, r^n \sin n\theta\}\,, \quad n = 1, 2, 3, \ldots\,, \end{aligned} \tag{3}$$

where r and θ are plane polar coordinates and j denotes the imaginary unit.

Besides, the outer normal derivatives of the T-complete functions are given as follows:

$$u_n^* = \{n r^{n-1} \mathrm{e}^{jn\theta}(n_r + jr n_\theta)\}\,, \quad n = 1, 2, 3, \ldots\,, \tag{4}$$

where $n_r \equiv \partial r/\partial n$ and $n_\theta \equiv \partial\theta/\partial n$. On an arbitrary boundary, the boundary profile is approximated by such functions as the spline function and Bezier function in order to calculate n_r and n_θ.

2.2. Element-free Trefftz formulation

The potential u is approximated by superposition of the T-complete functions:

$$u \simeq \tilde{u} = a_1 u_1^* + a_2 u_2^* + \ldots + a_N u_N^* = \mathbf{a}^{\mathrm{T}} \mathbf{u}^*\,, \tag{5}$$

where $\mathbf{u}^*$ and $\mathbf{a}$ are vectors of the T-complete functions and the unknown parameters, respectively. N is the total number of the T-complete functions. Differentiating the above equation in the direction normal to the boundary, we obtain the approximate equation of the flux:

$$q \simeq \tilde{q} \equiv \frac{\partial \tilde{u}}{\partial n} = a_1 q_1^* + a_2 q_2^* + \ldots + a_N q_N^* = \mathbf{a}^{\mathrm{T}} \mathbf{q}^*\,. \tag{6}$$

Equations (5) and (6) do not satisfy the boundary conditions. Substituting Eqs. (5) and (6) into Eq. (2), we have

$$\begin{aligned} R_u &\equiv \tilde{u} - \bar{u} = \mathbf{a}^{\mathrm{T}} \mathbf{u}^* - \bar{u} \neq 0 && \text{on } \Gamma_u\,, \\ R_q &\equiv \tilde{q} - \bar{q} = \mathbf{a}^{\mathrm{T}} \mathbf{q}^* - \bar{q} \neq 0 && \text{on } \Gamma_q\,. \end{aligned} \tag{7}$$

In the Element-Free Trefftz method, the residuals R_u and R_q must vanish at the collocation points P_i. From Eq. (7), we have

$$\begin{aligned} R_u(P_i) &= \mathbf{a}^{\mathrm{T}}\mathbf{u}^*(P_i) - \bar{u}(P_i) = 0\,, \quad i = 1,\ldots,M_1, \quad P_i \in \Gamma_u\,, \\ R_q(P_i) &= \mathbf{a}^{\mathrm{T}}\mathbf{q}^*(P_i) - \bar{q}(P_i) = 0\,, \quad i = 1,\ldots,M_2, \quad P_i \in \Gamma_q\,, \end{aligned} \tag{8}$$

where M_1 and M_2 are the total numbers of the collocation points placed on Γ_u and Γ_q, respectively. Rearranging the above equations, we obtain

$$\begin{aligned} \mathbf{a}^{\mathrm{T}}\mathbf{u}^*(P_i) &= \bar{u}(P_i)\,, \quad P_i \in \Gamma_u\,, \\ \mathbf{a}^{\mathrm{T}}\mathbf{q}^*(P_i) &= \bar{q}(P_i)\,, \quad P_i \in \Gamma_q\,, \end{aligned} \tag{9}$$

or

$$\begin{bmatrix} u^*_{11} & u^*_{12} & \cdots & u^*_{1N} \\ u^*_{21} & u^*_{22} & \cdots & u^*_{2N} \\ \vdots & \cdots & & \vdots \\ u^*_{M_1 1} & u^*_{M_1 2} & \cdots & u^*_{M_1 N} \\ q^*_{11} & q^*_{12} & \cdots & q^*_{1N} \\ q^*_{21} & q^*_{22} & \cdots & q^*_{2N} \\ \vdots & \cdots & & \vdots \\ q^*_{M_2 1} & q^*_{M_2 2} & \cdots & q^*_{M_2 N} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{Bmatrix} = \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_{M_1} \\ \bar{q}_1 \\ \bar{q}_2 \\ \vdots \\ \bar{q}_{M_2} \end{Bmatrix}, \tag{10}$$

where $u^*_j(P_i) \equiv u^*_{ij}$, $q^*_j(P_i) \equiv q^*_{ij}$, $\bar{u}(P_i) \equiv \bar{u}_i$ and $\bar{q}(P_i) \equiv \bar{q}_i$, respectively.

$$\mathbf{K}\mathbf{a} = \mathbf{f}\,. \tag{11}$$

Equation (11) is solved for $\mathbf{a}$, which is substituted into Eqs. (5) and (6) in order to calculate the potential and flux distributions.

2.3. Key-points for accurate analysis

2.3.1. Corner points

The boundary points at which the boundary conditions are not continuous are called corner points. Placement of the collocation points on the corner points strongly affects the computational accuracy. In this study, two kinds of placement schemes on the corner points are employed; the coincident and the near-by placement schemes. In the coincident scheme, two collocation points with different normal vectors are placed in the same corner point. According to the near-by scheme, two collocation points are placed on the boundary segments neighboring a corner point. The former scheme is employed in the cases when

- The type of boundary conditon changes at the corner point, or
- The Neumann conditions are specified on both sides of the corner points and moreover, normal vectors are not unique at those corner points.

In the other cases, the latter scheme is employed.

2.3.2. Condition number of matrices

T-complete functions are the power functions of the distance between the origin and the collocation point. Therefore, the condition numbers of the matrices become very large when the domains under consideraton are very small. In order to overcome this difficulty, this paper employs the following coordinates transformation technique. The same technique has been already presented by Jirousek [6].

- The domain is moved so that the origin is placed in the centroid.
- The domain is scaled down so that the average distance between the centroid and the collocation points is equal to unity.

Using this scheme, the coordinates of the collocation point in the original system (x_i, y_i) are transformed to:

$$\acute{x}_i = \frac{x_i - x_c}{D} \quad , \quad \acute{y}_i = \frac{y_i - y_c}{D} , \tag{12}$$

where x_c and y_c are the coordinates of the centroid in the original system, and D is the scaling parameter. Morover,

$$x_c = \frac{1}{M} \sum_{i=1}^{M} x_i ,$$

$$y_c = \frac{1}{M} \sum_{i=1}^{M} y_i ,$$

$$D = \frac{1}{M} \sum_{i=1}^{M} \sqrt{x_i^2 + y_i^2} .$$

3. Adaptive placement scheme of collocation points

3.1. Error estimators

3.1.1. Problems with Dirichlet or Neumann condition

Since in the Elemet-Free Trefftz method, the residuals vanish only on the boundary collocation points, residuals appear on the other boundary points. Therefore, the residual distributions on the boundary give a good estimation of the errors. In our previous study [9], we have defined the following error estimators for the Dirichlet problem:

$$\eta_i^2 = \frac{1}{L_i} \int_{\Gamma_i} R_u^2 \, \mathrm{d}\Gamma , \tag{13}$$

$$\eta^2 = \frac{1}{L} \int_{\Gamma} R_u^2 \, \mathrm{d}\Gamma = \frac{1}{L} \sum_{i=1}^{M} L_i \eta_i^2 , \tag{14}$$

where η_i is the local error estimator, which estimates the error distribution on the boundary segment, and η is the global error estimator. Γ_i denotes the boundary segment between the collocation points P_i and P_{i+1}. L and L_i denote the lengths of Γ and Γ_i, respectively. Numerical investigations show that these estimators lead to a good error estimation for the Dirichlet problem.

As the extention of the above estimators, we can define the estimators for the Neumann problem:

$$\eta_i^2 = \frac{1}{L_i} \int_{\Gamma_i} R_q^2 \, \mathrm{d}\Gamma , \tag{15}$$

$$\eta^2 = \frac{1}{L} \int_{\Gamma} R_q^2 \, \mathrm{d}\Gamma = \frac{1}{L} \sum_{i=1}^{M} L_i \eta_i^2 . \tag{16}$$

Although the numerical investigations are not performed, one can easily predict, from the results of the Dirichlet problem, that the above estimators will give good results for the error estimation of the Neumann problem.

3.1.2. Problem with complex condition

As the extention of the above estimators, we can define the estimators for the problem with the complex condition:

$$\eta_i^2 = \frac{1}{L_i} \begin{cases} \int_{\Gamma_i} R_u^2 \, \mathrm{d}\Gamma & \text{on } \Gamma_u \,, \\ \int_{\Gamma_i} R_q^2 \, \mathrm{d}\Gamma & \text{on } \Gamma_q \,. \end{cases} \tag{17}$$

In this case, there exist two errors: the potential and flux errors. Therefore, we must introduce the weight parameters.

The residuals of the boundary conditions are defined by Eq. (7), i.e.,

$$\begin{aligned} R_u &\equiv \mathbf{a}^{\mathrm{T}}\mathbf{u}^* - \bar{u} \,, \\ R_q &\equiv \mathbf{a}^{\mathrm{T}}\mathbf{q}^* - \bar{q} \,. \end{aligned} \tag{18}$$

The functions are approximated by N terms of the T-complete functions. Therefore, the truncated terms lead to the residuals and their maximum components are the $(N+1)$-th terms. We shall represent the residual by its maximum components:

$$R_u \simeq (R_u)_{\max} = a_{N+1} u_{N+1}^* \,, \tag{19}$$

$$R_q \simeq (R_q)_{\max} = a_{N+1} q_{N+1}^* \,. \tag{20}$$

Taking the square of both sides of the Eq. (19) and substituting Eq. (3), we have

$$\begin{aligned} |R_u|^2 \simeq |a_{N+1} u_{N+1}^*|^2 &= a_{N+1}^2 [r^{N+1} \mathrm{e}^{j(N+1)\theta}]^2 \\ &= a_{N+1}^2 r^{2(N+1)} \end{aligned}$$

and

$$|R_u| = |a_{N+1}| r^{N+1} \,. \tag{21}$$

In the same manner, taking the square of both sides of the Eq. (20) and substituting Eq. (4), we have

$$\begin{aligned} |R_q|^2 \simeq |a_{N+1} q_{N+1}^*|^2 &= a_{N+1}^2 [(N+1) r^N \mathrm{e}^{jN\theta} (n_r + jr n_\theta)]^2 \\ &= a_{N+1}^2 (N+1)^2 r^{2N} (1 + r^2) \end{aligned}$$

and

$$|R_q| = |a_{N+1}| (N+1) r^N \sqrt{1 + r^2} \,, \tag{22}$$

where $n_r \equiv \partial r / \partial n$ and $n_\theta \equiv \partial \theta / \partial n$.

From Eqs. (21) and (22), we obtain the following relation between R_u and R_q:

$$\frac{|R_u|}{|R_q|} = \alpha \frac{r}{(N+1)\sqrt{1+r^2}} = \alpha' \frac{r}{\sqrt{1+r^2}} \tag{23}$$

where α and α' are constants. In the present scheme, since the number of T-complete functions is constant during the adaptive process, $N+1$ is included in the constant α'.

Using the above relation, we can define the error estimators for the problem with the complex condition:

$$\eta_i = \frac{1}{L_i} \begin{cases} \displaystyle\int_{\Gamma_i} \sqrt{1+r^2}|R_u|\,\mathrm{d}\Gamma & \text{on } \Gamma_u\,, \\ \displaystyle\int_{\Gamma_i} r|R_q|\,\mathrm{d}\Gamma & \text{on } \Gamma_q\,, \end{cases} \tag{24}$$

$$\eta = \frac{1}{L}\sum_{i=1}^{M} L_i\eta_i\,. \tag{25}$$

3.2. Adaptive process

The flow of the adaptive process is as follows:

(1) Input initial data.

Input the following initial data; the profile and the boundary condition of the domain under consideration, the initial placement of the collocation points and the number of T-complete functions.

(2) Trefftz analysis.

Perform the Trefftz analysis to determine the potential and the flux distributions on the boundary.

(3) Error estimation.

Calculate the global and local error estimators from Eqs. (24) and (25).

(4) Convergence judgement.

Estimate the convergency criterion. The criterion is given as

$$\eta < \varepsilon\,, \tag{26}$$

where ε is a parameter, which must be specified by a user in advance.

If the criterion is satisfied, the process is terminated. If it is not so, the positions of the collocation points are refined.

(5) Refinement of placement of collocation points.

The refinement of the placement is performed so that the new collocation points are added in the center of the elements with the larger local error estimators than the global one. This operation is similar to the h-adaptive schemes of the finite and the boundary element methods. Therefore, we shall call this scheme the h-adaptive scheme of the Trefftz method.

After the refinement, the process goes to (2).

4. Numerical examples

We shall consider two-dimensional potential problem. The performance of the present method is estimated by the global error estimator η and the average residuals R_1 and R_2, which are defined as:

$$R_1 = \frac{1}{L_u}\int_{\Gamma_u} |R_u|\,\mathrm{d}\Gamma\,, \quad R_2 = \frac{1}{L_q}\int_{\Gamma_q} |R_q|\,\mathrm{d}\Gamma\,,$$

where L_u and L_q are the lengths of Γ_u and Γ_q, respectively.

The performance of the adaptive scheme is discussed by comparing with the results of uniform placement of the collocation points. In the following numerical results, the results obtained by means of the adaptive and the uniform schemes are referred to the labels "Adaptive" and "Uniform", respectively.

4.1. Example 1

The region and the boundary conditions of the first example are shown in Fig. 1. The initial placement is constructed by 12 collocation points: the coincident collocation points at each corner point, and the other collocation points in the center of each segment. The number of T-complete functions is 10, which is invariant during the process.

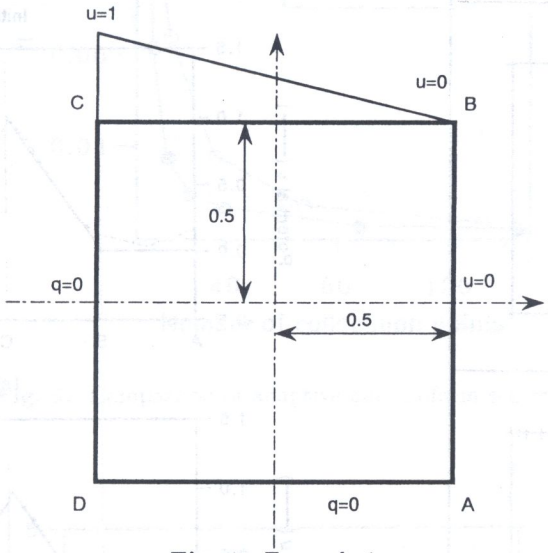

Fig. 1. Example 1

Figure 2 shows η, R_1 and R_2 versus the number of the collocation points. η decreases with the increase of the number of the collocation points. Although R_1 is larger than R_2 in the initial placement, R_1 converges faster than R_2. Figures 3 and 4 indicate the placements of the collocation

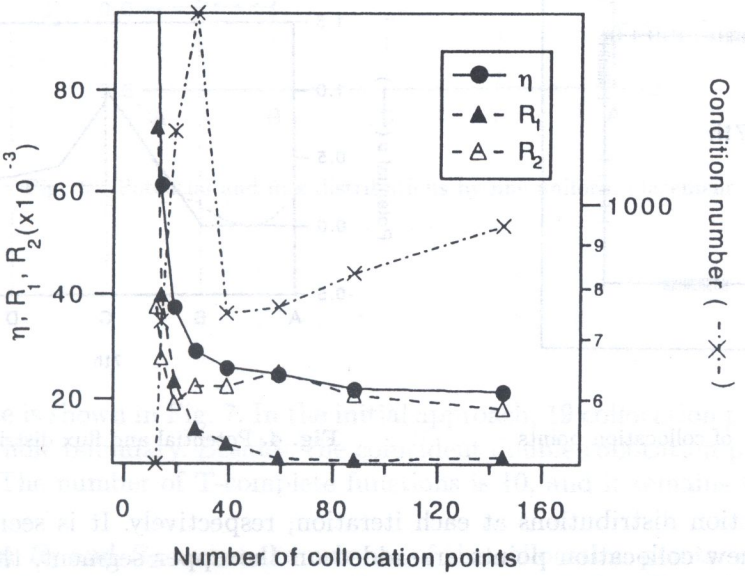

Fig. 2. Global error estimator and average residuals

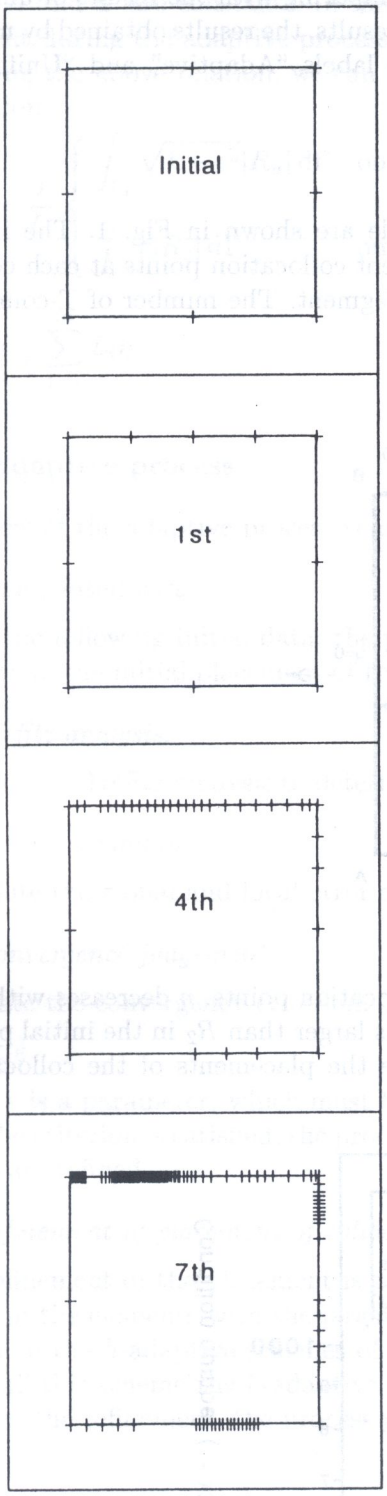

Fig. 3. Placement of collocation points

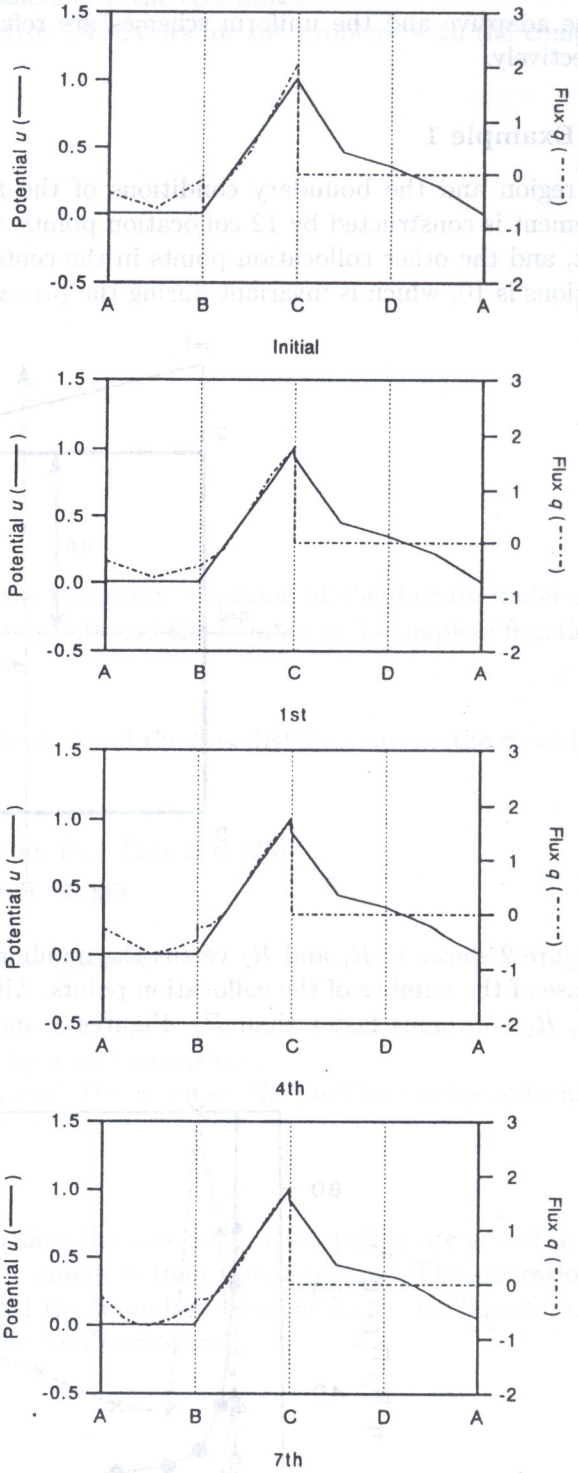

Fig. 4. Potential and flux distributions

points and the function distributions at each iteration, respectively. It is seen from Fig. 3, that when most of the new collocation points are added on the upper segment, their number on the left-hand segment is invariant during the refinement. Figure 5 shows the convergence curves of the global error estimators obtained by the adaptive and uniform schemes. This figure indicates that

the convergence of the adaptive scheme is faster than that of the uniform scheme. Besides, Figure 6 presents the potential and flux distributions estimated by the uniform placement of 132 collocation points on the whole boundary. Comparing Fig. 6 with the results obtained by the fourth placement shown in Fig. 4, we notice that both results are very similar, although only 39 collocation points were employed for computation in the adaptive process.

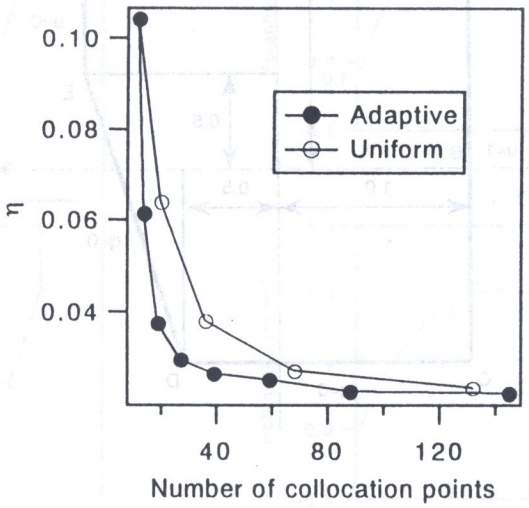

Fig. 5. Comparison of adaptive and uniform schemes

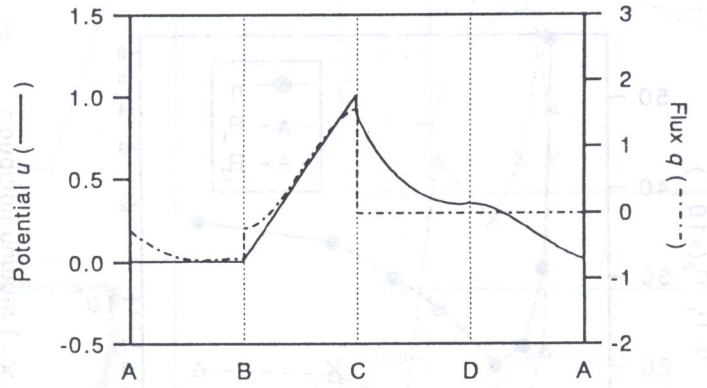

Fig. 6. Potential and flux distributions by fine uniform placement

4.2. Example 2

The second example is shown in Fig. 7. In the initial approach, 19 collocation points are distributed uniformly on the whole boundary. Besides, the coincident double collocation points are selected at the corner points. The number of T-complete functions is 10, and it remains invariant during the adaptive process.

Figure 8 shows η, R_1 and R_2 versus the number of the collocation points. η decreases until the third iteration and then, it begins to increase. This is because the coefficient matrices become ill-posed according to the increase of the collocation points and the computational errors increase.

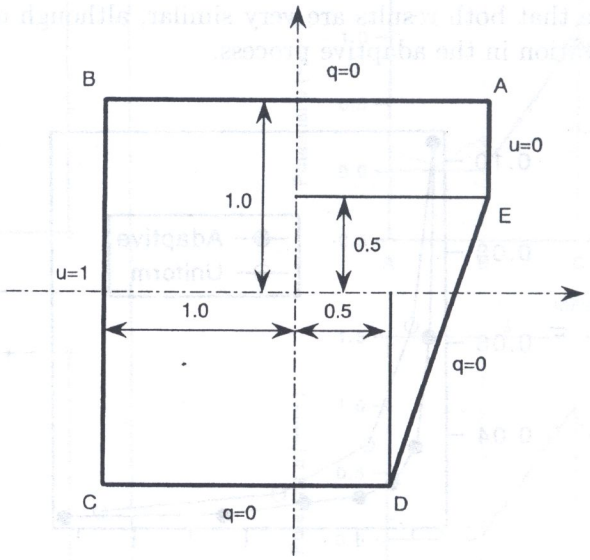

Fig. 7. Example 2

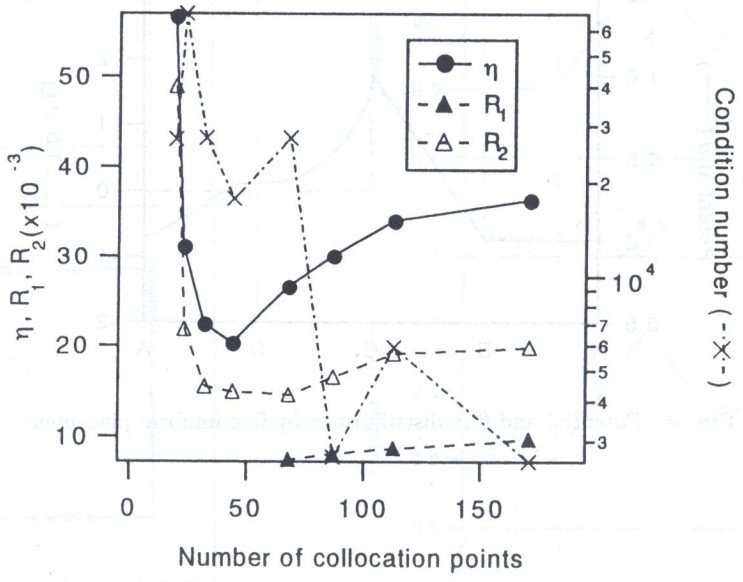

Fig. 8. Global error estimator and average residuals

Figures 9 and 10 indicate the positions of the collocation points and the function distributions at each iteration. Figure 11 shows the convergence curves of η obtained by the adaptive and uniform schemes. The adaptive scheme leads to more accurate results than the uniform scheme. Besides,

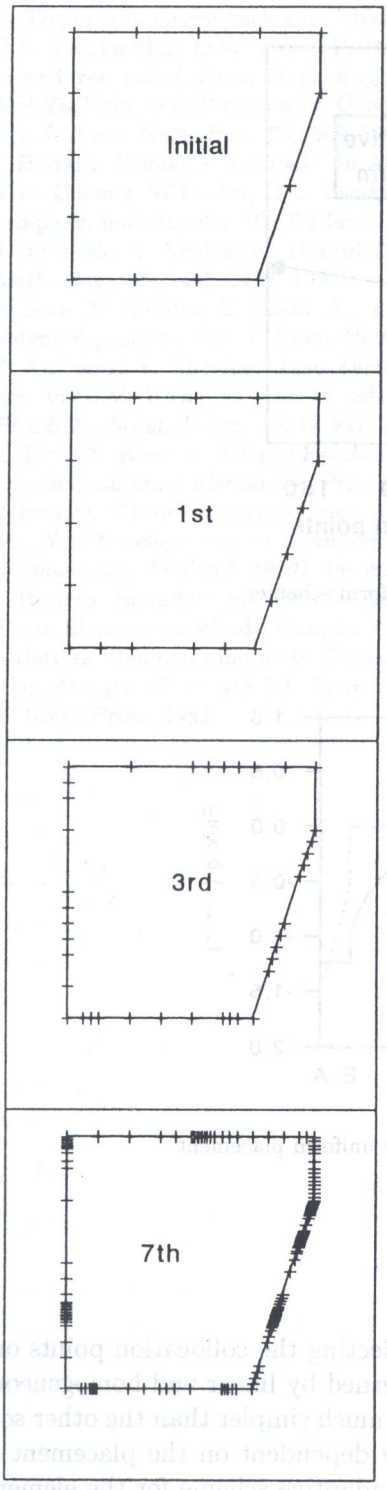

Fig. 9. Placement of collocation points

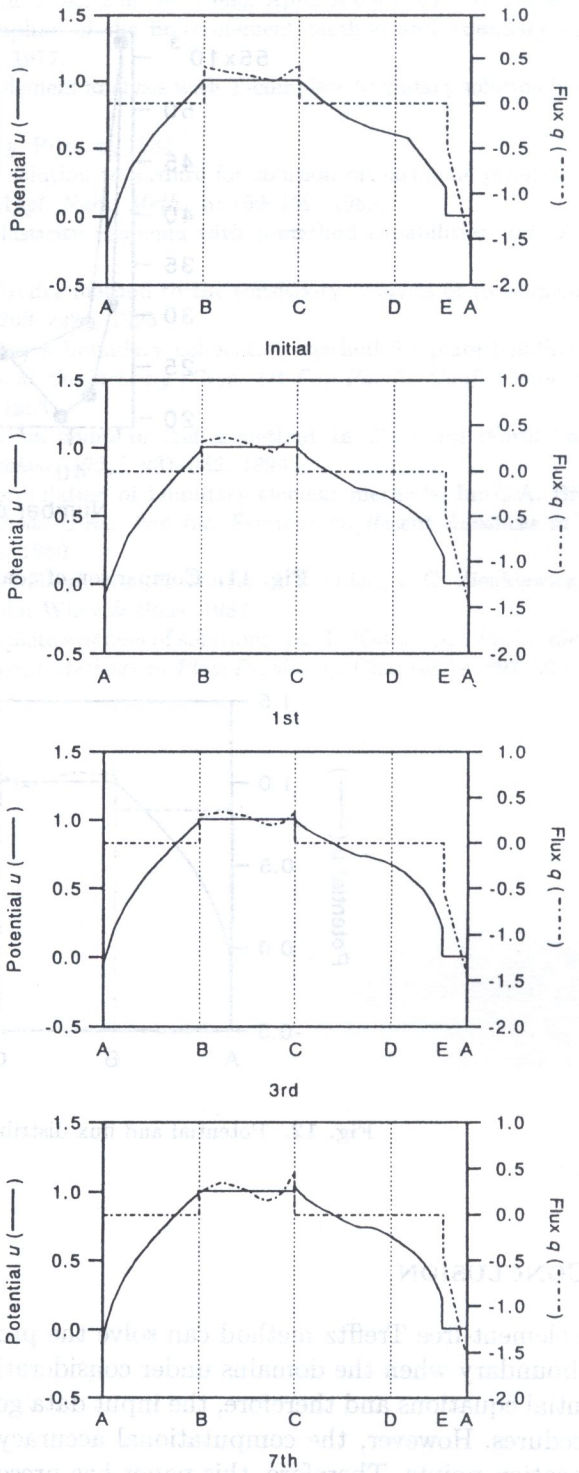

Fig. 10. Potential and flux distributions

Figure 12 indicates the potential and flux distributions by the uniform distribution of 116 collocation points on the whole boundary. Comparing Fig. 12 with the results due to the third placement shown in Fig. 10, we notice that both results are very similar, although the results of the adaptive process are determined by 44 collocation points only.

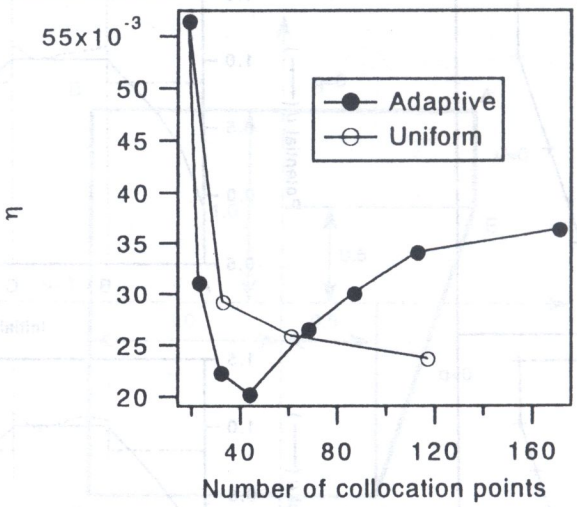

Fig. 11. Comparison of adaptive and uniform schemes

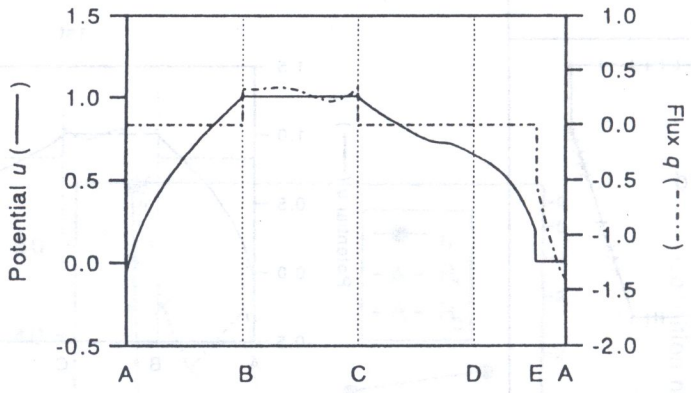

Fig. 12. Potential and flux distributions by fine uniform placement

5. Conclusion

The element-free Trefftz method can solve the problem by selecting the collocation points only on the boundary when the domains under consideration are governed by linear and homogeneous differential equations and therefore, the input data generation is much simpler than the other solution procedures. However, the computational accuracy is strongly dependent on the placement of the collocation points. Therefore, this paper has presented the h-adaptive scheme for the element-free Trefftz method. In this method, the global and local error estimators are defined by the residuals of the boundary conditions. The refinement of the placement is performed so that the new collocation points are added in the center of the boundary segments whose local estimators are larger than the global error estimator. The present method was applied to a two-dimensional potential problem. The numerical results are satisfactory and therefore, we can conclude that the validity of the present method was confirmed. We are planning to improve the refinement scheme in a further study.

REFERENCES

[1] E. Trefftz. Ein Gegenstück zum Ritzschen Verfahren. *Proc. 2nd Int. Cong. Appl. Mech.*, 131–137, Zurich, 1926.

[2] O.C. Zienkiewicz, D.W. Kelly, P. Bettes. The coupling of the finite element method and boundary solution procedures. *Int. J. Num. Meth. Eng.*, **11**: 355–375, 1977.

[3] A.P. Zieliński, O.C. Zienkiewicz. Generalized finite element analysis with T-complete boundary solution function. *Int. J. Num. Meth. Eng.*, **21**: 509–528, 1985.

[4] I. Herrera. *Boundary Methods: An Algebraic Theory.* Pitman, 1984.

[5] Y.K. Cheung, W.G. Jin, O.C. Zienkiewicz. Direct solution procedure for solution of harmonic problems using complete, non-singular, Trefftz functions. *Comm. Appl. Num. Meth.*, **5**: 159–169, 1989.

[6] J. Jirousek, A. Venkatesh. Hybrid-Trefftz plane elasticity elements with p-method capabilities. *Int. J. Num. Meth. Eng.*, **35**: 1443–1472, 1992.

[7] E. Kita, N. Kamiya, Y. Ikeda. An application of Trefftz method to the sensitivity analysis of two-dimensional potential problem. *Int. J. Num. Meth. Eng.*, **38**: 2209–2224, 1995.

[8] Z. Xiaoping, Y. Zhenhan, Liao Zhizhong. An adaptive boundary collocation method for plate bending problem. In: B.M. Kwak, M. Tanaka, eds., *Computational Engineering (Proc. 1st Pan-Pacific Conf. Comp. Eng. - PCCE'93, Seoul, Korea, 1993)*, 221–226. Elsevier, 1993.

[9] E. Kita, N. Kamiya. Adaptive placement of collocation points in Trefftz method. In: *Proc. 3rd World Congress on Computational Mechanics (Proc. WCCM III, Tokyo, 1994)*, 931–932, 1994.

[10] I. Herrera. Theory of connectivity: a systematic formulation of boundary element methods. In: C.A. Brebbia, ed., *New Developments in Boundary Element Methods (Proc. 2nd Int. Seminar on Recent Advances in BEM, Southampton, England, 1980)*, 45–58, Pentch Press, 1980.

[11] I. Herrera. Boundary methods for fluids. In: R.H. Gallagher, D.H. Norrie, J.T. Oden, O.C. Zienkiewicz, eds., Finite Elements in Fluids, Chapter 19, 403–432. John Wiley & Sons, 1982.

[12] I. Herrera. Boundary methods: Development of complete systems of solutions. In: T. Kawai, ed., *Finite Elements Flow Analysis (Proc. 4th Int. Symp. on Finite Element Methods in Flow Problems)*, Chapter 19, 897–906. Univ. of Tokyo Press, 1982.