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Modification of the structural tangent stiffness due to nonlinear configuration-dependent conservative loading

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This paper discusses the effect of deformation-sensitive loading devices. The nature of loading is generally not perfectly *dead*, namely, it is not perfectly independent of the occurring deflections. However, the surface tractions or body forces can show some *variable* characteristics, depending on the actual displacements and causing changes in the classical equilibrium and stability behaviour of the structure. The present analysis concerns the influence of deformation-sensitive loading devices on the structural tangent stiffness.

The configuration-dependent loading devices can be characterized by some load–deflection functions, similarly to the material behaviour characterized by stress–strain functions. The effect of loading seems to be similar to that of the material and consequently, the nonlinear loading processes can be handled similarly to the nonlinear materials in the equilibrium analyses of structures. Thus, we can find that in the tangent stiffness of the structure, beside the *tangent modulus of the material*, the *tangent modulus of the load* appears.

In this paper, the tensorial approach is followed by application to discrete model and the paper is concluded by numerical examples.

1. Introduction

Loading devices are analysed in [14, page 188] where dead, rigid and semi-rigid loading devices are distinguished. Dead load is a force type-loading, rigid load is a displacement-type loading, while the semi-rigid load means a mixed loading.

Fundamental aspects and classification of loading types are discussed in [1, pages 207 and 224] by distinguishing the term "dead" and "variable" load. Variable conservative loading process assumes the applied load to be dependent on the occurring deflections, but independent of the properties of the structure. This kind of loading device can be specified by a load–deflection function. Here we focus on that type of a variable load, when the force-type loading device is governed by a given load–deflection function.

Dead-type loading device assumes the applied load to be independent of the occurring deflections of the structure, namely, it is characterized by a constant load–deflection diagram. Naturally, in this case, during a loading process, the load F can be increased or decreased by a scalar load parameter λ, namely $F = F(\lambda) = \lambda F_0$, thus $\mathrm{d}F = (\mathrm{d}F/\mathrm{d}\lambda)\mathrm{d}\lambda = \mathrm{d}\lambda F_0$, while the constant characteristics of the load–deflection function is changeless (Fig. 1a).

Variable or *configuration-dependent type loading* assumes the applied load to be dependent on the occurring deflections, characterized by a variable, linear or nonlinear, load–deflection diagram. In this case, $F = F(\lambda, u) = \lambda F_0 + f(u)$, namely, the load is divided into two parts: the *controllable* part λF_0 governed by the load parameter λ, and the *deformation-sensitive* part $f(u)$ specified as a linear or nonlinear function. During a loading process, the controllable part of the load can be increased or decreased by the load parameter λ, while the original characteristics of the load–deflection function of the deformation-sensitive part is changeless (Fig. 1b and c). In the case of a linear variable load, the load function can be specified as $F = F(\lambda, u) = \lambda F_0 + f_l u$ in which $f_l = f$ is the constant *load modulus*, consequently $\mathrm{d}F = (\partial F/\partial\lambda)\mathrm{d}\lambda + (\partial F/\partial u)\mathrm{d}u = \mathrm{d}\lambda F_0 + f\mathrm{d}u$

(Fig. 1b). In the case of nonlinear variable load, the load function is $F = F(\lambda, u) = \lambda F_0 + f(u)$, thus, similarly to the nonlinear material, incremental analysis is needed, based on the *load tangent modulus* f_t (Fig. 1c).

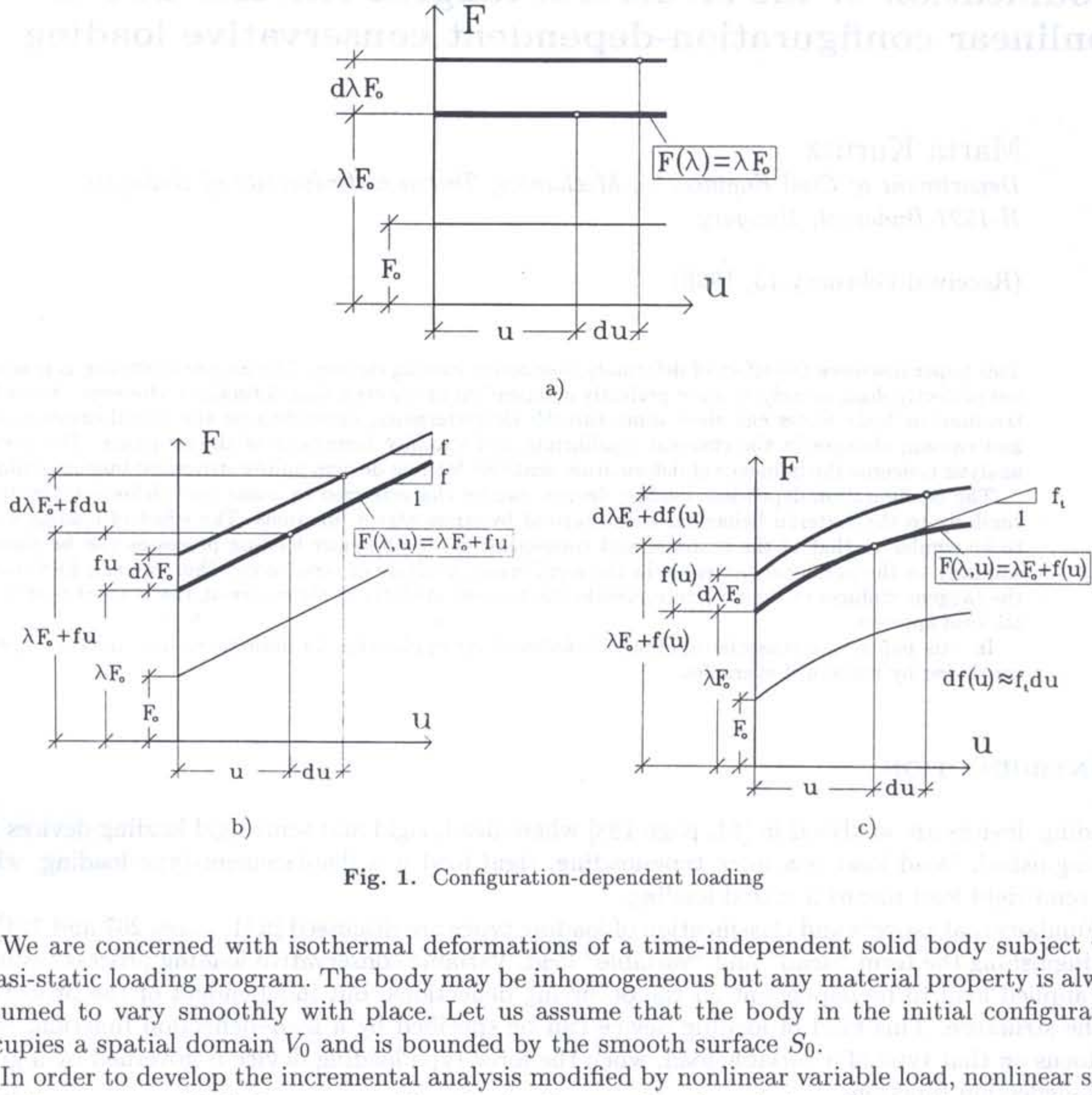


Fig. 1. Configuration-dependent loading

We are concerned with isothermal deformations of a time-independent solid body subject to a quasi-static loading program. The body may be inhomogeneous but any material property is always assumed to vary smoothly with place. Let us assume that the body in the initial configuration occupies a spatial domain V_0 and is bounded by the smooth surface S_0.

In order to develop the incremental analysis modified by nonlinear variable load, nonlinear state variables are specified. Let us consider, in the Lagrangian description, S_{ij} as the second Piola-Kirchhoff stress tensor and E_{ij} as the nonlinear Lagrange–Green strain tensor. Let the material be described by a *nonlinear material function* $S_{ij}(E_{mn})$, then

$$\mathrm{d}S_{ij} = \frac{\partial S_{ij}(E_{mn})}{\partial E_{kl}}\,\mathrm{d}E_{kl} = D^{\mathrm{t}}_{ijkl}(E_{mn})\,\mathrm{d}E_{kl}\,, \tag{1}$$

where $D^{\mathrm{t}}_{ijkl}(E_{mn})$ is the instantaneous *material tangent modulus tensor*, by means of which the nonlinear material behaviour can be handled in an *incrementally linear* form.

Let us assume that in the volume V_0 the body forces F_i are applied, on a nonzero part $S_{\mathrm{p}0}$ of the surface S_0 the surface tractions P_i are given, while on the remaining part $S_{\mathrm{u}0}$ the displacements ν_i are given. Let us assume the scalar loading parameter λ to be varied continuously and infinitely slowly in time. An initial equilibrium state of the body at a certain value of λ is regarded to be known. Thus, for dead load we have $F_i = F_i(\lambda) = \lambda F_{0i}$, and $P_i = P_i(\lambda) = \lambda P_{0i}$ consisting of controllable parts only, while for variable load these functions are completed by the linear or nonlinear functions of the deformation-sensitive parts of the loading, namely $F_i = F_i(\lambda, u_k) = \lambda F_{0i} + f_i(u_k)$ and $P_i = P_i(\lambda, u_k) = \lambda P_{0i} + p_i(u_k)$. In the case of linear loading functions, we can specify $f_i(u_k) = m_{ij}u_j$ and $p_i(u_k) = n_{ij}u_j$ where tensors m_{ij} and n_{ij} are independent of u_k, containing constant volume and surface load moduli, namely the load constants, like the Young modulus of the linear elastic materials. By applying nonlinear variable load, similarly to the material in (1), we need incremental form expressions

$$\begin{aligned} \mathrm{d}F_i &= \frac{\partial F_i(u_k,\lambda)}{\partial \lambda}\,\mathrm{d}\lambda + \frac{\partial F_i(u_k,\lambda)}{\partial u_j}\,\mathrm{d}u_j = \mathrm{d}\lambda\, F_{0i} + M^{\mathrm{t}}_{ij}(u_k)\,\mathrm{d}u_j\,, \\ \mathrm{d}P_i &= \frac{\partial P_i(u_k,\lambda)}{\partial \lambda}\,\mathrm{d}\lambda + \frac{\partial P_i(u_k,\lambda)}{\partial u_j}\,\mathrm{d}u_j = \mathrm{d}\lambda\, P_{0i} + N^{\mathrm{t}}_{ij}(u_k)\,\mathrm{d}u_j\,, \end{aligned} \tag{2}$$

where the tensors $M^{\mathrm{t}}_{ij}(u_k)$ and $N^{\mathrm{t}}_{ij}(u_k)$ represent the instantaneous *load tangent moduli* by which the structural tangent stiffness will be modified.

2. Principle of incremental virtual work applied to configuration-dependent loading

Based on the difference of the n-th and n+1-th equilibrium configurations, the linearized incremental form of the virtual work for any type of loading reads [2, page 129]

$$\begin{aligned} &\int_{V_0} \mathrm{d}S_{ij}\,\delta E_{ij}\,\mathrm{d}V_0 + \int_{V_0} S_{ij}\,\mathrm{d}\delta E_{ij}\,\mathrm{d}V_0 - \int_{V_0} \mathrm{d}F_i\,\delta u_i\,\mathrm{d}V_0 - \int_{V_0} F_i\,\mathrm{d}\delta u_i\,\mathrm{d}V_0 \\ &\qquad - \int_{S_{\mathrm{p}0}} \mathrm{d}P_i\,\delta u_i\,\mathrm{d}S_0 - \int_{S_{\mathrm{p}0}} P_i\,\mathrm{d}\delta u_i\,\mathrm{d}S_0 = 0\,. \end{aligned} \tag{3}$$

By taking the incrementally linear constitutive transformation (1) into account, in the case of *dead load*, the above expression takes the form

$$\begin{aligned} &\int_{V_0} \mathrm{d}E_{ij} D_{ijkl}\,\delta E_{kl}\,\mathrm{d}V_0 + \int_{V_0} S_{ij}\,\mathrm{d}\delta E_{ij}\,\mathrm{d}V_0 - \lambda\int_{V_0} F_{i0}\,\mathrm{d}\delta u_i\,\mathrm{d}V_0 - \lambda\int_{S_{\mathrm{p}0}} P_{i0}\,\mathrm{d}\delta u_i\,\mathrm{d}S_0 \\ &\qquad - \mathrm{d}\lambda\int_{V_0} F_{i0}\,\delta u_i\,\mathrm{d}V_0 - \mathrm{d}\lambda\int_{S_{\mathrm{p}0}} P_{i0}\,\delta u_i\,\mathrm{d}S_0 = 0\,, \end{aligned} \tag{4}$$

where the operations "d" and "δ" represent the first increments and variations, respectively.

If we apply any *configuration-dependent load*, the corresponding energy functionals can be defined provided the loading to be conservative in an overall sense [11]. This means that for a fixed value of the load parameter λ, the total work done by the body forces $F_i = \lambda F_{0i} + f_i(u_k)$ and the surface tractions $P_i = \lambda P_{0i} + p_i(u_k)$, in any virtual motion compatible with the kinematic constraints and leading from the given configuration u_k to any sufficiently close configuration $u_k + \mathrm{d}u_k$ is assumed to be path-independent, that is, for $\lambda =$ constant,

$$\int_{V_0}\int_{u_k}^{u_k+\mathrm{d}u_k} (\lambda F_{i0} + f_i(u_k))\,\mathrm{d}u_i\,\mathrm{d}V_0 + \int_{S_{\mathrm{p}0}}\int_{u_k}^{u_k+\mathrm{d}u_k} (\lambda P_{i0} + p_i(u_k))\,\mathrm{d}u_i\,\mathrm{d}S_0 = \Omega(u_k,\lambda) - \Omega(u_k + \mathrm{d}u_k, \lambda)\,, \tag{5}$$

where the functional $\Omega(u_k, \lambda)$ can be identified with the potential energy of the configuration-dependent loading device [10]. Consequently, any variational principles, or energetical approaches to stability analyses, moreover, the uniqueness and bifurcation conditions, can be investigated on the basis of the classical functionals modified by additive terms only.

By considering a general *nonlinear configuration-dependent loading* specified by the functions $F_i = \lambda F_{i0} + f_i(u_k)$ and $P_i = \lambda P_{i0} + p_i(u_k)$ and by using the incrementally linear load moduli (2), the incremental equilibrium equation (3) is modified to

$$\int_{V_0} \mathrm{d}E_{ij}\, D_{ijkl}\, \delta E_{kl}\, \mathrm{d}V_0 + \int_{V_0} S_{ij}\, \mathrm{d}\delta E_{ij}\, \mathrm{d}V_0$$

$$-\int_{V_0} (\lambda\, F_{i0} + f_i(u_k))\, \mathrm{d}\delta u_i\, \mathrm{d}V_0 - \int_{S_{\mathrm{p}0}} (\lambda\, P_{i0} + p_i(u_k))\, \mathrm{d}\delta u_i\, \mathrm{d}S_0$$

$$-\int_{V_0} (\mathrm{d}\lambda\, F_{i0} + M_{ij}\, \mathrm{d}u_j)\, \delta u_i\, \mathrm{d}V_0 - \int_{S_{\mathrm{p}0}} (\mathrm{d}\lambda\, P_{i0} + N_{ij}\, \mathrm{d}u_j)\, \delta u_i\, \mathrm{d}S_0 = 0\,. \tag{6}$$

By selecting the controllable and the deformation-sensitive parts of the loading, the classical and the additive terms can clearly be distinguished

$$\int_{V_0} \mathrm{d}E_{ij}\, D_{ijkl}\, \delta E_{kl}\, \mathrm{d}V_0 + \int_{V_0} S_{ij}\, \mathrm{d}\delta E_{ij}\, \mathrm{d}V_0 - \lambda \int_{V_0} F_{i0}\, \mathrm{d}\delta u_i\, \mathrm{d}V_0 - \lambda \int_{S_{\mathrm{p}0}} P_{i0}\, \mathrm{d}\delta u_i\, \mathrm{d}S_0$$

$$-\mathrm{d}\lambda \int_{V_0} F_{i0}\, \delta u_i\, \mathrm{d}V_0 - \mathrm{d}\lambda \int_{S_{\mathrm{p}0}} P_{i0}\, \delta u_i\, \mathrm{d}S_0$$

$$-\int_{V_0} \mathrm{d}u_j\, M_{ij}\, \delta u_i\, \mathrm{d}V_0 - \int_{S_{\mathrm{p}0}} \mathrm{d}u_j\, N_{ij}\, \delta u_i\, \mathrm{d}S_0 - \int_{V_0} f_i\, \mathrm{d}\delta u_i\, \mathrm{d}V_0 - \int_{S_{\mathrm{p}0}} p_i\, \mathrm{d}\delta u_i\, \mathrm{d}S_0 = 0\,, \tag{7}$$

where the effects of the deformation-sensitive parts of the loading are represented in the last row of the expression. Here $f_i = f_i(u_k)$ and $p_i = p_i(u_k)$, similarly to $S_{ij} = S_{ij}(E_{kl})$. In the case of *linear configuration-dependent loading* with $f_i(u_k) = m_{ij}u_j$ and $p_i(u_k) = n_{ij}u_j$, in the last row of expression (7) constant load moduli $M_{ij} = m_{ij}$ and $N_{ij} = n_{ij}$ appear.

Petryk in [10] summarizes the *conditions of bifurcation and stability* related to materially nonlinear bodies approximated by incrementally linear materials. In the case of dead load where the load increments are independent of the actual displacement field, in the current equilibrium state, the uniqueness of the first-order solution, that is, the bifurcation in the displacements is excluded when

$$\int_{V_0} \mathrm{d}E_{ij}\, D_{ijkl}\, \delta E_{kl}\, \mathrm{d}V_0 > 0\,. \tag{8}$$

In the case of configuration-dependent loading, the uniqueness condition is completed by the effect of the loading moduli

$$\int_{V_0} \mathrm{d}E_{ij}\, D_{ijkl}\, \delta E_{kl}\, \mathrm{d}V_0 - \int_{V_0} \mathrm{d}u_j\, M_{ij}\, \delta u_i\, \mathrm{d}V_0 - \int_{S_{\mathrm{p}0}} \mathrm{d}u_j\, N_{ij}\, \delta u_i\, \mathrm{d}S_0 > 0\,. \tag{9}$$

It seems to be evident that the deformation-sensitive part of the loading plays a similar role to that of the material, except for the signs. In contrast to the material terms, the load terms are negative. Thus, we can state that the material and loading tangent moduli can influence the uniqueness of the first-order solution, namely, the bifurcation of the displacements, and moreover, the stability of the equilibrium state and process, that is the definiteness characteristics of the structural tangent stiffness, in an opposite way. For example, while material softening has a destabilizing effect, load softening helps to maintain stability and conversely.

In order to modify the structural tangent stiffness with respect to the deformation-sensitive loading, the discrete structural model has to be introduced.

3. Discrete structural model

To construct the discrete model, generalized coordinates must be introduced. If the kinematic state of the structure can be characterized by number n of independent kinematic parameters $\mathbf{q} = \{q_i\}$, these parameters are used as *generalized coordinates* of the given numerical model. Then the displacements can be expressed in terms of the generalized parameters as follows

$$\underset{(3)}{\mathbf{u}} = \underset{(3)}{\mathbf{u}}(\mathbf{X}, \mathbf{q}) = \begin{bmatrix} u_1(\mathbf{X}, \mathbf{q}) \\ u_2(\mathbf{X}, \mathbf{q}) \\ u_3(\mathbf{X}, \mathbf{q}) \end{bmatrix} = \begin{bmatrix} u_1(X_1, X_2, X_3; q_1, q_2, \ldots, q_n) \\ u_2(X_1, X_2, X_3; q_1, q_2, \ldots, q_n) \\ u_3(X_1, X_2, X_3; q_1, q_2, \ldots, q_n) \end{bmatrix}, \tag{10}$$

where $\mathbf{X}$ are the coordinates of the geometric space and $\mathbf{q}$ are the coordinates of the function space. Since nonlinear geometry is considered, functions (10) are nonlinear in $\mathbf{q}$.

In the case of a *finite element model*, after dividing the body into subdomains named *elements*, the generalized coordinates, as a consequence of the division, form the displacement values at the nodal points. In the case of *geometric linearity*, the variables $\mathbf{X}$ and $\mathbf{q}$ in (10) can be separated by the linear combination

$$u_i = \sum_{k=1}^{n} q_k \, \varphi(\mathbf{X}) \tag{11}$$

which leads to the classical basic expression of the *linear finite element displacement method*

$$\underset{(3)}{\mathbf{u}(\mathbf{X}, \mathbf{q})} = \underset{(3,n)}{\mathbf{N}(\mathbf{X})} \, \underset{(3)}{\mathbf{q}}, \tag{12}$$

where matrix $\mathbf{N}$ contains the basis functions $\varphi(\mathbf{X})$. In the case of *geometric nonlinearity*, the direct separation (11) can not be applied, and the solution generally leads to iteration. In such cases, *incrementally linear analysis* is needed, so that the increments $\mathrm{d}\mathbf{u}$ of the displacements $\mathbf{u}$ can be expressed in the form of a linear relation in terms of the increments $\mathrm{d}\mathbf{q}$.

The first order increment and variation of the displacements are

$$\underset{(3)}{\mathrm{d}\mathbf{u}} = \left\{ \frac{\partial u_i(\mathbf{X}, \mathbf{q})}{\partial q_j} \right\} \mathrm{d}\mathbf{q} = \underset{(3,n)}{\mathbf{H}(\mathbf{X}, \mathbf{q})} \, \underset{(n)}{\mathrm{d}\mathbf{q}}, \quad \underset{(3)}{\delta\mathbf{u}} = \left\{ \frac{\partial u_i(\mathbf{X}, \mathbf{q})}{\partial q_j} \right\} \delta\mathbf{q} = \underset{(3,n)}{\mathbf{H}(\mathbf{X}, \mathbf{q})} \, \delta\mathbf{q}, \tag{13}$$

where the matrix $\mathbf{H}$ of $3 \times n$ elements contains the first derivatives of the displacement functions with respect to the generalized coordinates $\mathbf{q}$. Thus, the incrementally linear relation in (13) can be considered as the basic relation of the *nonlinear finite element displacement method.*

Taking into account that the generalized coordinates $\mathbf{q}$ are scalar parameters, thus, their increments have only the first order terms, that is $\Delta\mathbf{q} = \mathrm{d}\mathbf{q}$ and $\mathrm{d}^2\mathbf{q} = \mathbf{0}$, we can assume that $\delta\mathrm{d}\mathbf{q} = \delta\mathbf{q}$ and $\mathrm{d}\delta\mathbf{q} = \mathrm{d}\mathbf{q}$, consequently

$$\underset{(3)}{\mathrm{d}\delta\mathbf{u}} = \underset{(3)}{\delta\mathrm{d}\mathbf{u}} = \delta\mathbf{q}^{\mathrm{T}} \left\{ \frac{\partial^2 u_i(\mathbf{X}, \mathbf{q})}{\partial q_j \partial q_k} \right\} \mathrm{d}\mathbf{q} = \underset{(n)}{\delta\mathbf{q}^{\mathrm{T}}} \, \underset{(n,3,n)}{\mathbf{W}(\mathbf{X}, \mathbf{q})} \, \underset{(n)}{\mathrm{d}\mathbf{q}}, \tag{14}$$

where the three-dimensional matrix $\mathbf{W}$ of measure $n \times 3 \times n$ has a cubic arrangement, by containing the second derivatives of the displacement functions with respect to the generalized coordinates $\mathbf{q}$. This matrix represents the nonlinear geometry, since for the variation of the displacement increments the second derivatives of the displacement functions are needed.

By expressing the nonlinear Green–Lagrange strains in terms of the displacement gradients, for the discrete version we use the form

$$\underset{(6)}{\mathbf{E}} = \underset{(6)}{\mathbf{E}}(\mathbf{u}(\mathbf{X}, \mathbf{q})) = \underset{(6,3)}{\mathbf{A}} \cdot \underset{(3)}{\mathbf{u}} + \tfrac{1}{2} \, \underset{(6,9)}{\mathbf{C}(\mathbf{u})} \cdot \underset{(9)}{\mathbf{B}(\mathbf{u})}, \tag{15}$$

where $\mathbf{A}$, $\mathbf{C}(.)$ and $\mathbf{B}(.)$ are differential operators with respect to $\mathbf{X}$, since $\mathbf{A}\cdot\mathbf{u}$, $\mathbf{C}(\mathbf{u})$ and $\mathbf{B}(\mathbf{u})$ represent the displacement gradients. Namely, expression (15) in a detailed form is as follows:

$$\mathbf{E}=\begin{bmatrix} E_{11}\\ E_{22}\\ E_{33}\\ 2E_{12}\\ 2E_{13}\\ 2E_{23}\end{bmatrix}=\begin{bmatrix} \dfrac{\partial}{\partial X_1} & 0 & 0\\ 0 & \dfrac{\partial}{\partial X_2} & 0\\ 0 & 0 & \dfrac{\partial}{\partial X_3}\\ \dfrac{\partial}{\partial X_2} & \dfrac{\partial}{\partial X_1} & 0\\ \dfrac{\partial}{\partial X_3} & 0 & \dfrac{\partial}{\partial X_1}\\ 0 & \dfrac{\partial}{\partial X_3} & \dfrac{\partial}{\partial X_2}\end{bmatrix}\cdot\mathbf{u}+\frac{1}{2}\begin{bmatrix} \dfrac{\partial\mathbf{u}^{\mathrm{T}}}{\partial X_1} & 0 & 0\\ 0 & \dfrac{\partial\mathbf{u}^{\mathrm{T}}}{\partial X_2} & 0\\ 0 & 0 & \dfrac{\partial\mathbf{u}^{\mathrm{T}}}{\partial X_3}\\ \dfrac{\partial\mathbf{u}^{\mathrm{T}}}{\partial X_2} & \dfrac{\partial\mathbf{u}^{\mathrm{T}}}{\partial X_1} & 0\\ \dfrac{\partial\mathbf{u}^{\mathrm{T}}}{\partial X_3} & 0 & \dfrac{\partial\mathbf{u}^{\mathrm{T}}}{\partial X_1}\\ 0 & \dfrac{\partial\mathbf{u}^{\mathrm{T}}}{\partial X_3} & \dfrac{\partial\mathbf{u}^{\mathrm{T}}}{\partial X_2}\end{bmatrix}\cdot\begin{bmatrix} \dfrac{\partial\mathbf{u}}{\partial X_1}\\ \dfrac{\partial\mathbf{u}}{\partial X_2}\\ \dfrac{\partial\mathbf{u}}{\partial X_3}\end{bmatrix}. \tag{16}$$

The first term $\mathbf{A}\cdot\mathbf{u}$ represents the linear, while the second term $\mathbf{C}(\mathbf{u})\cdot\mathbf{B}(\mathbf{u})$ forms the nonlinear part of the strains, since the strain nonlinearity is related to the displacement gradients. Note that in the case of a linear finite element approach (12), the first term $\mathbf{Au}=\mathbf{ANq}$ is used only.

Consider now the required increments and variations of the strains. The first order terms of them are as follows

$$\underset{(6)}{\mathrm{d}\mathbf{E}}=[\underset{(6,3)}{\mathbf{A}}\cdot\underset{(3,n)}{\mathbf{H}}+\underset{(6,9)}{\mathbf{C}(\mathbf{u})}\cdot\underset{(9,n)}{\mathbf{B}(\mathbf{H})}]\,\underset{(n)}{\mathrm{d}\mathbf{q}} \quad\text{and}\quad \underset{(6)}{\delta\mathbf{E}}=[\underset{(6,3)}{\mathbf{A}}\cdot\underset{(3,n)}{\mathbf{H}}+\underset{(6,9)}{\mathbf{C}(\mathbf{u})}\cdot\underset{(9,n)}{\mathbf{B}(\mathbf{H})}]\,\underset{(n)}{\delta\mathbf{q}} \tag{17}$$

which do not vanish even if the strains are linear. However, the variation of the increment (or, equally, the increment of the variation)

$$\underset{(6)}{\mathrm{d}\delta\mathbf{E}}=\underset{(6)}{\delta\mathrm{d}\mathbf{E}}=\underset{(n)}{\delta\mathbf{q}^{\mathrm{T}}}[\underset{(6,3)}{\mathbf{A}}\cdot\underset{(3,n,n)}{\mathbf{W}}+\underset{(n,6,9)}{\mathbf{C}(\mathbf{H})}\cdot\underset{(9,n)}{\mathbf{B}(\mathbf{H})}+\underset{(6,9)}{\mathbf{C}(\mathbf{u})}\cdot\underset{(9,n,n)}{\mathbf{B}(\mathbf{W})}]\,\underset{(n)}{\mathrm{d}\mathbf{q}} \tag{18}$$

needs either nonlinear strains or nonlinear displacements. Namely, in the case of strain linearity with geometric nonlinearity, if $\mathbf{E}=\mathbf{Au}$ and $\mathbf{W}\neq\mathbf{0}$, the strain increments and their variation take the form $\mathrm{d}\mathbf{E}=\mathbf{A}\cdot\mathbf{H}\,\mathrm{d}\mathbf{q}$ and $\mathrm{d}\delta\mathbf{E}=\delta\mathrm{d}\mathbf{E}=\delta\mathbf{q}^{\mathrm{T}}\mathbf{A}\cdot\mathbf{W}\,\mathrm{d}\mathbf{q}$. On the contrary, in the case of strain nonlinearity with geometric linearity, we have $\mathrm{d}\mathbf{E}=[\mathbf{A}\cdot\mathbf{H}+\mathbf{C}(\mathbf{u})\cdot\mathbf{B}(\mathbf{H})]\,\mathrm{d}\mathbf{q}$ and $\mathrm{d}\delta\mathbf{E}=\delta\mathrm{d}\mathbf{E}=\delta\mathbf{q}^{\mathrm{T}}\mathbf{C}(\mathbf{H})\cdot\mathbf{B}(\mathbf{H})\,\mathrm{d}\mathbf{q}$. Obviously, the case of both strain and geometric linearity results in $\mathrm{d}\mathbf{E}=\mathbf{A}\cdot\mathbf{H}\,\mathrm{d}\mathbf{q}$, and consequently $\mathrm{d}\delta\mathbf{E}=\delta\mathrm{d}\mathbf{E}=\mathbf{0}$.

Using the constitutive law $\mathbf{S}(\mathbf{E})$, the stresses can be expressed in terms of the generalized coordinates, too. Representing a nonlinear material by the nonlinear function $\mathbf{S}=\mathbf{S}(\mathbf{E}(\mathbf{u}(\mathbf{X},\mathbf{q})))$, for the incrementally linear constitutive law (1) we obtain

$$\underset{(6)}{\mathrm{d}\mathbf{S}}=\left\{\frac{\partial S_i(\mathbf{E})}{\partial E_j}\right\}\mathrm{d}E_j=\underset{(6,6)}{\mathbf{D}_{\mathrm{t}}(\mathbf{E})}\,\underset{(6)}{\mathrm{d}\mathbf{E}}, \tag{19}$$

where the quadratic matrix $\mathbf{D}_{\mathrm{t}}(\mathbf{E})$ contains the actual material tangent moduli, the scalar elements of the fourth order tensor $D^{\mathrm{t}}_{ijkl}(E_{mn})$, as a consequence of the vector-arranged stress and strain tensors.

Let us consider now the load variability. By having a configuration-dependent load prescribed by a nonlinear loading function, for the discrete form of the loads we obtain the vectors

$$\underset{(3)}{\mathbf{F}}=\underset{(3)}{\mathbf{F}(\mathbf{u}(\mathbf{X},\mathbf{q}))}=\lambda\underset{(3)}{\mathbf{F}_0}+\underset{(3)}{\mathbf{f}(\mathbf{u}(\mathbf{X},\mathbf{q}))} \quad\text{and}\quad \underset{(3)}{\mathbf{P}}=\underset{(3)}{\mathbf{P}(\mathbf{u}(\mathbf{X},\mathbf{q}))}=\lambda\underset{(3)}{\mathbf{P}_0}+\underset{(3)}{\mathbf{p}(\mathbf{u}(\mathbf{X},\mathbf{q}))}, \tag{20}$$

where both functions are nonlinear in $\mathbf{u}$. By applying the concept of incremental linearity to the loading as well, the incrementally linear loading (2) can be introduced in a discrete version

$$\underset{(3)}{\mathrm{d}\mathbf{F}}=\mathrm{d}\lambda\,\underset{(3)}{\mathbf{F}_0}+\underset{(3,3)}{\mathbf{M}_{\mathrm{t}}}\,\underset{(3)}{\mathrm{d}\mathbf{u}} \quad\text{and}\quad \underset{(3)}{\mathrm{d}\mathbf{P}}=\mathrm{d}\lambda\,\underset{(3)}{\mathbf{P}_0}+\underset{(3,3)}{\mathbf{M}_{\mathrm{t}}}\,\underset{(3)}{\mathrm{d}\mathbf{u}} \tag{21}$$

where the matrices

$$\mathbf{M}_t = \left\{ \frac{\partial f_i(\mathbf{u})}{\partial u_j} \right\} = \mathbf{M}_t(\mathbf{u}) \quad \text{and} \quad \mathbf{N}_t = \left\{ \frac{\partial p_i(\mathbf{u})}{\partial u_j} \right\} = \mathbf{N}_t(\mathbf{u}) \tag{22}$$

represent the loading tangent moduli related to the deformation-sensitive part of the loading.

By using the matrix form of the state variables, the incremental equilibrium condition (7) can be obtained in a matrix form

$$\begin{aligned}
&\int_{V_0} \mathrm{d}\mathbf{E}^{\mathrm{T}}\mathbf{D}_t\,\delta\mathbf{E}\,\mathrm{d}V_0 + \int_{V_0} \mathbf{S}^{\mathrm{T}}\,\mathrm{d}\delta\mathbf{E}\,\mathrm{d}V_0 \\
&\quad -\int_{V_0} \mathrm{d}\mathbf{u}^{\mathrm{T}}\mathbf{M}_t\,\delta\mathbf{u}\,\mathrm{d}V_0 - \lambda\int_{V_0} \mathbf{F}_0^{\mathrm{T}}\,\mathrm{d}\delta\mathbf{u}\,\mathrm{d}V_0 - \int_{V_0} \mathbf{f}^{\mathrm{T}}\,\mathrm{d}\delta\mathbf{u}\,\mathrm{d}V_0 \\
&\quad -\int_{S_{\mathrm{p}0}} \mathrm{d}\mathbf{u}^{\mathrm{T}}\mathbf{N}_t\,\delta\mathbf{u}\,\mathrm{d}S_0 - \lambda\int_{S_{\mathrm{p}0}} \mathbf{P}_0^{\mathrm{T}}\,\mathrm{d}\delta\mathbf{u}\,\mathrm{d}S_0 - \int_{S_{\mathrm{p}0}} \mathbf{p}^{\mathrm{T}}\,\mathrm{d}\delta\mathbf{u}\,\mathrm{d}S_0 \\
&\quad -\mathrm{d}\lambda\int_{V_0} \mathbf{F}_0^{\mathrm{T}}\,\delta\mathbf{u}\,\mathrm{d}V_0 - \mathrm{d}\lambda\int_{S_{\mathrm{p}0}} \mathbf{P}_0^{\mathrm{T}}\,\delta\mathbf{u}\,\mathrm{d}S_0 = 0
\end{aligned} \tag{23}$$

from which, by substituting the expressions of the state variables in terms of the generalized coordinates and their increments, the tangent stiffness matrix modified by variable loading can be obtained.

4. Modification of the tangent stiffness due to configuration-dependent loading

By substituting (13), (14), (17) and (18) into (23), the linearized incremental equilibrium equation modified by the configuration-dependent loading is obtained as

$$\begin{aligned}
&\delta\mathbf{q}^{\mathrm{T}}\left\{ \int_{V_0} \left[\left(\mathbf{H}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} + \mathbf{B}(\mathbf{H})^{\mathrm{T}}\mathbf{C}(\mathbf{u})^{\mathrm{T}} \right) \mathbf{D}_t \left(\mathbf{C}(\mathbf{u})\,\mathbf{B}(\mathbf{H}) + \mathbf{A}\,\mathbf{H} \right) \right] \mathrm{d}V_0 \right. \\
&\quad + \int_{V_0} \left[\mathbf{S}^{\mathrm{T}} \left(\mathbf{A}\,\mathbf{W} + \mathbf{C}(\mathbf{H})\,\mathbf{B}(\mathbf{H}) + \mathbf{C}(\mathbf{u})\,\mathbf{B}(\mathbf{W}) \right) \right] \mathrm{d}V_0 \\
&\quad - \int_{V_0} \mathbf{H}^{\mathrm{T}}\mathbf{M}_t\,\mathbf{H}\,\mathrm{d}V_0 - \int_{S_{\mathrm{p}0}} \mathbf{H}^{\mathrm{T}}\mathbf{N}_t\,\mathbf{H}\,\mathrm{d}S_0 \\
&\quad \left. - \int_{V_0} \mathbf{f}^{\mathrm{T}}\mathbf{W}\,\mathrm{d}V_0 - \int_{S_{\mathrm{p}0}} \mathbf{p}^{\mathrm{T}}\mathbf{W}\,\mathrm{d}S_0 - \lambda\int_{V_0} \mathbf{F}_0^{\mathrm{T}}\mathbf{W}\,\mathrm{d}V_0 - \lambda\int_{S_{\mathrm{p}0}} \mathbf{P}_0^{\mathrm{T}}\mathbf{W}\,\mathrm{d}S_0 \right\} \mathrm{d}\mathbf{q} \\
&\quad - \delta\mathbf{q}^{\mathrm{T}}\left\{ \mathrm{d}\lambda\int_{V_0} \mathbf{H}^{\mathrm{T}}\mathbf{F}_0\,\mathrm{d}V_0 + \mathrm{d}\lambda\int_{S_{\mathrm{p}0}} \mathbf{H}^{\mathrm{T}}\mathbf{P}_0\,\mathrm{d}S_0 \right\} = 0
\end{aligned} \tag{24}$$

which is equivalent to the scalar equation

$$\delta\mathbf{q}^{\mathrm{T}}\,\mathbf{K}_t\,\mathrm{d}\mathbf{q} - \delta\mathbf{q}^{\mathrm{T}}\,\mathrm{d}\mathbf{Q} = 0\,, \tag{25}$$

and if we assume $\delta\mathbf{q}$ to be arbitrary, it leads to the system of incremental equilibrium equations

$$\mathbf{K}_t\,\mathrm{d}\mathbf{q} = \mathrm{d}\mathbf{Q}\,. \tag{26}$$

Here $\mathbf{K}_t$ is the *tangent stiffness matrix*, namely

$$\underset{(n,n)}{\mathbf{K}_t} = \int_{V_0} \left[\left(\underset{(n,3)}{\mathbf{H}^T} \underset{(3,6)}{\mathbf{A}^T} + \underset{(n,9)}{\mathbf{B}(\mathbf{H})^T} \underset{(9,6)}{\mathbf{C}(\mathbf{u})^T} \right) \underset{(6,6)}{\mathbf{D}_t} \left(\underset{(6,9)}{\mathbf{C}(\mathbf{u})} \underset{(9,n)}{\mathbf{B}(\mathbf{H})} + \underset{(6,3)}{\mathbf{A}} \underset{(3,n)}{\mathbf{H}} \right) \right] \mathrm{d}V_0$$

$$+ \int_{V_0} \left[\underset{(6)}{\mathbf{S}^T} \left(\underset{(6,3)}{\mathbf{A}} \underset{(3,n,n)}{\mathbf{W}} + \underset{(6,n,9)}{\mathbf{C}(\mathbf{H})} \underset{(9,n)}{\mathbf{B}(\mathbf{H})} + \underset{(6,9)}{\mathbf{C}(\mathbf{u})} \underset{(9,n,n)}{\mathbf{B}(\mathbf{W})} \right) \right] \mathrm{d}V_0$$

$$-\lambda \int_{V_0} \underset{(3)}{\mathbf{F}_0^T} \underset{(3,n,n)}{\mathbf{W}} \, \mathrm{d}V_0 - \lambda \int_{S_{p0}} \underset{(3)}{\mathbf{P}_0^T} \underset{(3,n,n)}{\mathbf{W}} \, \mathrm{d}S_0$$

$$- \int_{V_0} \underset{(n,3)}{\mathbf{H}^T} \underset{(3,3)}{\mathbf{M}_t} \underset{(3,n)}{\mathbf{H}} \, \mathrm{d}V_0 - \int_{S_{p0}} \underset{(n,3)}{\mathbf{H}^T} \underset{(3,3)}{\mathbf{N}_t} \underset{(3,n)}{\mathbf{H}} \, \mathrm{d}S_0 - \int_{V_0} \underset{(3)}{\mathbf{f}^T} \underset{(3,n,n)}{\mathbf{W}} \, \mathrm{d}V_0 - \int_{S_{p0}} \underset{(3)}{\mathbf{p}^T} \underset{(3,n,n)}{\mathbf{W}} \, \mathrm{d}S_0 \tag{27}$$

and $\mathrm{d}\mathbf{Q}$ is the load increment, that is

$$\underset{(n)}{\mathrm{d}\mathbf{Q}} = \mathrm{d}\lambda \int_{V_0} \underset{(n,3)}{\mathbf{H}^T} \underset{(3)}{\mathbf{F}_0^T} \, \mathrm{d}V_0 + \mathrm{d}\lambda \int_{S_{p0}} \underset{(n,3)}{\mathbf{H}^T} \underset{(3)}{\mathbf{P}_0^T} \, \mathrm{d}S_0 \,. \tag{28}$$

This is the basis for the Newton–Raphson iteration procedure to be discussed later, according to [12].

Here we interpret the *structural tangent matrix* in the sense of the *system gradient matrix* described in [3], containing the terms related not only to the material and geometry but also to the loading. In this meaning, the first three rows in (27) concern the classical tangent stiffness (system gradient) matrix of full nonlinearity, while the last one represents the modification due to configuration-dependent loading. The first row in (27) concerns the material, named *material tangent stiffness*, represented by the material tangent modulus

$$\underset{(n,n)}{\mathbf{K}_{t,\mathrm{mat}}} = \int_{V_0} \left[\left(\underset{(n,3)}{\mathbf{H}^T} \underset{(3,6)}{\mathbf{A}^T} + \underset{(n,9)}{\mathbf{B}(\mathbf{H})^T} \underset{(9,6)}{\mathbf{C}(\mathbf{u})^T} \right) \underset{(6,6)}{\mathbf{D}_t} \left(\underset{(6,9)}{\mathbf{C}(\mathbf{u})} \underset{(9,n)}{\mathbf{B}(\mathbf{H})} + \underset{(6,3)}{\mathbf{A}} \underset{(3,n)}{\mathbf{H}} \right) \right] \mathrm{d}V_0 \,, \tag{29}$$

while the second row concerns the geometry, named *geometric tangent stiffness*, represented by the geometric and strain nonlinearities, namely

$$\underset{(n,n)}{\mathbf{K}_{t,\mathrm{geom}}} = \int_{V_0} \left[\underset{(6)}{\mathbf{S}^T} \left(\underset{(6,3)}{\mathbf{A}} \underset{(3,n,n)}{\mathbf{W}} + \underset{(6,n,9)}{\mathbf{C}(\mathbf{H})} \underset{(9,n)}{\mathbf{B}(\mathbf{H})} + \underset{(6,9)}{\mathbf{C}(\mathbf{u})} \underset{(9,n,n)}{\mathbf{B}(\mathbf{W})} \right) \right] .\mathrm{d}V_0 \tag{30}$$

The third row in (27) contains the effect of the classical controllable (dead) part of the loading, named *dead load tangent stiffness*, as

$$\underset{(n,n)}{\mathbf{K}_{t,\mathrm{dload}}} = \lambda \int_{V_0} \underset{(3)}{\mathbf{F}_0^T} \underset{(3,n,n)}{\mathbf{W}} \, \mathrm{d}V_0 + \lambda \int_{S_{p0}} \underset{(3)}{\mathbf{P}_0^T} \underset{(3,n,n)}{\mathbf{W}} \, \mathrm{d}S_0 \,. \tag{31}$$

The last row represents the effect of the deformation-sensitive part of the loading, named *variable load tangent stiffness*, as

$$\underset{(n,n)}{\mathbf{K}_{t,\mathrm{vload}}} = \int_{V_0} \underset{(n,3)}{\mathbf{H}^T} \underset{(3,3)}{\mathbf{M}_t} \underset{(3,n)}{\mathbf{H}} \, \mathrm{d}V_0 + \int_{S_{p0}} \underset{(n,3)}{\mathbf{H}^T} \underset{(3,3)}{\mathbf{N}_t} \underset{(3,n)}{\mathbf{H}} \, \mathrm{d}S_0 + \int_{V_0} \underset{(3)}{\mathbf{f}^T} \underset{(3,n,n)}{\mathbf{W}} \, \mathrm{d}V_0 + \int_{S_{p0}} \underset{(3)}{\mathbf{p}^T} \underset{(3,n,n)}{\mathbf{W}} \, \mathrm{d}S_0 \,. \tag{32}$$

Notice that the first two terms in (32) with the *load tangent moduli* are similar to the material tangent stiffness (29) with the *material tangent moduli*, while the last two terms in (32) are similar to the geometric stiffness (30) with the actual value of the material function and the geometric nonlinearities. Through these expressions, it is obvious that the variable load plays a similar role to that of the material. *Thus, the classical tangent stiffness matrix of dead-loaded structure is completed by the also symmetric terms (32) of a deformation-sensitive loading program.*

The tangent stiffness matrix consists basically of three types of matrices. The first group, matrices $\mathbf{A}$, $\mathbf{C}(.)$ and $\mathbf{B}(.)$ are differential operators with respect to the *geometric space* $\mathbf{X}$, applied to the second group, to the matrices $\mathbf{u}$, $\mathbf{H}$, $\mathbf{W}$ which are generated from each other by derivatives with respect to the *function space* $\mathbf{q}$. The third group, matrices $\mathbf{D}_t$, $\mathbf{M}_t$, $\mathbf{N}_t$ concern the material and loading characteristics.

In the case of a fully nonlinear approach, the various types of nonlinearities are represented by the following matrices: geometric type by $\mathbf{W}$, strain type by $\mathbf{C}(.)\cdot\mathbf{B}(.)$, material type by $\mathbf{D}_t$, similarly to the loading nonlinearity represented by $\mathbf{M}_t$ and $\mathbf{N}_t$. Matrix $\mathbf{D}_t$ is a diagonal one if we have one-dimensional material functions. Similarly, if we assume each load functions in (20) to be one-dimensional, $\mathbf{M}_t$ and $\mathbf{N}_t$ are diagonal matrices too.

In the case of *strain linearity with geometric nonlinearity* at the same time, the tangent stiffness matrix has a simpler form

$$\begin{aligned}\mathbf{K}_t = &\int_{V_0}\mathbf{H}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{D}_t\,\mathbf{A}\,\mathbf{H}\,\mathrm{d}V_0 + \int_{V_0}\mathbf{S}^{\mathrm{T}}\mathbf{A}\,\mathbf{W}\,\mathrm{d}V_0\\ &-\lambda\int_{V_0}\mathbf{F}_0^{\mathrm{T}}\mathbf{W}\,\mathrm{d}V_0 - \lambda\int_{S_{p0}}\mathbf{P}_0^{\mathrm{T}}\mathbf{W}\,\mathrm{d}S_0\\ &\qquad -\int_{V_0}\mathbf{H}^{\mathrm{T}}\mathbf{M}_t\,\mathbf{H}\,\mathrm{d}V_0 - \int_{S_{p0}}\mathbf{H}^{\mathrm{T}}\mathbf{N}_t\,\mathbf{H}\,\mathrm{d}S_0 - \int_{V_0}\mathbf{f}^{\mathrm{T}}\mathbf{W}\,\mathrm{d}V_0 - \int_{S_{p0}}\mathbf{p}^{\mathrm{T}}\mathbf{W}\,\mathrm{d}S_0\,. \end{aligned}\tag{33}$$

However, for *geometric linearity with strain nonlinearity*, among other load terms, the control terms would be lost, since in this case

$$\begin{aligned}\mathbf{K}_t = &\int_{V_0}\left\{\left(\mathbf{H}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} + \mathbf{B}(\mathbf{H})^{\mathrm{T}}\mathbf{C}(\mathbf{u}^{\mathrm{T}})\right)\mathbf{D}_t\left(\mathbf{C}(\mathbf{u})\,\mathbf{B}(\mathbf{H}) + \mathbf{A}\,\mathbf{H}\right)\right\}\mathrm{d}V_0\\ &+\int_{V_0}\mathbf{S}^{\mathrm{T}}\mathbf{C}(\mathbf{H})\,\mathbf{B}(\mathbf{H})\,\mathrm{d}V_0 - \int_{V_0}\mathbf{H}^{\mathrm{T}}\mathbf{M}_t\,\mathbf{H}\,\mathrm{d}V_0 - \int_{S_{p0}}\mathbf{H}^{\mathrm{T}}\mathbf{N}_t\,\mathbf{H}\,\mathrm{d}S_0\,. \end{aligned}\tag{34}$$

Moreover, in the case of *both geometric and strain linearity*, (27) is simplified to

$$\mathbf{K}_t = \int_{V_0}\mathbf{H}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{D}_t\,\mathbf{A}\,\mathbf{H}\,\mathrm{d}V_0 - \int_{V_0}\mathbf{H}^{\mathrm{T}}\mathbf{M}_t\,\mathbf{H}\,\mathrm{d}V_0 - \int_{S_{p0}}\mathbf{H}^{\mathrm{T}}\mathbf{N}_t\,\mathbf{H}\,\mathrm{d}S_0 \tag{35}$$

depending exclusively on the material and loading muduli, since in this case matrix $\mathbf{H}$ contains constant elements only. Evidently, to investigate stability, in the case of both dead and variable load, geometric nonlinearity is strictly required.

Generally, from the tangent stiffness matrix, stability conclusions are drawn: *stability of an equilibrium state* and *of an equilibrium path*; these two terminologies are essentially different [10]. Stability of an equilibrium state belongs to a constant load level, while stability of an equilibrium path, or a *deformation process* in an equivalent terminology, assumes a variable load parameter.

When an *equilibrium state* is analysed, for a *fixed load parameter* $\lambda = \hat{\lambda}$, generally by means of the iteration process based on the system gradient matrix, we determine the equilibrium state $\hat{\mathbf{q}} = \mathbf{q}(\hat{\lambda})$. If the solution $\hat{\mathbf{q}}$, by chance, coincides with a bifurcation point, the determinant of the tangent stiffness matrix $\hat{\mathbf{K}}_t = \mathbf{K}_t(\hat{\mathbf{q}})$ associated with the state $\hat{\mathbf{q}}$ vanishes. If $\hat{\mathbf{K}}_t$ is positive definite, the equilibrium state $\hat{\mathbf{q}}$ is stable. If it is negative definite, the equilibrium state $\hat{\mathbf{q}}$ is unstable and practically unrealizable in a physical system, since in this case a *dynamic departure (a snap-through)* occurs from the state. Thus, in the case of *state instability*, the fundamental pre-bifurcation path is left dynamically at a *fixed* λ [10].

In the case of a *deformation process*, the *load parameter* λ is *variable* during the quasi-static loading program. In this case, even the variations of the load parameter, the *functions* $\lambda(\mathbf{q})$ *of the equilibrium paths* are required. In the case of a n-dimensional discrete problem, the equilibrium paths form hypercurves in the n-dimensional space. Namely, to each equilibrium equation a function

$\lambda_i(\mathbf{q})$ belongs, forming a hypersurface in the n-dimensional space of the generalized coordinates $\mathbf{q}$. Since the equilibrium paths $\lambda(\mathbf{q})$ have to fulfil all the equilibrium conditions, they are obtained as the intersection of all the hypersurfaces $\lambda_i(\mathbf{q})$, resulting in a hypercurve in the space $\mathbf{q}$. Then, by substituting $\lambda(\mathbf{q})$ to $\mathbf{K}_t(\mathbf{q},\lambda)$ in (27), the associated *functions* $\mathbf{K}_t(\mathbf{q})$ *of the tangent stiffness matrix* can be obtained, for qualifying the stability of the equilibrium paths. Thus, on the basis of the function of the tangent stiffness matrix, the stability of the deformation process can be controlled. We can find that for the same loading program, there exists a bifurcating branch of quasi-static deformations at varying λ. Thus, in the case of *path instability*, the fundamental pre-bifurcation path is left quasi-statically at *varying* λ [10].

In the case of any type of nonlinearity, for the analysis of both an equilibrium state (at constant loading), or a deformation process (at varying loading), the *iteration method* is used, based on the tangent stiffness (system gradient) matrix (27). For classical *dead-loaded nonlinear finite element problems*, the Newton–Raphson type iteration methods are discussed by Stein *et al.* in [12] or by Doltsinis in [3]. An iteration process consists of different individual iteration sub-processes, each aims to find the subsequent equilibrium state $\mathbf{q}_B$ at configuration B, based on the preceding equilibrium state $\mathbf{q}_A$ at A, as seen in Fig. 2a. The general iterative scheme to obtain the geometry of the deformed structure at the subsequent state is as follows

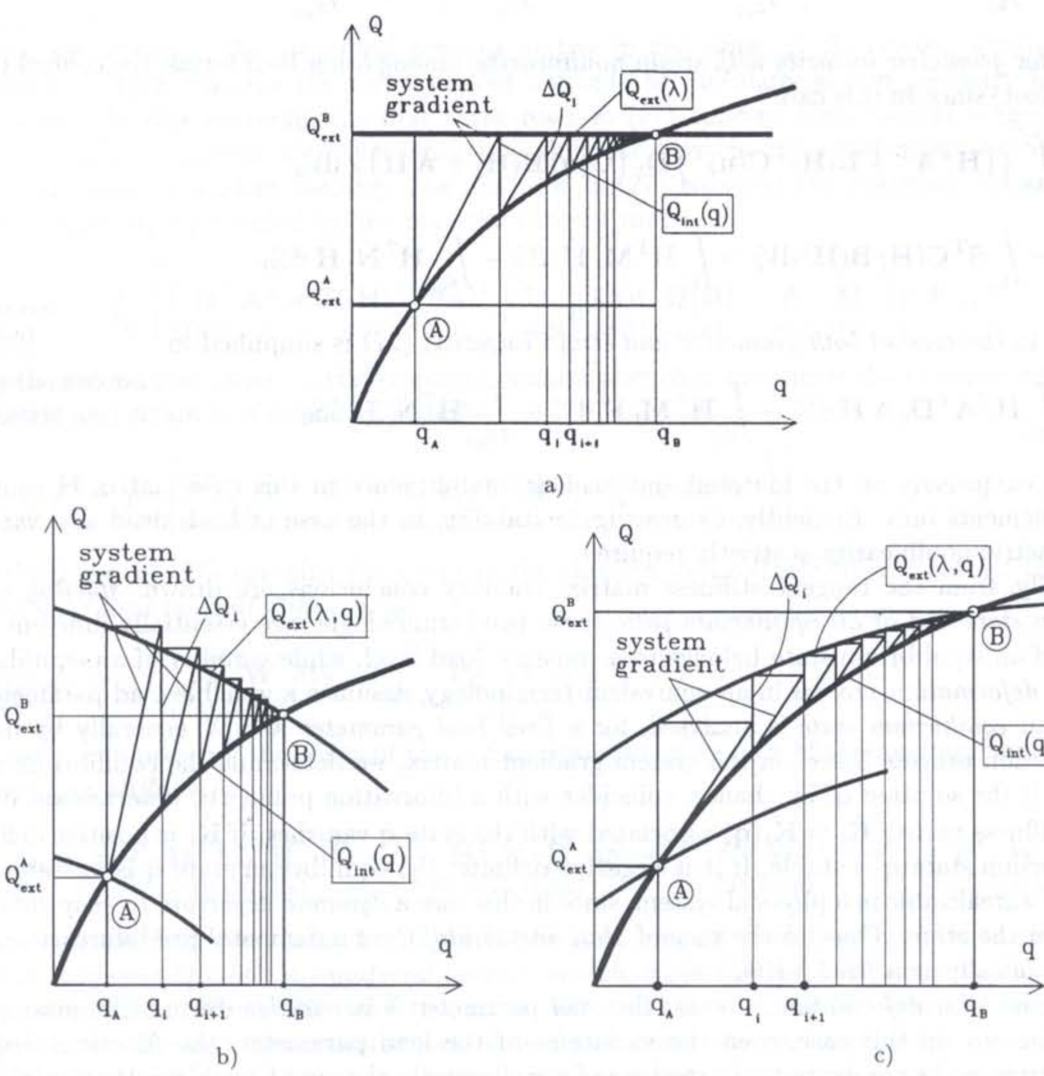


Fig. 2. Iteration processes for dead and variable loading devices

$$\mathbf{q}_{i+1} = \mathbf{q}_i + \mathbf{K}_t(\mathbf{q}_i)^{-1} \cdot \Delta\mathbf{Q}(\mathbf{q}_i)\,, \tag{36}$$

where $\mathbf{K}_t(\mathbf{q}_i)$ is the classical system gradient matrix calculated at the last known state $\mathbf{q}_i$. In order to avoid the calculation of a new tangent stiffness matrix in each step, generally, during an iteration sub-cycle (between the two equilibrium states A and B), this matrix is kept constant as $\mathbf{K}_t(\mathbf{q}_A)$, calculated at the last equilibrium state $\mathbf{q}_A$. Thus the *recurrence formula* reads

$$\mathbf{q}_{i+1} = \mathbf{q}_i + \mathbf{K}_t(\mathbf{q}_A)^{-1} \cdot \Delta\mathbf{Q}(\mathbf{q}_i) \tag{37}$$

represented by parallel system gradient lines in Fig. 2a. For controlling the convergence, the *residual non-equilibrium forces* $\Delta\mathbf{Q}(\mathbf{q}_i)$ are to be calculated in each step, as the difference of the external and internal forces

$$\Delta\mathbf{Q}(\mathbf{q}_i) = \mathbf{Q}_{\mathrm{ext}}(\mathbf{q}_i) - \mathbf{Q}_{\mathrm{int}}(\mathbf{q}_i) \tag{38}$$

illustrated also in Fig. 2a as the vertical distances between the functions $\mathbf{Q}_{\mathrm{ext}}(\mathbf{q}_i)$ and $\mathbf{Q}_{\mathrm{int}}(\mathbf{q}_i)$. For dead load, in a sub-process, the external load is fixed as $\mathbf{Q}_{\mathrm{ext}}(\mathbf{q}_i) = \mathbf{Q}^B_{\mathrm{ext}}$, thus $\Delta\mathbf{Q}(\mathbf{q}_i) = \mathbf{Q}^B_{\mathrm{ext}} - \mathbf{Q}_{\mathrm{int}}(\mathbf{q}_i)$.

Consider now the *iteration process of a configuration-dependent loading program.* While the loading functions in each iteration sub-cycle are constant, represented by horizontal straight lines in Fig. 1a, in the case of variable loading they are changing, as seen in Fig. 2b and c for nonlinearly hardening and softening loading devices, respectively. Even in this case we can use the same recurrence formula (37) and convergence criterion (38), but the terms appearing in them are modified. Namely, in the iterative scheme (37) the system gradient matrix $\mathbf{K}_t(\mathbf{q}_A)$ contains the additive terms (32) concerning the variable loading; moreover, the residual non-equilibrium forces $\Delta\mathbf{Q}(\mathbf{q}_i)$ in (31) are extended to the effect of variable loading, represented by the vertical distances between the functions $\mathbf{Q}_{\mathrm{ext}}(\mathbf{q}_i)$ and $\mathbf{Q}_{\mathrm{int}}(\mathbf{q}_i)$ in Fig. 2b and c.

In the nonlinear finite element realization, the external load and internal stress terms in the recurrence formula are as follows:

$$\begin{aligned}\mathbf{Q}_{\mathrm{ext}}(\mathbf{q}_i) &= \int_{V_0} \lambda\mathbf{F}_0^{\mathrm{T}}\,\mathbf{H}(\mathbf{X},\mathbf{q}_i)\,\mathrm{d}V_0 + \int_{S_{\mathrm{p}0}} \lambda\mathbf{P}_0^{\mathrm{T}}\,\mathbf{H}(\mathbf{X},\mathbf{q}_i)\,\mathrm{d}S_0 \\ &+ \int_{V_0} \mathbf{f}(\mathbf{u}(\mathbf{X},\mathbf{q}_i))^{\mathrm{T}}\,\mathbf{H}(\mathbf{X},\mathbf{q}_i)\,\mathrm{d}V_0 + \int_{S_{\mathrm{p}0}} \mathbf{p}(\mathbf{u}(\mathbf{X},\mathbf{q}_i))^{\mathrm{T}}\,\mathbf{H}(\mathbf{X},\mathbf{q}_i)\,\mathrm{d}S_0\,,\end{aligned} \tag{39}$$

in which the first two terms are related to the classical controllable part, while the last two terms concern the effect of sensitive part of the loading. The internal effects are the same as in the classical case, namely

$$\mathbf{Q}_{\mathrm{int}}(\mathbf{q}_i) = \int_{V_0} \mathbf{S}(\mathbf{E}(\mathbf{u}(\mathbf{X},\mathbf{q}_i)))^{\mathrm{T}} \left(\mathbf{A}\mathbf{H}(\mathbf{X},\mathbf{q}_i) + \mathbf{C}(\mathbf{u}(\mathbf{X},\mathbf{q}_i))\,\mathbf{B}(\mathbf{H}(\mathbf{X},\mathbf{q}_i))\right)\,\mathrm{d}V_0\,. \tag{40}$$

An iteration sub-process is continued until the displacements stabilise or diverge. Displacement stabilisation will occur if the structure is loaded below its smallest critical load factor, and diverge above this level. The iteration process can be continued with the applied load progressively increasing, and it is terminated if the structure arrives at its primary eigenstate, by reaching the generalized limit point of the load–deflection diagram. Petryk in [10] mentioned the term *generalized limit point* related to configuration-dependent loading devices. In a dead-loaded program the equilibrium path has a classical limit point if it becomes tangent to the horizontal characteristic of the dead-loaded device (Fig. 3a). According to Petryk, in the case of variable load, *the generalized limit point is met when the classical equilibrium path of dead-loaded structure becomes tangent to the linear (or possible nonlinear) characteristic of the loading device* (Fig. 3b and c). It seems to be obvious, since in the case of configuration-dependent load, the deformation-sensitive parts of the loading is added (with opposite sign) to the material-dependent parts of the classical tangent

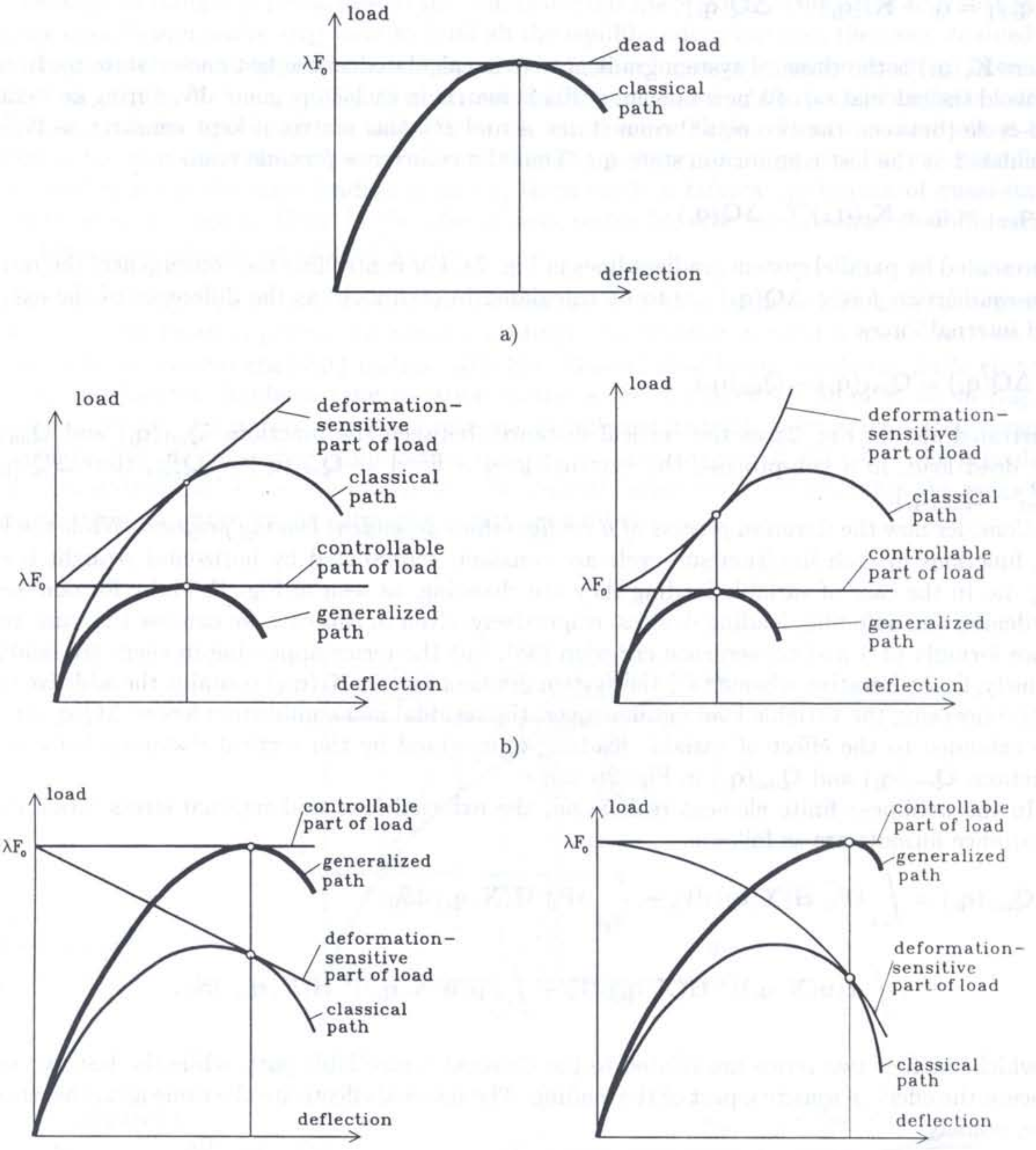


Fig. 3. Generalized limit point

stiffness matrix. Thus, at the generalized limit point, the tangent stiffness tends to vanish, if the material and loading characteristics have the same tangent. Consequently, the definition of Petryk for the generalized limit point can be written in an alternative form. That is, *the generalized limit point is met when the generalized equilibrium path of configuration-dependently loaded structure becomes tangent to the horizontal characteristic of the controllable part of the loading device.* Both these equivalent statements concern the tangent stiffness relations (27) and (28), respectively. Figure 3b and c show the cases of linear and nonlinear hardening and softening loading devices, respectively.

An alternative definition of the generalized limit point can be given on the basis of an eigen-value analysis related to the structural tangent stiffness matrix. Also the classical critical load and bifurcation analyses lead to eigen-value analysis, dividing the tangent stiffness matrix into two parts as

$$\mathbf{K}_t = \mathbf{K}_{t,str} - \lambda \mathbf{K}_{t,load} , \tag{41}$$

separated into structural and loading effect, respectively, according to Milner in [9], for a dead-loaded program. To detect bifurcation instability, or the critical load level, the condition

$$(\mathbf{K}_{t,str} - \lambda \mathbf{K}_{t,load})\, d\mathbf{q} = \mathbf{0} \tag{42}$$

is fulfilled by performing an eigen-value analysis [9]. The symmetric matrix $\mathbf{K}_{t,str}$ in the eigen-problem consists of $\mathbf{K}_{t,mat}$ and $\mathbf{K}_{t,geom}$ (29) and (30), respectively, and in the case of configuration-dependent loading it is completed by the matrix $\mathbf{K}_{t,vload}$ of the deformation-sensitive loading effects in (32),

$$\mathbf{K}_{t,str} = \mathbf{K}_{t,mat} + \mathbf{K}_{t,geom} - \mathbf{K}_{t,vload} , \tag{43}$$

while the term $\mathbf{K}_{t,load}$ is the same as for dead load (31), namely

$$\mathbf{K}_{t,load} = \mathbf{K}_{t,dload} . \tag{44}$$

Consequently, (27) leads to

$$(\mathbf{K}_{t,mat} + \mathbf{K}_{t,geom} - \mathbf{K}_{t,vload} - \lambda \mathbf{K}_{t,dload})\, d\mathbf{q} = \mathbf{0} . \tag{45}$$

Milner in [9] analyses the component matrices in (42) for a dead-loaded program, in the aspect of the definiteness characteristics of them and the Rayleigh quotients and the minimum eigen-value. Milner assumes the stiffness $\mathbf{K}_{t,str}$ to be positive definite due to the positive definite strain energy. In the case of configuration-dependent load, the positive definiteness of matrix $\mathbf{K}_{t,str}$ is not ensured. On the basis of (45), in harmony with Fig. 3, we can conclude that material and load exert opposite effects on the structural stability. Namely, material softening has a destabilizing effect, while the load softening has a stabilizing effect, and conversely, material hardening helps to maintain stability, while load hardening in itself can cause the loss of stability. The case of simultaneous material and load variability can result in, for example, a stable state, in spite of the material softening.

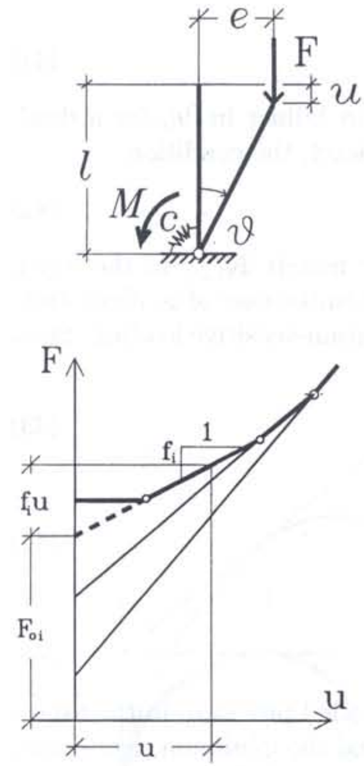
Numerically, determination of the equilibrium states or the equilibrium paths, and moreover, of the tangent stiffness matrix, leads to enormous mathematical difficulties. In the case of full nonlinearity, for the instantaneously changing material and loading moduli, an approximation is needed. As for the material, it has been proved by Kurutz in papers [4, 5, 6] that by using nonsmooth analysis for material nonlinearity, if the material functions are approximated by polygons, the equilibrium paths can be obtained as the envelope of the individual linear solutions related to each segment of the material polygon. As we have seen in (27), (29), (30) and (32), the loading and the material variability play a similar role in the tangent stiffness; consequently, we can extend the results of nonlinear material to the case of nonlinear configuration-dependent loading.

In order to illustrate the effect of configuration-dependent load on the equilibrium state and path, let us present several one-dimensional numerical examples.

5. Applications: modified post-bifurcation equilibrium paths and tangent stiffness due to configuration-dependent loading

To compare the effect of dead and variable load, we consider the simple structural models applied usually in stability analyses. To avoid the integration in the potential functions, these models are composed of perfectly rigid elements connected to each other and to the supports by springs by which the material properties, the relations between the stresses and strains are represented.

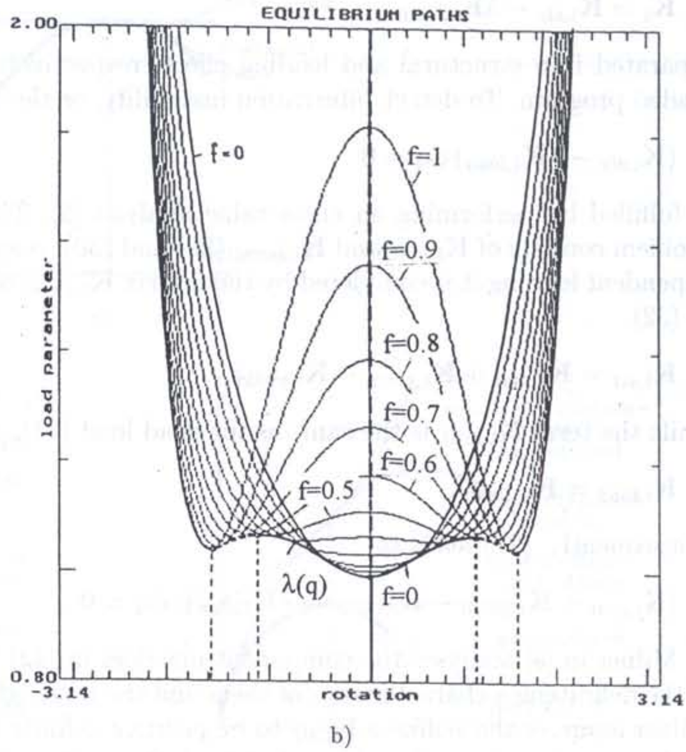
Consider first the classical example of *stable symmetric bifurcation* described in [13, page 5]. This type of dead-loaded simple structure seen in Fig. 4a is assumed to be composed of a perfectly rigid element of length l, pinned to a rigid foundation and connected to the support by a linear



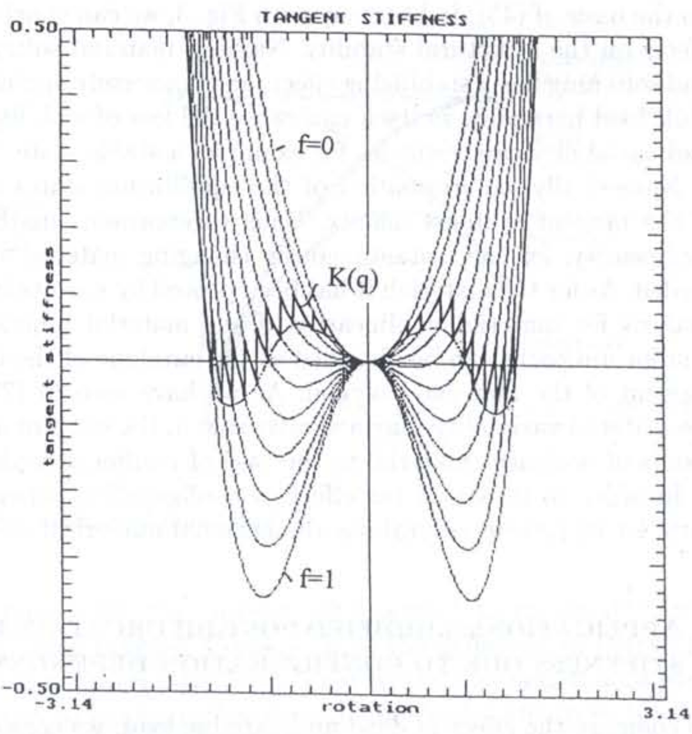
$$F = \lambda F_{0i} + f_i u$$

i	F_{0i}	f_i
1	1.00	0.0
2	0.99	0.1
3	0.98	0.2
4	0.96	0.3
5	0.94	0.4
6	0.90	0.5
7	0.85	0.6
8	0.79	0.7
9	0.72	0.8
10	0.64	0.9
11	0.55	1.0

a)



b)



c)

Fig. 4. Equilibrium path and tangent stiffness due to nonlinearly hardening loading

elastic spring. The material behaviour is concentrated in this spring by a spring constant c in both tension and compression. The spring moment M represents the stress variable, while the rotational deformation θ in the spring represents the strain variable. We suppose a linear elastic material by the linear material function $M(\theta) = c\theta$, but we can prescribe a nonlinear constitutive law $M(\theta)$ as well.

By applying a one parameter load controlled by λ, we consider $F(\lambda, u)$ as the given external load acting on the top of the cantilever; furthermore, consider u as the associated displacement. For *dead load* we have $F(\lambda) = \lambda F_0$, while for *linear configuration-dependent load* we have $F(\lambda, u) = \lambda F_0 + fu$. In the case of *nonlinear load*, we apply polygonal approximation for the instantaneously changing tangent load modulus f_t. In this case, the load polygon is specified by the associated load moduli f_i and initial load values F_{0i} segment by segment, thus $F_i(\lambda, u) = \lambda F_{0i} + f_i u$, $i = 1, 2, \ldots, n$ for n segments, as shown in Fig. 4a.

The functional discretization needs the generalized coordinates. In our simple discrete model, let the single generalized coordinate be the angle of rotation $q = \theta$ at the support hinge. We use perfect nonlinear geometry, namely, the trigonometric function $u = l(1 - \cos q)$ as the compatibility conditions. Here we assume that the system is perfect in the sense that the spring is unstrained when the link is vertical.

In the case of *dead load*, from the incremental form of the virtual work, the tangent stiffness matrix

$$K_t(\lambda, q) = c - \lambda F_0 \, l \cos q \tag{46}$$

and the load increment

$$\mathrm{d}Q = \mathrm{d}\lambda \, F_0 \, l \sin q \tag{47}$$

are obtained. Since we want to deal with the stability of the deformation process, we need the function of the equilibrium paths. In the case of *dead load*, the pre- and postbifurcation equilibrium paths are given by the classical expression

$$\lambda(q) = \frac{c}{F_0 l} \frac{q}{\sin q}, \quad \lambda_{\mathrm{cr}} = \frac{c}{F_0 l}, \tag{48}$$

with the stable symmetric point of bifurcation at the critical equilibrium state at $q = 0$ in Figure 4b for $f = 0$, by supposing $c = 1$, $l = 1$, and $F_0 = 1$. By substituting $\lambda(q)$ into $K_t(\lambda, q)$, the associated function $K_t(q)$ of the tangent stiffness is obtained

$$K_t(q) = c\left(1 - \frac{q}{\tan q}\right) \tag{49}$$

shown in Fig. 4c for $f = 0$.

In the case of *linear configuration-dependent load*, where $f_i = f = \text{const}$, and $F_{0i} = F_0 = \text{const}$, the modified postbifurcation equilibrium path takes the form

$$\lambda(q) = \frac{1}{F_0}\left(\frac{c}{l}\frac{q}{\sin q} - fl(1 - \cos q)\right), \quad \lambda_{\mathrm{cr}} = \frac{c}{F_0 l}, \tag{50}$$

with the same symmetric point of bifurcation at the same critical point at $q = 0$ of dead load, shown also in Fig. 4b for the individual cases of different values f_i. The associated functions

$$K_t(q) = c\left(1 - \frac{q}{\tan q}\right) - fl^2 \sin^2 q \tag{51}$$

of the modified tangent stiffness can also be seen in Fig. 4c for the individual cases of different values f_i.

Let us consider now the case of *nonlinear configuration-dependent load*. It has been proved by Kurutz in [4, 5, 6] that by approximating the material functions by polygons, the equilibrium paths can be obtained as the envelope of the individual equilibrium paths related to each segment of the material polygon. By extending the results to the case of nonlinear variable load, without any details of the nonsmooth approach presented in [8], the approximate equilibrium path can be obtained as the envelope of the paths

$$\lambda_i(q) = \frac{1}{F_{0i}}\left(\frac{c}{l}\frac{q}{\sin q} - f_i l(1-\cos q)\right), \quad i = 1, 2, \ldots, n \tag{52}$$

by keeping constant the Young modulus c of the material. The equilibrium paths of the given nonlinearly hardening loading device is seen in Fig. 4b, as the *lower* envelope of the individual paths.

In the case of such polygonal loading, from the modified incremental virtual work

$$\delta q\left[c - f_i l^2(\sin^2 q - \cos^2 q + \cos q) - \lambda F_{0i}\, l \cos q\right] \mathrm{d}q - \delta q\, \mathrm{d}\lambda\, F_{0i}\, l \sin q = 0, \quad i = 1, 2, \ldots, n \tag{53}$$

applied to each of the actual loading segment, the modified tangent stiffness

$$K_{ti}(\lambda, q) = c - f_i l^2(\sin^2 q - \cos^2 q + \cos q) - \lambda F_{0i}\, l \cos q, \quad i = 1, 2, \ldots, n \tag{54}$$

can be obtained, in which we can discover the components of (45) as

$$K_{ti,\mathrm{mat}} = c, \quad K_{ti,\mathrm{vload}} = f_{ti} l^2(\sin^2 q - \cos^2 q + \cos q), \quad K_{ti,\mathrm{dload}} = F_{0i} l \cos q, \tag{55}$$

since in (45)

$$K_{ti} = K_{ti,\mathrm{mat}} - K_{ti,\mathrm{vload}} - \lambda K_{ti,\mathrm{dload}}, \quad i = 1, 2, \ldots, n. \tag{56}$$

By substituting the equilibrium paths (52) into (54), the function of the associated tangent stiffness can be obtained also as the envelope of the following functions

$$K_{ti}(q) = c\left(1 - \frac{q}{\tan q}\right) - f_i l^2 \sin^2 q, \quad i = 1, 2, \ldots, n \tag{57}$$

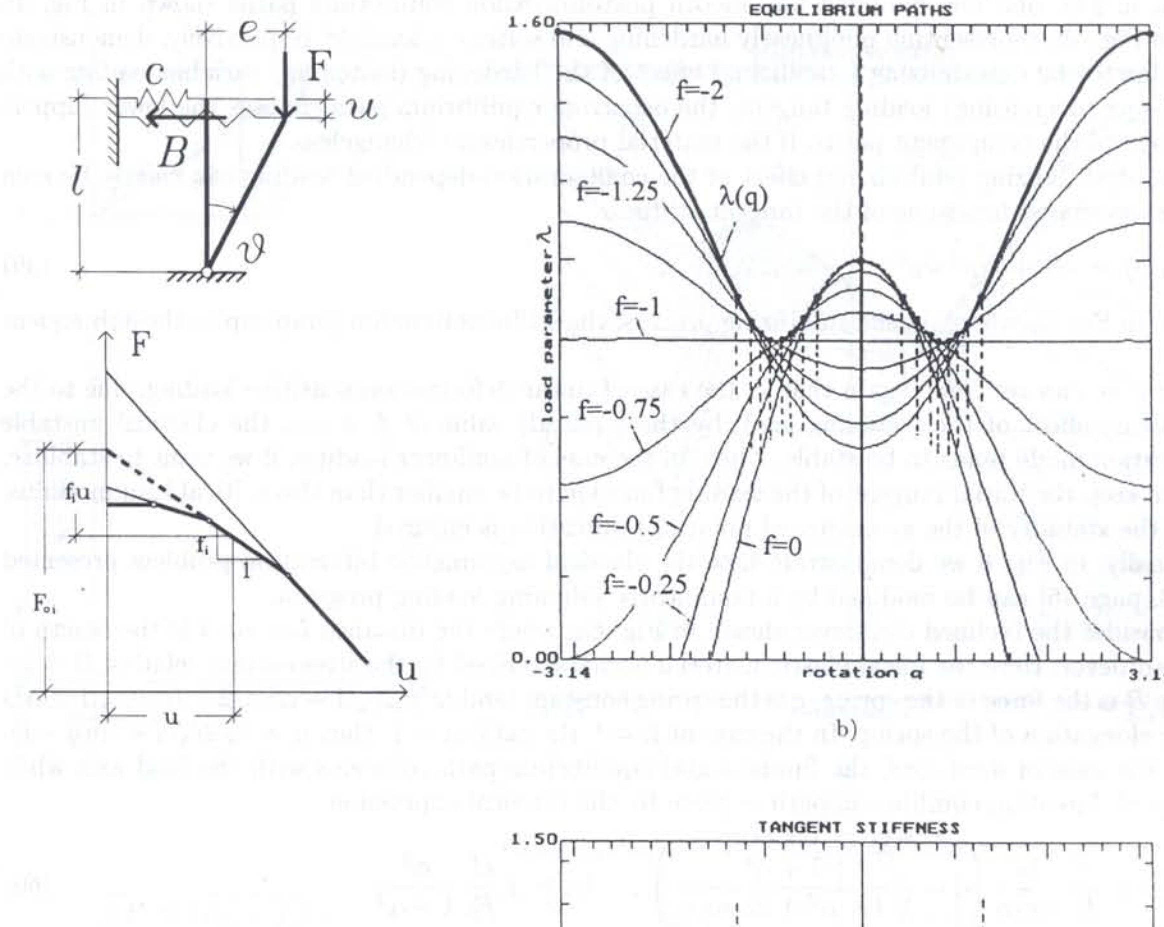
shown in Fig. 4c.

Figure 4b illustrates the approximate generalized equilibrium paths due to a *nonlinearly hardening configuration-dependent load* seen in Fig. 4a. Here we can see that due to the destabilizing effect of the hardening variable load, the equilibrium path is represented by the *lower envelope* of the component paths. In comparison with the dead-loaded program, in the case of a nonlinearly hardening loading process, the equilibrium forces are smaller. If the loading function is specified to be initially tangent to the dead load function, then the stable symmetrical point of bifurcation remains stable but the post-bifurcation equilibrium path turns to be unstable causing a dynamic snap-through to the nearest stable path.

In Fig. 4b and c the effects of dead and variable loads can be compared. From the graphs of the functions, the stabilizing (destabilizing) effect of the softening (hardening) effect of the load modulus can be seen. In the case of a linear deformation-sensitive loading, at a certain value of f, the classical stable bifurcation mode turns to be unstable. In the papers of the author [7,8] it has been shown that the value $f = c/3l^2$, by which the bifurcation mode changes, can be considered as a *critical load modulus.* Thus, in the case of nonlinear loading, if the initial tangent of the loading function is smaller than this critical load modulus, the stability of the symmetrical point of bifurcation is ensured.

On the contrary, in Fig. 5, we introduce the case of the *nonlinearly softening configuration-dependent loading*, applied to the classical unstable symmetric bifurcation problem seen in Fig. 5a and discussed in [13, page 9]. By specifying the loading program given also in Fig. 5a and by supposing a constant linearly elastic material at the same time, the postbifurcation equilibrium path can be composed as the *upper* envelope of the component paths given by the functions

$$\lambda_i(q) = \frac{l}{F_{0i}}\left(c \cos q - f_i l(1-\cos q)\right), \quad i = 1, 2, \ldots, n \tag{58}$$



$$F = \lambda F_{0i} + f_i u$$

i	f_i	F_{0i}
1	0.00	1.000
2	-0.25	1.025
3	-0.50	1.075
4	-0.75	1.150
5	-1.00	1.250
6	-1.25	1.375
7	-1.50	1.525
8	-1.75	1.700
9	-2.00	1.900

Fig. 5. Equilibrium path and tangent stiffness due to nonlinearly softening loading

shown in Fig. 5b. The modified nonsmooth postbifurcation equilibrium paths shown in Fig. 4b and in Fig. 5b, representing nonlinearly hardening and softening loading, respectively, demonstrate that due to the destabilizing (stabilizing) effect of the hardening (softening) variable loading with increasing (decreasing) loading tangent, the occurring equilibrium path chooses the lower (upper) envelope of the component paths, if the material properties are changeless.

The destabilizing (stabilizing) effect of the configuration-dependent loading can clearly be seen in the associated functions of the tangent stiffness

$$K_i(q) = -(c + f_i)l^2 \sin^2 q\,, \quad i = 1, 2, \ldots, n \tag{59}$$

shown in Fig. 5c where, in the stabilizing process, the stiffness function jumps up to the subsequent curve.

Here we can conclude again that in the case of linear deformation-sensitive loading, due to the stabilizing effect of the softening load, by the (critical) value of $f = -c$, the classical unstable bifurcation mode tends to be stable. Thus, in the case of nonlinear loading, if we want to stabilize, we can keep the initial tangent of the loading function to be smaller than this critical load modulus; thus, the stability of the symmetrical point of bifurcation is ensured.

Finally, in Fig. 6 we demonstrate how the classical asymmetric bifurcation problem presented in [13, page 15] can be modified by a nonlinearly softening loading program.

Consider the inclined cantilever shown in Fig. 6a, where the distance $L = \alpha l$, l is the length of the cantilever. Here the linear elastic material is characterized by the stress-strain relation $B = cw$ where B is the force in the spring, c is the spring constant, and $w = l(\sqrt{1+\alpha^2+2\alpha\sin q} - \sqrt{1+\alpha^2})$ is the elongation of the spring. In the case of $L = l$, the ratio $\alpha = 1$, thus $w = l\sqrt{2}(\sqrt{1+\sin q} - 1)$.

In the case of *dead load*, the fundamental equilibrium path coincides with the load axis while the postbifurcation equilibrium path is given by the classical expression

$$\lambda(q) = \frac{cl}{F_0}\frac{\alpha}{\tan q}\left(1 - \sqrt{\frac{1+\alpha^2}{1+\alpha^2+2\alpha\sin q}}\right), \quad \lambda_{\rm cr} = \pm\frac{cl}{F_0}\frac{\alpha^2}{1+\alpha^2} \tag{60}$$

representing the asymmetric point of bifurcation at $q = 0$, and $q = \pm\pi$, respectively, seen in Fig. 6b for $f_i = 0$, in the case of $c = 1$, $l = 1$, $F_0 = 1$ and by choosing $\alpha = 2$. For the associated classical functions $K(q)$ of the tangent stiffness, we obtain

$$K(q) = cl^2\left[\alpha^2\cos^2 q\sqrt{\frac{1+\alpha^2}{(1+\alpha^2+2\alpha\sin q)^3}} - \frac{\alpha}{\sin q}\left(1 - \sqrt{\frac{1+\alpha^2}{1+\alpha^2+2\alpha\sin q}}\right)\right] \tag{61}$$

shown in Fig. 6c as a curve for $f = 0$.

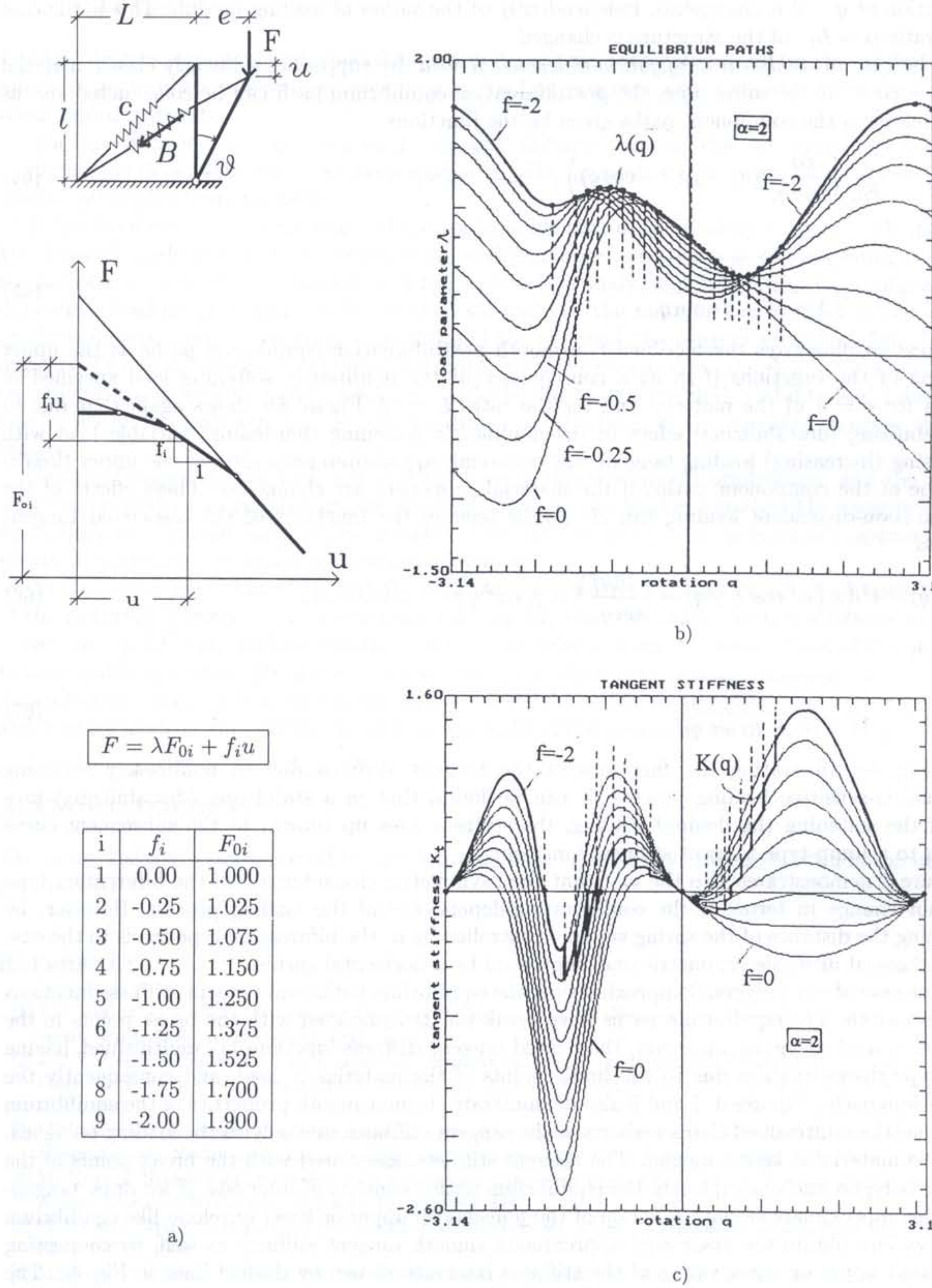
In the case of *linear variable load*, the postbifurcation equilibrium paths are

$$\begin{aligned} \lambda(q) &= \frac{l}{F_0}\left[\frac{c\alpha}{\tan q}\left(1 - \sqrt{\frac{1+\alpha^2}{1+\alpha^2+2\alpha\sin q}}\right) - f(1-\cos q)\right], \\ \lambda_{\rm cr} &= \frac{cl}{F_0}\frac{\alpha^2}{1+\alpha^2}, \quad \lambda_{\rm cr} = \frac{l}{F_0}\left(c\frac{\alpha^2}{1+\alpha^2} - 2f\right), \end{aligned} \tag{62}$$

with the same asymmetric point of bifurcation at the critical point at $q = 0$, and different asymmetric points of bifurcation at $q = \pm\pi$, respectively, shown in Fig. 6b for the individual values of f_i. The associated tangent stiffness function reads

$$K(q) = cl^2\left[\alpha^2\cos^2 q\sqrt{\frac{1+\alpha^2}{(1+\alpha^2+2\alpha\sin q)^3}} - \frac{\alpha}{\sin q}\left(1 - \sqrt{\frac{1+\alpha^2}{1+\alpha^2+2\alpha\sin q}}\right)\right] - fl^2\sin^2 q \tag{63}$$

illustrated in Fig. 6c for the individual values f_i. On the graphs of the functions of the equilibrium paths and the associated tangent stiffness we can observe that the asymmetric characteristic of the



$$F = \lambda F_{0i} + f_i u$$

i	f_i	F_{0i}
1	0.00	1.000
2	-0.25	1.025
3	-0.50	1.075
4	-0.75	1.150
5	-1.00	1.250
6	-1.25	1.375
7	-1.50	1.525
8	-1.75	1.700
9	-2.00	1.900

Fig. 6. Asymmetric bifurcation problem modified by configuration-dependent loading Case $L = 2l$

bifurcation at $q = 0$ is changeless, independently of the values of loading moduli. This is the case if the ratio $\alpha = L/l$ of the structure is changed.

In the case of *nonlinear configuration-dependent load*, by supposing a linearly elastic material with $c = \text{const}$ at the same time, the postbifurcation equilibrium path can be constructed also as the envelope of the component paths given by the functions

$$\lambda_i(q) = \frac{l}{F_{0i}}\left(\frac{c\alpha}{\tan q}\beta(q) - f_i(1-\cos q)\right), \quad 1,2,\ldots,n, \tag{64}$$

in which

$$\beta(q) = 1 - \sqrt{\frac{1+\alpha^2}{1+\alpha^2+2\alpha\sin q}}. \tag{65}$$

Figure 6b illustrates the modified nonsmooth postbifurcation equilibrium paths as the upper envelope of the functions (64), as a consequence of the nonlinearly softening load specified in Fig. 6a for $c = 1$ of the material and for the ratio $L = 2l$. Figure 6b shows again that due to the stabilizing (destabilizing) effect of the nonlinearly softening (hardening) variable load with decreasing (increasing) loading tangent, the occurring equilibrium path chooses the upper (lower) envelope of the component paths, if the material properties are changeless. These effects of the configuration-dependent loading can clearly be seen in the functions of the associated tangent stiffness

$$K_i(q) = l^2\left(c\left(\alpha^2\cos^2 q\,\gamma(q) - \frac{\alpha\,\beta(q)}{\sin q}\right) - f_i\sin^2 q\right), \quad 1,2,\ldots,n, \tag{66}$$

in which

$$\gamma(q) = \sqrt{\frac{1+\alpha^2}{(1+\alpha^2+2\alpha\sin q)^3}}. \tag{67}$$

In Fig. 6c illustrating the functions of the tangent stiffness due to nonlinearly softening deformation-sensitive loading process, it can be found that in a stabilizing (destabilizing) process of the softening (hardening) loading, the stiffness goes up (down) to the subsequent curve leading to a jump-type tangent stiffness function.

Figure 6 demonstrates also the fact that the asymmetric characteristic of the bifurcation type does not change in terms of the configuration-dependence of the loading process. However, by increasing the distance of the spring support controlled by α, the bifurcation type tends to the case of the classical unstable symmetric one, supported by a horizontal spring.

In the case of any polygonal approximation, the equilibrium paths and tangent stiffness functions are nonsmooth. The equilibrium paths have break points associated with the break points in the material or load polygons. Moreover, the related tangent stiffness functions are multivalued, having jump-type discontinuities due to the break points of the material or load, and consequently the equilibrium paths. Figures 4, 5 and 6 also demonstrate the nonsmooth properties of the equilibrium paths and the multivalued characteristics of the tangent stiffness, due only to the loading polygons, since the material is kept constant. The tangent stiffness, associated with the break points of the loading polygon and consequently the equilibrium paths, consists of intervals. If we draw tangentially the approximate smooth function of the nonsmooth upper or lower envelope-like equilibrium paths, we can obtain the associated approximate smooth tangent stiffness as well, by connecting the related upper or lower values of the stiffness intervals, shown by dashed lines in Fig. 4c. The nonsmooth approaches are discussed in [8].

By the method presented here, the effect of the material can also be analysed. The interaction between the nonlinear material and nonlinear loading; moreover, some other interesting analyses, for example, information about the imperfection-sensitivity with respect to the material, geometry, supports, etc., in a deformation-sensitive loading process can also be analysed by the presented method. The corresponding results will be published elsewhere.

6. Conclusions

Loading devices were analysed, special attention being paid to the structural tangent stiffness. The effects of configuration-dependent conservative loading program were compared with the classical dead-loaded programs.

The modification of the structural tangent stiffness matrix due to linear or nonlinear configuration-dependent load has been discussed. The presented method is based on the incremental principle of virtual work.

It has been proved that the effect of the controllable part of the loading is equal to the effect of the classical dead loading, while the effect of the deformation-sensitive part of the loading is similar to that of the material. Consequently, in the equilibrium analysis of structures, the configuration-dependent loading programs can be handled similarly to the material ones. Thus, in the system gradient matrix, besides the tangent modulus of the material, the tangent modulus of the loading appears. The modified structural tangent stiffness matrix is obtained by completing the classical one by a deformation-sensitive part. Iteration methods can be investigated, stability and bifurcation conditions and conclusions can equally be drawn on the basis of the modified tangent stiffness matrix.

Tensorial investigations were followed by discrete (finite element) application, and the results were illustrated by numerical examples. The nonlinear variable load was analyzed by using polygonal approximation, on the basis of the results of the author, related to polygonally approximated nonlinear materials, by applying nonsmooth analysis.

As a conclusion, we can state that the deformation-sensitive load has an effect opposite to that of the material. Namely, while a softening material has a destabilizing effect, a softening load stabilizes the equilibrium, and conversely. This fact can be seen from the construction of the modified tangent stiffness matrix. By using polygonal approximation, any type of nonlinear configuration-dependent conservative loading process, moreover, interaction between linear or nonlinear materials and loading devices, can also be studied on the basis of the presented method.

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