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Multi-disciplinary shape optimization of notches in 2-D machine components

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A multi-disciplinary, numerical approach to shape optimization of notches is presented. The design of the optimal shape of notches in 2-D elastic machine (structural) components is formulated using the Fictitious Stress Method. The design objective is to minimize the maximum effective stress for a given load. Formulation is based on constant stress boundary element. A special concept of segmented Bezier interpolants is adopted for defining geometry of the machine component, and the Sequential Linear Programming is used as optimization procedure.

1. Introduction

An extensive literature has been developed on optimization of structures and structural elements which are defined by cross-section and thickness variables (size optimization). A more important problem, from the point of view of mechanical design, is determination of the shape of 2-D, or 3-D structural elements (shape optimization). For such problems, the shape of the structural element must be treated as design variables.

There are some constructional reasons to design holes, fillets, grooves, undercuts, cut-outs etc. (known collectively as "notches"), which perturb and change any uniform stress state. In the cases where the increase in the stress results from the existence of notches, the notion of the Stress Concentration Factor (SCF) is introduced. It is demonstrated in literature that a decrease in the SCF significantly increases the fatigue life of the components. To decrease the SCF, possibility exists to minimize stress by changing the shape of the notch. Such a class of optimization problems is reffered to as the notch shape optimization of construction parts.

Recent work on optimal shape design can be categorized as follows [14, 17]: (1) weight minimization with stress constraints, and (2) stress minimization. In most cases, the weight of the machine component is chosen as the objective function. The notch shape optimization problem with respect to stresses was initiated by Tvergaard [47]. Since the first numerical treatment of the notch shape optimization problem by Tvergaard, numerous papers have appeared [3–7, 10, 12, 15, 16, 22, 23, 25–32, 34, 37–43, 46–55, 58, 60].

The focus of this paper is to illustrate the usage of an integrated shape optimal design of notches in a multi-disciplinary design environment [24, 33, 36]. A general methodology for the shape optimal design is developed by linking a geometric modeler (shape definition) of the notch in a machine component, stress analysis by the Boundary Element Method (BEM), called for short analyzer, the sensitivity analysis, which is related to the analysis method and the optimization procedure (optimizer).

The paper consists of eight sections. After the Introduction, components of the notch shape optimization algorithm are described in section 2. Section 3 presents the shape definition of notches by modified Bezier curves, and sections 4 and 5 contain the description of the Fictitious Stress Method (FSM) used for analysis of the stress field and stress gradients computations. In section 6 the chosen optimization procedure is briefly discused. Section 7 contains several numerical examples. In section 8 conclusions are presented.

2. Components of notch shape optimization algorithm

Generally, a shape optimization problem of notches in machine components can be stated mathematically as one of minimizing a specific objective function subject to certain behavioural constraints and bounds on the design variables. Shape optimization using numerical methods of analysis of stress field requires the sequential use of structural and sensitivity analysis combined with a numerical optimizer. The success of the shape optimization process depends on the proper choices with respect to the finite element or boundary element model, behaviour sensitivity analysis, objective function, constraints, design variables and method of solution of the optimization task.

2.1. Selection of the objective

The objective of the shape optimization of notches is to decrease the peak of stress concentration and to obtain shapes giving rise to uniform maximum reference stress along the single or multiple notch boundaries. Different choices of stress functions as the objective function are demostrated in literature. The objective function can be selected as: the maximum effective stress (von Mises, tangential) along a boundary of the notch [3, 4, 6, 16, 22, 27, 28, 31, 32, 37–42, 44, 46–48, 50–55, 58], the diference between maximum and minimum tangential stresses [5], maximum shear stresses [25, 26], stress levelling (integral stress function) [4–6, 30, 43, 46] and mixed weight-stress levelling objective [6].

The maximum effective stress and stress levelling objective functions have been compared in the paper [47]. It has appeared from the tests that the best objective function is the maximum effective stress. A problem of form finding of notches in 2-D machine components with respect to the maximum effective stress is considered in this paper.

2.2. Problem definition

Let us consider a machine component occupying a domain Ω with varying shape of internal $\Gamma \equiv \Gamma_i$ and/or external $\Gamma \equiv \Gamma_e$ notches (Fig. 1). The component is subjected to the boundary tractions $\mathbf{T}_0$ on a part Γ_T and to prescribed displacements $\mathbf{u}_0$ on a part Γ_u. As in every stress analysis problem,

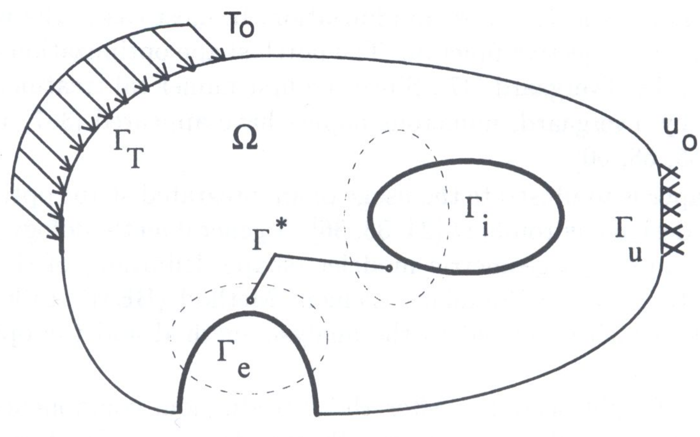

Fig. 1. 2-D machine component with varying shape of internal Γ_i and external Γ_e notches, Γ^* — variation domain

the stress, strain and displacements fields $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, $\mathbf{u}$ satisfy equilibrium, compatibility and boundary conditions, as well as stress-strain relations.

A shape optimization (form finding) problem of notches can be stated mathematically as one of minimizing a maximum effective (von Mises or tangential) stress σ_{e} in Ω, ie.:

$$\min\,[\max_{\Omega} \sigma_{\mathrm{e}}] = \min\,[\max_{\Gamma} \sigma_{\mathrm{e}}] \tag{1}$$

with the geometrical constraints:

$$\Gamma \subset \Gamma^{*} \tag{2}$$

for a given load.

The min-max problem, which is discontinuos and nondifferentiable is solved in this work by using a bound formulation. This original min-max problem can be converted to a simple min problem in terms of an unknown bound on the stresses, and can be written as:

$$\min \sigma_{\mathrm{emax}} \tag{3}$$

subject to the constraints (2) and to the additional constraints:

$$\sigma_{\mathrm{e}j}(\mathbf{D}) - \sigma_{\mathrm{emax}} \leq 0\,, \quad j = 1, \ldots, nc\,, \tag{4}$$

where nc is the number of critical points, $\mathbf{D} = [D_1, D_2, \ldots, D_n]^{\mathrm{T}} \in \{\mathbf{D}_{\mathrm{l}} \leq \mathbf{D} \leq \mathbf{D}_{\mathrm{u}}\} \subset R^n$ is a vector of design variables, $\mathbf{D}_{\mathrm{l}}$ and $\mathbf{D}_{\mathrm{u}}$ are the lower and upper limits on $\mathbf{D}$ respectively.

2.3. Shape definition and selection of design variables

The use of coordinates for boundary nodes in the finite element model as a shape design variables was the earliest used method [14, 17, 21]. Although this approach is simple and associated with the numerical methods of analysis of stress fields, it has severe drawbacks: it involves a large number of the design variables, and leads to an undesirable or impractical shapes.

In general, it is desirable to define the shape of the boundary by means of a reasonably small number of design variables to reduce the dimension of the problem. Some researches have defined the boundary of the structure by straight, or circular segments [25, 26, 32, 46]. Much more popular is use of polynomials [4–6, 16, 58], or orthogonal polynomials [27, 31, 50] to locate the boundary shape. The coefficients of polynomials treated as design variables significantly decrease their number. More general than the orthogonal polynomials description is a basic function concept [49]. Althought this polynomial boundary representation guarantees the smoothness of the boundary (one of the basic requirements for shape definition), it can also give an impractical oscillatory boundary shape when the polynomial order is too high. The shape oscilations eliminate spline curves [48], Hermitian splines [30, 42], Bezier curves [26, 33, 51–55]. Other ways for shape definition, for instance, B-splines, rational B-splines, or the use of the magnitudes of a set of fictitious loads as the design variables and the deformation produced by those loads to update the shape are reported in literature.

In this paper a special concept of Bezier's curves is used to locate the boundary of the notches.

2.4. Analysis of the stress field

It is characteristic for optimal shape design, that the mathematical model of a machine component cannot be expressed in terms of classical functions, and it is necessary to use numerical methods for response analysis. For the shape optimization of machine components with respect to stress, the Finite Element Method (FEM) has been used extensively [5, 16, 22, 27, 28, 31, 32, 34, 37–41, 45, 46, 48, 49, 51]. It is well known that the maximum stress is usually attained on the boundary of the machine component. Because the Boundary Element Method (BEM) , which has been recognized as an effective method of analysis of stress field, requires only the modelling of the boundary, it seems to be ideally suited for notch boundary shape optimization. The essential advantages of BEM are the following: (1) the method permits us to discretize only the boundary of the machine component and for this reason is much easier to prepare and control the data input, (2) the method also assures a more accurate solution, especially for stress concentration, (3) infinite regions are easy to solve, (4) not so much experience is necessary as in FEM to decide on the element size needed, (5) coupling to graphic computer program is much easier. Numerous applications of BEM to notch shape optimization are reported [3, 15, 30, 42, 43, ref. [3] in [45], 50, 52–54, 58]. The use of FEM coupling with BEM is reported in [23].

In this paper the Fictitious Stress Method (FSM), indirect variant of BEM is used.

2.5. Sensitivity analysis of the shape of the notch

Although many approaches to sensitivity analysis exist, there are two fundamentally different ones, namely, the discretized approach, called the Direct Sensitivity Analysis (DSA), and the continuous approach, called the Variational Design Sensitivity (VDSA) [7, 8, 12, 13, 20]. The well known Finite Difference Method (FDM) can be ranked among the discretized approach.

In the case of DSA approach the sensitivities are obtained by a direct implicit differentiation of the discrete analytic equations with respect to each design variable. There are several formulations of this method: 1) analytical techniques like: a) the Direct Differentiation Method (DDM), the Adjoint System Method (ASM), and 2) semi-analytical (quasi-analytical) technique [3, 7, 8, 12, 13, 20]. The last one is used, when analytical derivatives are complicated, and when the required derivatives are obtained using finite differences. The DDM method is more efficient than ASM method if the number of design variables is less than the number of constraints in the case of one load condition [20]. This is the case in the notch shape optimization problems considered.

In this paper design sensitivities are obtained by the direct implicit differentiation of the discretized boundary equations.

2.6. Optimization procedures

Two philosophically different approaches have been followed for notch shape optimization problem [1]. In the first approach, known as the optimality criteria methods (variational approach, see [7, 10, 12, 26, 29]), necessary conditions for optimality are derived. Because these conditions are generally nonlinear, numerical methods must be used to solve the nonlinear necessary conditions. The second approach is iterative, in which numerical algorithms, grouped into primal and transformation methods [35] are used to improve a starting estimate of the optimum solution.

Several numerical methods have been used for the solution of stress minimization problems: Sequential Linear Programming (SLP) [3, 4–6, 27, 31, 34, 43, 47, 50–55, 58], Sequential Unconstrained Minimization Technique (SUMT) [16, 25, 26, 30, 32, 42], feasible direction method [15, 28], multiplier methods [45, 46]. Schnack [22, 37–41] originally has developed the hypothesis about the stress reduction in the neighboourhood of a notch helped by dynamic programming method.

The notch shape optimization problem considered in this paper is characterized by: (1) relatively small number of design variables, (2) a large numer of stress constraints growing in subsequent iterations, and (3) linear objective function.

In this paper, the SLP procedure based on variable move limits is used.

3. Shape definition by modified Bezier curves

It is assumed that the variable boundary Γ of a notch can be formed by: (1) a curvilinear part and, if it is necessary (2), a straight line. In this paper Bezier's curves will be used to shape representation of the curvilinear part of the boundary (Figs. 2, 3). A standard, one-segmented Bezier curve (polynomial) of order $m+1$ (degree m) is defined by [59]

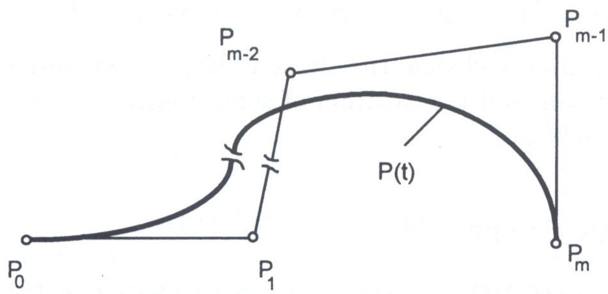

Fig. 2. Standard Bezier curve

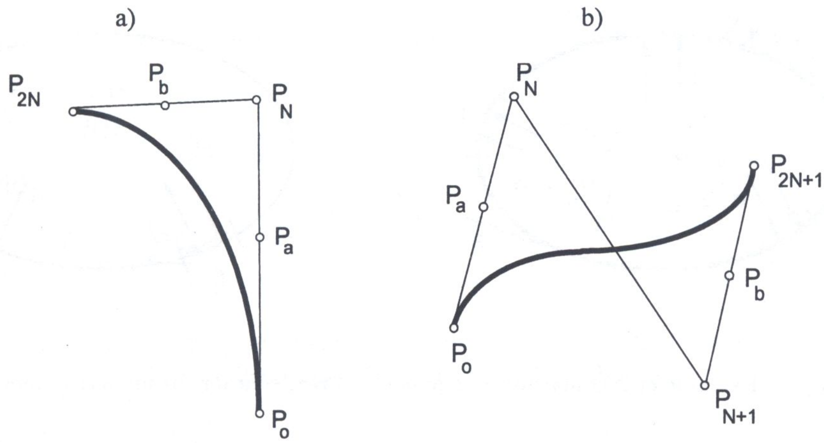

Fig. 3. Bezier curve inside: a) the CT, b) the ZCP

$$P(t) = \sum_{i=0}^{m} P_i B_{i,m}(t)\,, \tag{5}$$

where

$$B_{i,m} = \frac{m!}{i!(m-i)!}\, t^i (1-t)^{m-i}\,, \quad \text{and} \quad t \in [0,1]\,, \tag{6}$$

is the Bernstein polynomial, P_i, $i = 0, 1, \ldots, m$ are the Bezier points (control points) in R^2 (or R^3), and t is a parameter representive of a curve. Bezier curves offer many interesting properties: (1) each curve lies within the convex hull of the control points that define it, (2) the Bezier curve passes only through control endpoints, (3) at the end points, the curve is tangent to the corresponding edge of the polygon formed by the control points.

In the papers [18, 19] some modifications on Bezier curves are presented. A Bezier curve (Fig. 3a, 3b) is defined by so called characteristic triangle (CT) or Z-shaped polygon (ZCP) and, for more flexible control, two shape parameters are introduced. The power representation of Bezier curves is given in Appendix.

The vertices composing a CT satisfy:

$$P_a - P_0 = c_1(P_N - P_0)\,, \quad P_b - P_{2N} = c_2(P_N - P_{2N})\,, \quad 0 < c_1, c_2 < 1\,. \tag{7}$$

The vertices composing a ZCP satisfy:

$$P_a - P_0 = c_1(P_N - P_0)\,, \quad P_b - P_{2N+1} = c_2(P_{N+1} - P_{2N+1})\,, \quad 0 < c_1, c_2 < 1\,. \tag{8}$$

Parameters c_1 and c_2 continuosly change the shape of segmented Bezier curve inside the CT and ZCP. These shape parameters and the position of some control nodes (master nodes, key nodes) are assumed as design variables.

4. FSM METHOD AS ANALYSER

The Fictitious Stress Method (FSM) presented in [11] is used for an analysis of the stress distribution in 2-D structural elements. This is an indirect boundary element method. Figure 4 shows a cavity

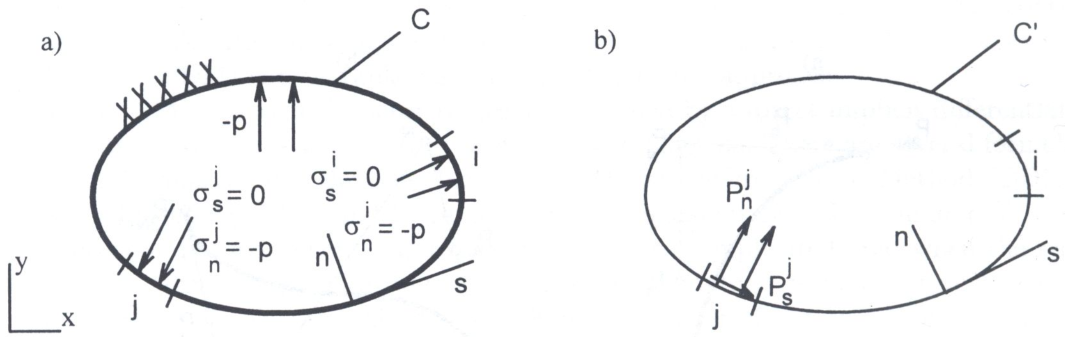

Fig. 4. Example of 2-D machine component: a) real contour, b) numerical model

(very long in z direction) in an infinite elastic body. The boundary of the cavity is labelled C in Fig. 4a. The dashed curve C' shown in Fig. 4b has the same shape as the curve C. Both curves are approximated by straight line segments (elements), joined end to end. The difference between curves is, that curve C' represents the location of these line segments in an inifinite body (without cavity), which are coincident with the real boundary C. The shear P^j_{s} and normal P^j_{n} stresses applied to the segment j induce the actual stresses σ^j_{s} and σ^j_{s} at the midpoints of each element of curve C', $i = 1$ to n. The stresses P^j_{s} and P^j_{n} are fictitious quantities and should be determined. The relation between the actual stresses σ^j_{s} and σ^j_{n} and the fictitious stresses P^j_{s} and P^j_{n} is based on the analytical singular solution to the problem of constant normal and shear stresses applied to an arbitrarily oriented, finite line segment in an infinite body (Kelvin solution), what leads directly to the system of $2n * 2n$ equations

$$\mathbf{C}\mathbf{P} = \mathbf{b}\,, \tag{9}$$

where: $\mathbf{C}$ — the influence coefficient matrix, $\mathbf{P}$ — unknown fictitious stress components, $\mathbf{b}$ — given tractions (or displacements).

If the fictitious stresses are known, the tangential stresses can be found

$$\sigma_t = \mathbf{A}_{\mathrm{ts}}\mathbf{P}_{\mathrm{s}} + \mathbf{A}_{\mathrm{tn}}\mathbf{P}_{\mathrm{n}}\,, \tag{10}$$

where: $\mathbf{A}_{\mathrm{ts}}$ and $\mathbf{A}_{\mathrm{tn}}$ are $n*n$ matrices of the influence coefficients for tangential stresses, $\mathbf{P}_{\mathrm{s}}$ and $\mathbf{P}_{\mathrm{n}}$ are $n*1$ vectors of the shear and normal fictitious stress components, respectively. Because constant stress elements are assumed to be on the boundary, there is no need for numerical integration. The accuracy of results can be increased by increasing the number of elements. The accuracy of the FSM for the solution of stress concentration problem in 2-D machine components has been examined in

[56]. It has been found that there is no remarkable difference between the results of the FSM notch stress analysis and the analytical, and numerical results widely presented in literature.

The main motivation for the selection of FSM as the analyser is the relative simplicity of stress gradient computations.

5. Direct sensitivity analysis

It should be mentioned that the objective function is linear, and only there is a need to evaluate the gradients of the stresses with respect to design variables. Design sensitivity analysis of a boundary element of a discretized structural element with the notch is based on the analytical implicit differentiation of the tangential stress equation (10) and of the global equilibrium equation (9). Differentiating Eq. (10) with respect to design variable D_j, we obtain the equation

$$\frac{\partial \sigma_t}{\partial D_j} = \frac{\partial \mathbf{A}_{\mathrm{ts}}}{\partial D_j}\mathbf{P}_{\mathrm{s}} + \mathbf{A}_{\mathrm{ts}}\frac{\partial \mathbf{P}_{\mathrm{s}}}{\partial D_j} + \frac{\partial \mathbf{A}_{\mathrm{tn}}}{\partial D_j}\mathbf{P}_{\mathrm{n}} + \mathbf{A}_{\mathrm{tn}}\frac{\partial \mathbf{P}_{\mathrm{n}}}{\partial D_j}\,. \tag{11}$$

The fictitious stress derivatives can be obtained by differentiation of equilibrium equations (7)

$$\frac{\partial \mathbf{C}}{\partial D_j}\mathbf{P} + \mathbf{C}\frac{\partial \mathbf{P}}{\partial D_j} = \frac{\partial \mathbf{b}}{\partial D_j}\,. \tag{12}$$

Rearranging yields

$$\frac{\partial \mathbf{P}}{\partial D_j} = \mathbf{C}^{-1}\left[-\frac{\partial \mathbf{C}}{\partial D_j}\mathbf{P} + \frac{\partial \mathbf{b}}{\partial D_j}\right]. \tag{13}$$

The above sensitivities are derived analitically by differentiation with respect to D_j [57]. For comparison of the results the Finite Difference Method (FDM) is also used

$$\frac{\partial \sigma_{\mathrm{t}}}{\partial D_j} \cong \frac{\sigma_{\mathrm{t}}(D_j + \Delta D_j) - \sigma_{\mathrm{t}}(D_j)}{\Delta D_j}\,. \tag{14}$$

6. SLP as optimizer

It is well known, that the cost of computations is extremaly high when the FEM or BEM is used as component of optimization algorithm. To overcome these difficulties suitable approximation concept is introduced, where implicit original optimization problem is replaced by explicit, approximatelly equivalent one, but which is easier to solve [2]. This concept has found wide application in the optimization technology.

The simplest is the linear (first-order) approximation based on the Taylor expansion of the objective function and constraints, what leads to the Sequential Linear Programming (SLP) method used in this paper. The following facts were the main motivation for the selection of SLP as the optimization procedure: (1) Earlier [44] and latest investigations [35] showed that the SLP code is quite robust, efficient and can be applied to solve large real life design problem succesfully. (2) The SLP method is one of easiest algorithms to program, and efficient linear program algorithms are available on most computer systems. (3) The objective function because of the use of "bound formulation" is linear, and only stress constraints should be linearized.

After linearization of stress constraints, the problem (2) can be rewritten as:

$$\text{minimize: } \sigma_{\mathrm{emax}} \tag{15}$$

subject to the constraints (2) and to the linearized additional constraints:

$$\sigma_{\mathrm{e}} + \nabla^{\mathrm{T}}\sigma_{\mathrm{e}}\Delta\mathbf{D} - \mathbf{1}\sigma_{\mathrm{emax}} \leq \mathbf{0}\,, \tag{16}$$

and

$$\Delta \mathbf{D}_{\mathrm{l}} \leq \Delta \mathbf{D} \leq \Delta \mathbf{D}_{\mathrm{u}}\,, \tag{17}$$

where: $\Delta\mathbf{D}$ are design changes and $\Delta\mathbf{D}_{\mathrm{l}}$ and $\Delta\mathbf{D}_{\mathrm{u}}$ are the lower and upper move limits [9] on $\Delta\mathbf{D}$ respectively, which can change in each iteration. They play an important role because they control convergence and allow to avoid oscillations if a solution does not lie at the vertex of the design space. The "move limits" are also used for transforming designs into non-negative designs (it is required by the LP algorithm that all variables are limited to the non-negative range).

7. Numerical examples

A computer program written in TurboPascal was developed for the analysis of 2-D machine components and for the Sequential Linear Programming method. This program was implemented using IBM PC/AT computer. The optimization procedure is considered to have converged to the final minimum stress concentration when the condition $|\sigma_i^{\max} - \sigma_{i-1}^{\max}|/\sigma_{i-1}^{\max} \leq \varepsilon$ is satisfied for the two succesive iterations, where σ_i and σ_{i-1} are the effective stresses after i and $i-1$ iterations, and ε is a user-defined tolerance value (e.g. $\varepsilon = 0.01$ corresponds to a 1% convergence criterion). The tolerance $\varepsilon = 0.0001$ is assumed in this paper. Seven examples are chosen to illustrate the use of the computer program.

Example 1. The classical fillet problem

The fillet problem (Fig. 5) is a classical test problem for shape optimization. The objective is to find the shape of the transition zone Γ in the variation domain Γ^* that minimizes the maximum of effective stress.

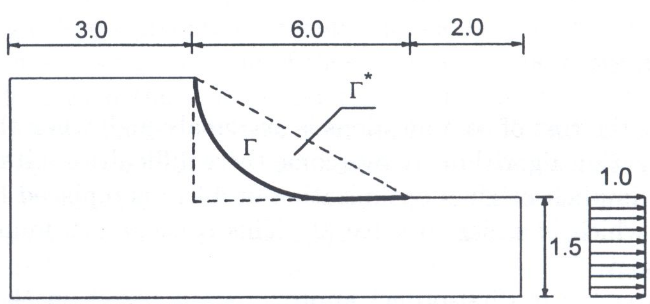

Fig. 5. Fillet problem

Due to the symmetry of the problem, only a quarter of the fillet is analyzed. Figure 6 presents the boundary element mesh (subsequent values of parameters t_j and t_{j-1} define the endpoints of the boundary element) and the shape definition of the boundary notch. A boundary of the fillet is modelled by 21 boundary elements. The shape of the fillet defined by CT is controlled through the the position of the control node $D_1 = P_6$, and control shape parameters $D_2 = c_1$, $D_3 = c_2$ (in fact the position of control nodes $P_{1,2}$ and $P_{4,5}$). The control nodes are only allowed to move in the directions specified by the arrows. Starting with initial values $\mathbf{D} = (0.5,\ 0.5,\ 0.5)$ after 6 iterations the optimal fillet shape (Fig. 7) is achieved for design variables $\mathbf{D} = (2.400,\ 0.0624,\ 0.7161)$ with

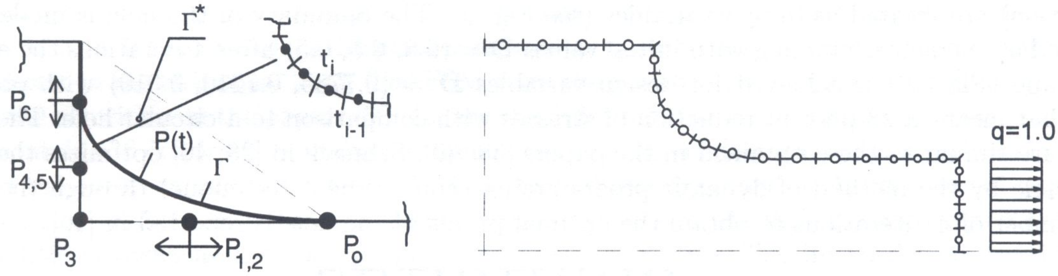

Fig. 6. Shape definition and boundary element model for fillet

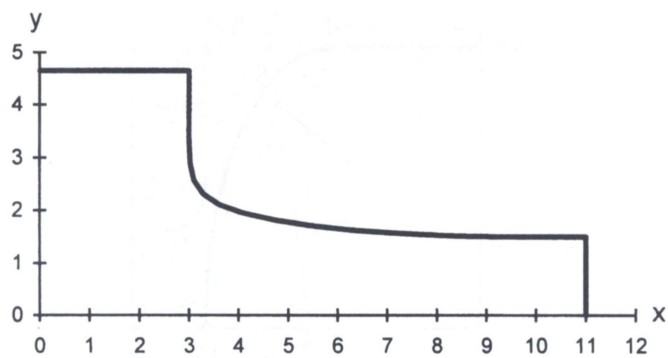

Fig. 7. Fillet problem — optimal contour

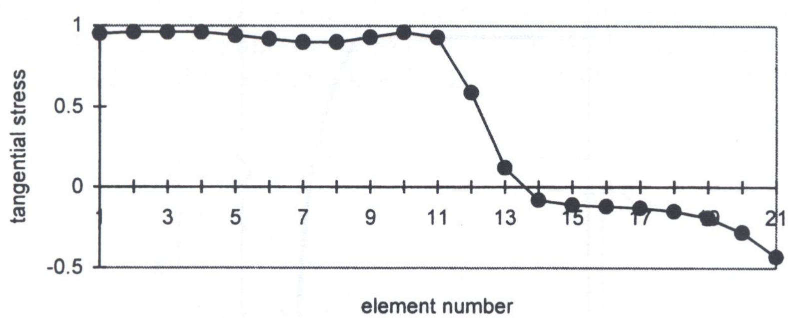

Fig. 8. Fillet problem — stress distribution around transition zone

objective 0.9649, what means that the stress concentration is completely removed. The result 1.0098 after 7 iterations has been obtained in [34] using SLP algoritm with FEM. The optimal profile of the fillet and the optimal stress distribution (the numbering of nodes on the notch boundary is local) along the fillet are shown in Fig. 8.

Example 2. Optimum shape of quasi-square hole in infinite plate under uniaxial tension

The next example is also treated as the test problem. The infinite plate is under uniaxial loading. The problem is symmetric about both coordinate axes, so only the quarter of the plate needs to be modelled. A boundary of the hole is described by the seven-order Bezier curve ($N = 3$). Control shape parameters $D_2 = c_1$, $D_3 = c_2$, and the position of the control node $D_1 = P_6$ (measured in x direction) are treated as design variables (see Fig. 9). The boundary of the hole is modelled by 28 boundary elements. Starting with initial values $\mathbf{D} = (0.5, 0.5, 0.5)$ after 4 iterations the optimal hole shape (Fig. 10) is achieved for design variables $\mathbf{D} = (0.7785, 0.1210, 0.010)$ with objective 2.16. That means a 28 percent reduction of stresses with comparison to a circular hole. The above results are similar to these obtained in the papers [38, 40]. Schnack in [38, 40] optimizes the shape of the hole by the method of dynamic programming (non gradient technique) through the FEM. The number of 48 iterations to obtain the optimal profile of the hole is reported in [40].

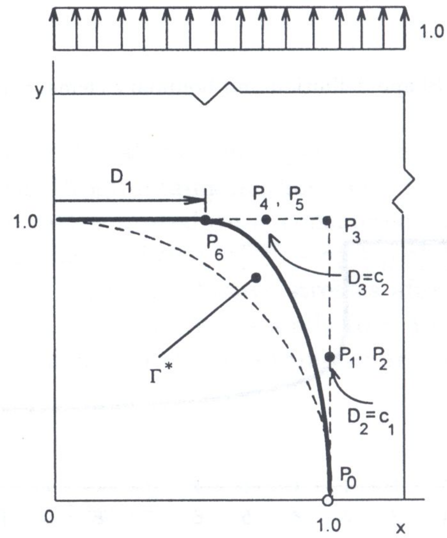

Fig. 9. Single hole problem

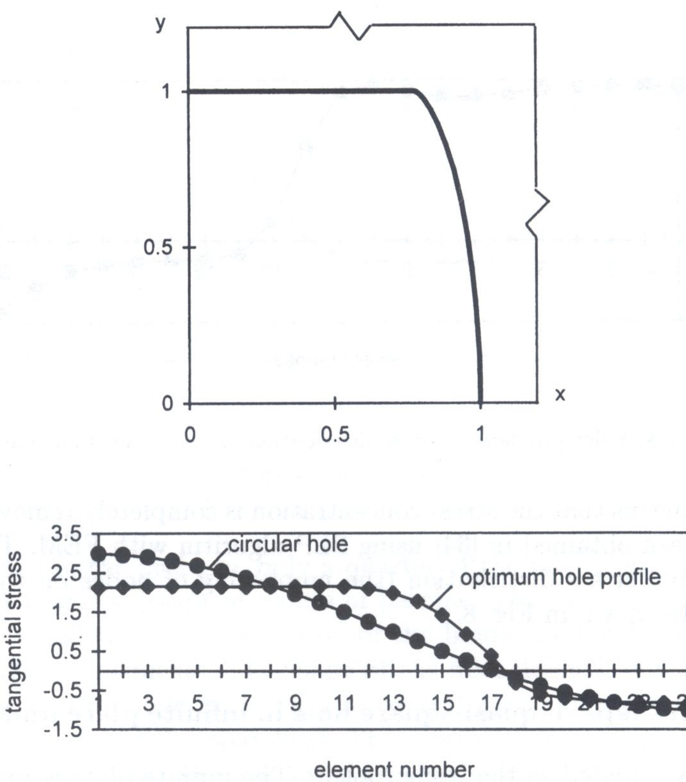

Fig. 10. Single hole problem — optimal contour and stress distribution for optimal hole profile and circular hole

Example 3. Optimum shape of a hole in an infinite plate under biaxial tension

This example comes from the papers [30, 42]. The plate is under biaxial field with the ratio $\sigma_2/\sigma_1 = 1/2$. In the cited papers the plate is treated as finite with ratio of the width of the plate to the diameter of the hole equal 20/1. The FSM gives the possibility to consider an infinite plate. Details of the algorithm in the papers [30, 42] are the following: the direct variant of BEM is used, the model contains 36 quadratic boundary elements, the hermitian cubic spline defined the hole boundary, and the optimization method used is the extended penalty function. In the present paper: the indirect BEM (FSM), 25 constant elements, Bezier curve, and the SLP method respectively. The circular hole is taken as the starting profile in both algorithms. The initial design variables and the movements directions are shown in Fig. 11. The standard Bezier curve order $m = 3$ with

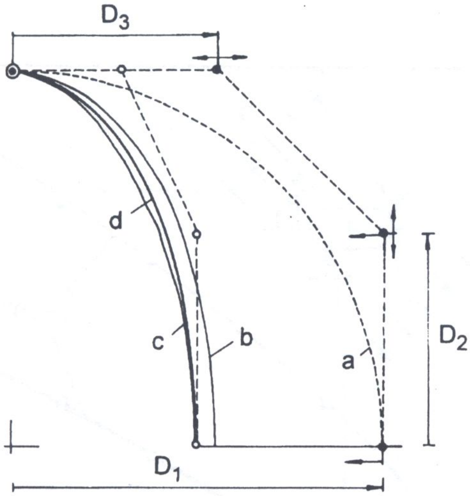

Fig. 11. Optimum shape of hole: a) initial shape, b) optimum shape [30,42], c) Author's optimum profile, d) analytical solution

$D_2 = D_3 = 4/3(\sqrt{2} - 1) \cong 0.55$ gives a good approximate expression of the circle. Starting with initial values $\mathbf{D} = (1.0,\ 0.55,\ 0.55)$ and the objective $\sigma_e = 2{,}4995$, after 8 iterations (25 iterations in [30]) the final shape is achieved for design variables $\mathbf{D} = (0.4999,\ 0.5581,\ 0.2738)$ with $\sigma_e = 1.5021$.

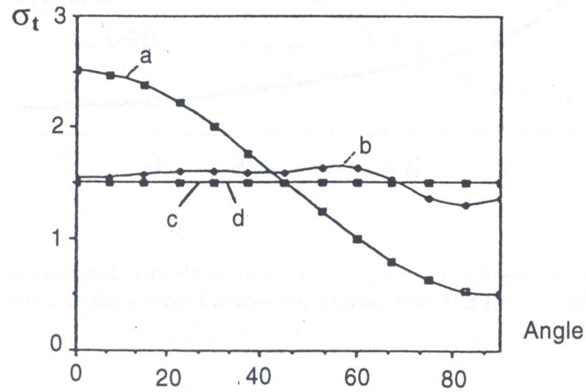

Fig. 12. Stress distribution around hole: a) for initial shape, b) for optimal solution [30], c) present solution, d) analytical solution

Figure 11 also shows the optimal analytical solution ($\sigma_e = 1.50$) and that obtained in the papers [30, 42]. Figure 12 shows the stress distribution along the hole before and after optimization with the comparison to the results in the above mentioned papers.

Example 4. Optimum shape of a quasi-ovaloid hole in infinite plate

The stress concentration can be reduced by making the hole oblong in the direction of loading (so-called oblong, ovaloid hole). If the loading acts perpendicularly to the oblongity, the SCF increases). A CT concept is used for the shape definition of the boundary of the quasi-ovaloid hole. Only three (practically two) design variables are needed to control the shape of the boundary. Figure 13 shows the SCF's for the holes with the circular rounded boundaries, and the SCF's corresponding to the optimum shapes of quasi-ovaloid holes, for two loading cases $\beta = 0^o$, and $\beta = 90^o$, and for different values of the parameter t. About 28–39% reduction in stresses is obtained in comparison with the ovaloid hole with circular rounded boundaries, when the oblongity is in the direction of loading. More details are presented in the paper [54].

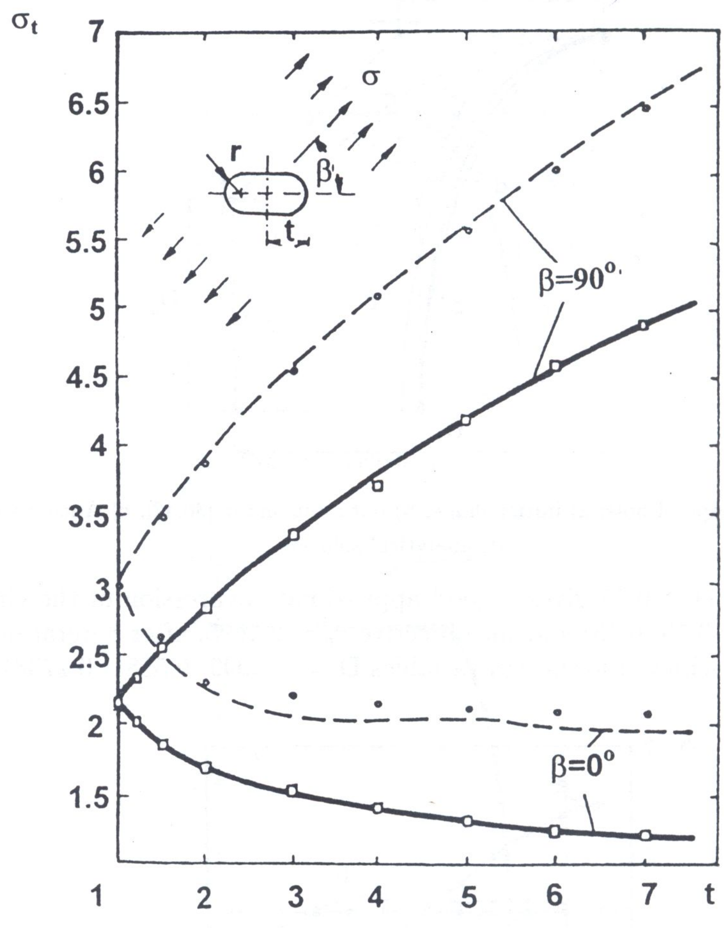

Fig. 13. Stress concentration factors for: (- - -) ovaloid hole with circular rounded boundaries (Ref. [1] in [54]), (—) optimal hole profile, (o) FSM test results for ovaloid hole with circular rounded boundaries

Example 5. Optimum shape of quasi-square hole in finite tensioned strip

In this example the shape of the hole in finite tensioned strip is optimized. For details of the problem description see Example 2. Table 1 presents the optimal results for a single hole in finite tensioned strip for different ratios r/w, where r is the dimension of the hole, and w is the width of the strip. Figure 14. shows the optimal contour of the hole and the corresponding stress distribution around the hole in finite strip for $r/w = 0.5$.

Table 1

No.	r/w	Optimal design variables D_1	D_2	D_3	Optimal objective	Number of iterations
1	0.10	0.7829	0.1264	0.0100	2.2338	5
2	0.25	0.7998	0.1207	0.0100	2.3125	6
3	0.50	0.8601	0.2754	0.1762	2.9745	14
4	0.75	0.9394	0.7298	0.3688	4.9989	16

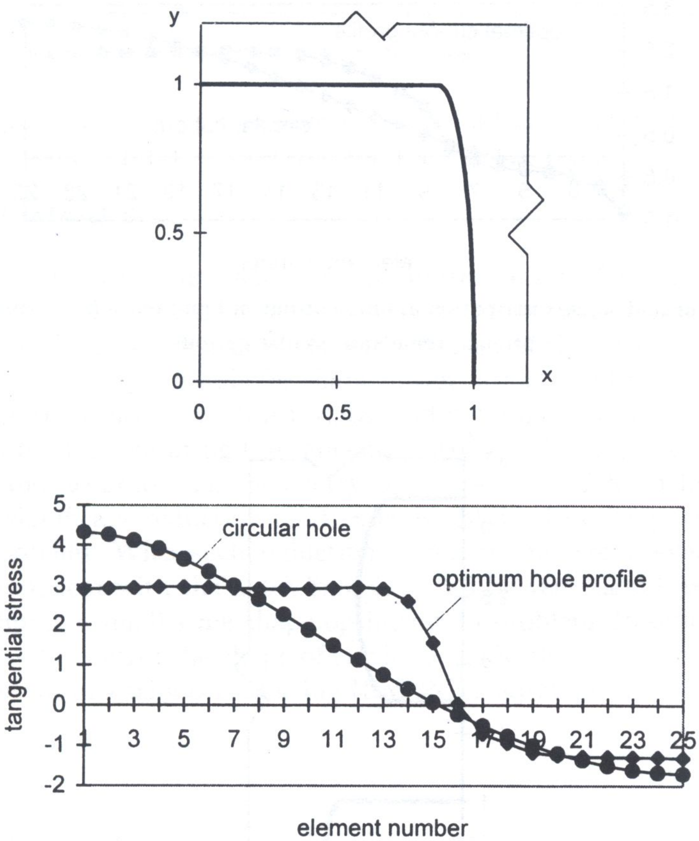

Fig. 14. Optimal contour and stress distribution around single hole in finite tensioned strip for $r/w = 0.5$ for optimal hole profile and circular hole

Example 6. Optimum shape of quasi-square cut-out in finite tensioned strip

In this example the shape of the external notch cut-out in finite tensioned strip is optimized. Figure 15 shows the optimal contour of the cut-out and the corresponding stress distribution around the cut-out for $r/w = 0.5$.

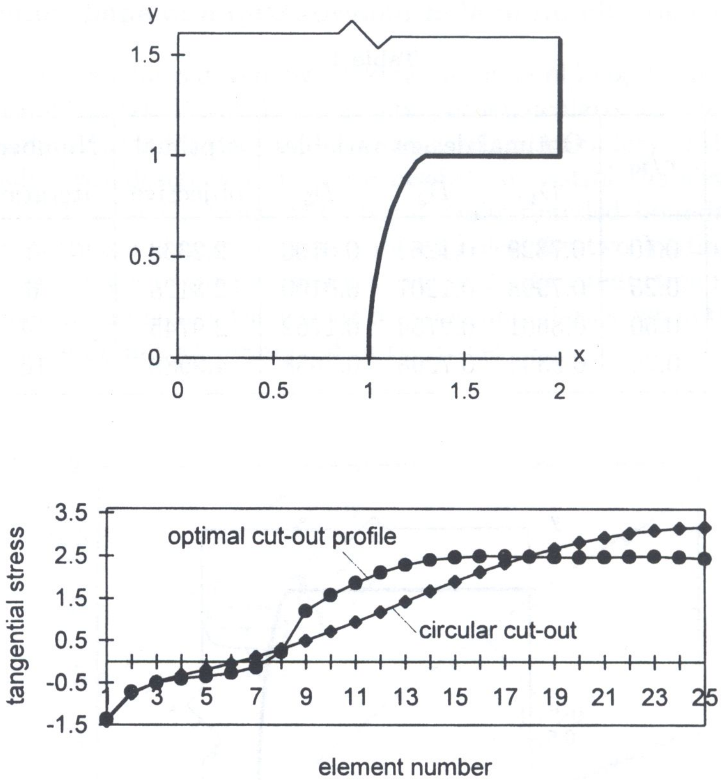

Fig. 15 Optimal contour and stress distribution around cut-out in finite tensioned strip for $r/w = 0.5$ for optimal profile and circular cut-out

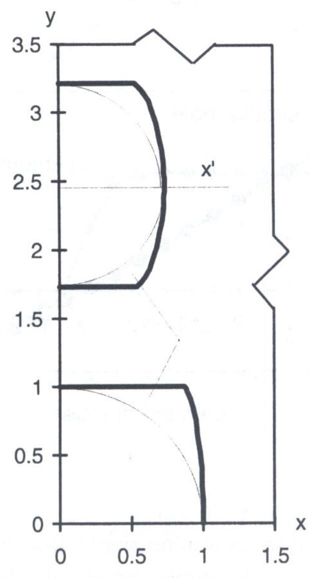

Fig. 16. Optimal contours for central and defense holes

Example 7. Optimum shape of internal notches (holes) in finite strip under uniaxial tension

One method for reducing the static stress concentration is introducing additional (defense) holes in the direction of loading on either side of the central hole. Further reduction of the stresses is possible with simultaneous shape optimization of both kinds of holes. The problem is symmetric about both coordinate axes, so only the quarter of the strip needs to be modelled (Fig. 16). The boundary of the central hole was modelled using 21 boundary elements, and the boundary of the additional hole using 42 boundary elements. The exterior boundary of the strip is divided into 60 boundary elements. Figure 16 shows the optimal design of the central end defense holes, and Fig. 17 shows the optimal stress distribution along the design boundaries. The 37% reduction in stresses in comparison with the single circular hole is obtained by the presented numerical approach.

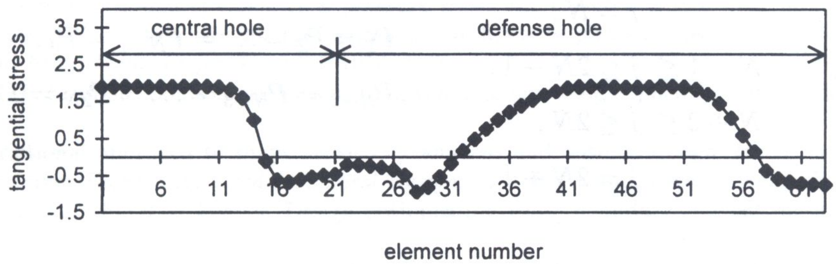

Fig. 17. Stress distribution along central and defense design boundaries

8. Concluding remarks

A multi-disciplinary, numerical approach to shape optimization of notches is presented. Shape optimization of notches in 2D machine components to minimize stress concentrations is formulated as the sequential linear programming problem with the use of the Fictitious Stress method. The Fictitious Stress Method is very suitable for shape optimization problems, and in comparison with the Finite Element Method, needs much less data, and gives more accurate stress solution. Stress derivatives are found by differentiating the tangential stress and the global equilibrium equations, which is less costly and accurate. The geometry of the notch is defined by a special concept of Bezier technique. A significant reduction in stresses is obtained in comparison with traditionally used shapes of the notches. With such reduction in maximum stress level, the improvement in fatigue life of the component with the notch (notches) can be very significant.

Presented algorithm is a small scale shape optimization problem. Practically only a few design variables are necessary to control the shape of single, or multiple notches, which means that these and similar problems can be easily solved using IBM PC computers.

Appendix

Power representation of Bezier interpolants [18, 19]

The Bezier segment inside CT is:

$$P(t) = \sum_{i=0}^{2N} a_i t^i \,, N \geq 2 \,, \qquad a_i = \begin{cases} P_0 \,, & i = 0 \,, \\ (-1)^i \binom{2N}{i} \left[P_b + (-1)^i \binom{i}{j} P_j \right] \,, & 1 \leq i \leq 2N \,, \end{cases}$$

$$P_j = \begin{cases} P_a & 1 \leq j \leq N-1 \,, \\ P_N & j = N \,, \qquad\qquad P_1 = P_2 = \ldots = P_{N-1} = P_a \,, \\ P_b & N+1 \leq j \leq 2N-1 \,, \qquad P_{N+1} = P_{N+2} = \ldots = P_{2N-1} = P_b \,. \\ P_{2N} & j = 2N \,, \end{cases}$$

The Bezier segment inside a ZCP is described as:

$$P(t)=\sum_{i=0}^{2N+1} a_i t^i\,,\qquad a_i=\begin{cases} P_0\,, & i=0\,,\\ (-1)^i\binom{2N+1}{i}\left[P_0+\sum_{j=1}^{i}(-1)^i\binom{i}{j}P_j\right], & 1\le i\le 2N+1\,,\end{cases}$$

$$P_j=\begin{cases} P_a & 1\le j\le N-1\,,\\ P_N & j=N\,,\\ P_{N+1} & N+1\le j\le 2N-1\,,\\ P_b & N+2\le j\le 2N\,,\\ P_{2N+1} & j=2N+1\,,\end{cases}\qquad \begin{aligned}&P_1=P_2=\ldots=P_{N-1}=P_a\,,\\ &P_{N+2}=P_{N+3}=\ldots=P_{2N}=P_b\,.\end{aligned}$$

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