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S–continued fractions for complex transport coefficients of two-phase composites

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We examine S–continued fraction bounds on the effective dielectric constant ε_e of a two-phase composite for the case where the dielectric coefficients ε_1 and ε_2 are complex. The starting point for our study is a power series expansion of $\varepsilon_e(z)$ at $z = 0$, $z = \varepsilon_2/\varepsilon_1 - 1$. The S–continued fractions to the expansions of $\varepsilon_e(z)$ have an interesting mathematical structure. Its convergents represent the best bounds derived earlier by Milton [24–25], and independently by Bergman [5]. Specific examples of calculation of complex S–continued fraction bounds on ε_e are provided.

1. Introduction

Transport coefficients of composite materials may be evaluated effectively by the method of bounds [6,17,18,24,25]. The bounds become increasingly accurate when more information concerning the geometrical properties of the medium is available.

From Bergman's paper [4] it follows that the bulk effective dielectric constant $\varepsilon_e(\varepsilon_1, \varepsilon_2)$ of two-components composite with dielectric constants ε_1, ε_2 has a special Stieltjes function representation. The analytical properties of this Stieltjes function have been used to develop a number of methods for evaluation of bounds on $\varepsilon_e(\varepsilon_1, \varepsilon_2)$ for cases, where: 1) the microstructure of the composite is known exactly [21,25]; 2) only partial information about microstructure [5–11] is available.

The analytical properties of Stieltjes functions were studied extensively in mathematical literature, *cf.* [1–3]. Consequently, the methods applied in the theory of inhomogeneous media [4–11,24,25] are directly related to the Padé approximants and continued fractions approaches developed in [1–3].

In the present paper we shall apply the S–continued fraction method for evaluation of the effective dielectric constant ε_e of a two-phase medium. This method was originally developed by Baker for calculation of the errors of Padé approximants to Stieltjes functions [13].

Our paper is organized as follows. In Section 2 we introduce a characteristic geometrical function $zf_1(z)$ representing ε_e and derive the S–continued fraction to $zf_1(z)$. In Section 3 the definitions of the inclusion regions and bounds for a hierarchy of special Stieltjes functions $zf_1(z)$ $(p = 1, 2, \ldots)$ are introduced. Analytical expressions for the auxiliary inclusion regions and bounds are derived in Section 4. In Section 5 the results of numerical calculations of complex bounds on bulk dielectric constants in the case of square and hexagonal arrays of cylinders are presented. Summary and discussion complete the paper.

2. Basic assumptions, definitions and notations

We consider the effective dielectric constant E_e of a two-phase medium for the case where the dielectric constants ε_1 and ε_2 are complex numbers. The bulk dielectric coefficient E_e is defined by the linear relationship between the volume–averaged electric field $\langle \mathbf{U} \rangle$ and the volume-averaged displacement field $\langle \mathbf{D} \rangle$

$$\langle \mathbf{D} \rangle = E_{\mathrm{e}} \langle \mathbf{U} \rangle . \tag{1}$$

By φ_1 and φ_2 we denote the matrix and the inclusion volume fractions, respectively. In general, E_{e} will be a second-rank symmetric tensor, even when ε_1 and ε_2 are both scalars. Our study will be focused upon one of the principal values of E_{e} denoted by $\varepsilon_{\mathrm{e}}(\varepsilon_1,\varepsilon_2)$. The remaining principal values of E_{e} can be studied similarly. The analytic properties of the bulk dielectric coefficient $\varepsilon_{\mathrm{e}}(\varepsilon_1,\varepsilon_2)$ were examined by Bergman in [3]. He proved that $\varepsilon_{\mathrm{e}}(\varepsilon_1,\varepsilon_2)/\varepsilon_1 = \varepsilon_{\mathrm{e}}(1,\varepsilon_2/\varepsilon_1)$ is a Stieltjes function of $\varepsilon_2/\varepsilon_1$, analytical outside the negative part of the real axis. Consequently, we can write

$$\frac{\varepsilon_{\mathrm{e}}(z)}{\varepsilon_1} - 1 = z f_1(z) , \tag{2}$$

where

$$f_1(z) = \int_0^1 \frac{\mathrm{d}\gamma(u)}{1+zu} , \quad z = h - 1 , \quad h = \varepsilon_2/\varepsilon_1 , \tag{3}$$

is a Stieltjes function defined in the cut $(-\infty \leq z \leq -1)$ complex plane by means of the real, bounded and non-decreasing spectrum $\gamma_1(u)$, *cf.* [1–3].

Consider now the power expansions of (2) at $z = 0$:

$$z f_1(z) = \sum_{n=1}^{\infty} c_n^{(1)} z^n . \tag{4}$$

The coefficients c_n are determined as moments

$$c_n^{(1)} = (-1)^{n+1} \int_0^1 u^{n-1} \mathrm{d}\gamma_1(u) , \quad n = 1,2,\ldots . \tag{5}$$

Let us introduce an infinite sequence of functions $f_p(z)$ $(p = 1,2,\ldots)$ defined by

$$f_{p-1}(z) = \frac{c_1^{(p-1)}}{1 + z f_p(z)} , \quad c_1^{(p-1)} = f_{p-1}(0) , \quad p = 2,3,\ldots . \tag{6}$$

Here $f_p(z)$, given by (3), is an input for relation (6), while $c_1^{(p-1)}$ denotes the first coefficient of the power expansion of the function $z f_{p-1}(z)$. We call $z f_1(z)$ *the basic function*, while $z f_p(z)$ $(p = 2,3,\ldots)$, *the auxiliary functions.* The functions $f_p(z)$ $(p = 1,2,\ldots)$ generated by $f_1(z)$ via recurrence formula (6) are Stieltjes functions of type (3), see [1, Lemma 15.3 and Chap. 17A]. Thus we have

$$f_p(z) = \int_0^1 \frac{\mathrm{d}\gamma_p(u)}{1+zu} , \quad p = 1,2,\ldots . \tag{7}$$

The power expansions of $z f_p(z)$ at $z = 0$ can be written as follows

$$z f_p(z) = \sum_{n=1}^{\infty} c_n^{(p)} z^n , \quad p = 1,2,\ldots , \tag{8}$$

where

$$c_n^{(p)} = (-1)^{n+1} \int_0^{\infty} u^{n-1} \mathrm{d}\gamma_p(u) , \quad n = 1,2,\ldots ; \quad p = 1,2,\ldots . \tag{9}$$

The fractional transformation (6) applied $(p-1)$ times to the Stieltjes function $f_1(z)$ leads to a continued fraction relationship between $f_1(z)$ and $f_p(z)$

$$f_1(z) = \frac{g_1}{1} \, \frac{}{+} \, \frac{g_2 z}{1} \, \frac{}{+} \cdots \frac{}{+} \, \frac{g_{p-2} z}{1} \, \frac{}{+} \, \frac{g_{p-1} z}{1} \, \frac{}{+} \, \frac{z f_p(z)}{1} , \quad p = 1,2,\ldots , \tag{10}$$

where

$$g_k = c_1^{(k)}\,, \quad k = 1,2,\ldots,p-1\,. \tag{11}$$

Since the first power series coefficients $c_1^{(k)}$ $(k = 1,2,\ldots,p-1)$, given by (9) satisfy the condition $c_1^{(k)} = f_k(0) = \int_0^1 \mathrm{d}\gamma_k(u) > 0$, hence

$$g_k > 0\,, \quad k = 1,2,\ldots,p-1\,. \tag{12}$$

Now we are in the position to propose the algorithm for finding the S–continued fraction coefficients g_k from the coefficients $c_k^{(1)}$ $(k = 1,2,\ldots,p-1)$, given by (5). By applying the linear fractional transformation (6) to (8), we obtain

$$\frac{g_{k+1}z}{1+\sum\limits_{n=1}^{p-2-k} c_n^{(k+2)} z^n} = \sum_{n=1}^{p-1-k} c_n^{(k+1)} z^n\,, \quad k = 0,1,\ldots,p-3\,, \tag{13}$$

where $g_{k+1} = c_1^{(k+1)}$. Simple rearrangements yield

$$\begin{cases} k = 0,1,2,\ldots,p-2\,, \\ \begin{cases} n = 1,2,\ldots,p-2-k\,, \\ c_0^{(2+k)} = 1\,, \quad c_n^{(2+k)} = -\dfrac{1}{c_1^{(1+k)}}\left(\sum\limits_{j=0}^{n-1} c_j^{(2+k)} c_{n+1-j}^{(1+k)}\right), \end{cases} \\ g_{2+k} = c_1^{(2+k)}\,, \end{cases} \tag{14}$$

where

$$c_m^{(1)}\,, \quad m = 1,2,\ldots,p-1\,; \quad g_1 = c_1^{(1)} \tag{15}$$

are input data for (14). By starting from $(p-1)$ terms of the Stieltjes series $f_1(z)$ we generate successively, with the aid of (14)–(15), the power expansions of $f_2(z)$, $f_3(z)$, $f_4(z)$, $\ldots$, $f_{p-1}(z)$ with steadily decreasing numbers of terms, see (15). At some point we will be left with the function $f_p(z)$, for which no terms of its series expansion are given.

3. Inclusion regions and bounds

Our subsequent considerations are based on the assumption that a number of $p-1$ $(p = 1,2,\ldots)$ coefficients of the power expansion of the Stieltjes function $zf_1(z)$, given by (4), is known. Let us rewrite (10) as follows:

$$zf_k(z) = \frac{g_k z}{1} \, \underset{+}{} \, \frac{g_{k+1} z}{1} \, \underset{+}{} \cdots \underset{+}{} \, \frac{g_{p-2} z}{1} \, \underset{+}{} \, \frac{g_{p-1} z}{1} \, \underset{+}{} \, \frac{z f_p(z)}{1}\,, \quad 1 \le k \le p\,. \tag{16}$$

Now, for z given by $(3)_{2,3}$ and for fixed parameters u_1, u_2 and v_1, v_2, and for a fixed k $(1 \le k \le p)$, we introduce a region $\mathcal{F}_k^{(p)}(z)$ defined in the $\mathcal{F}_k^{(p)}$ — complex plane as follows

$$\mathcal{F}_k^{(p)}(z) = \{\mathcal{F}_k^{(p)}(z,u,v) \mid u_1 \le u \le u_2\,,\ v_1 \le v \le v_2\}\,. \tag{17}$$

We may now introduce three definitions, convenient for our subsequent considerations.

Definition 1. We call $\mathcal{F}_k^{(p)}(z)$ the inclusion region for $zf_k(z)$, while $\mathcal{F}_k^{(p)}(z,u,v)$ the including function, if

$$zf_k(z) \in \mathcal{F}_k^{(p)}(z) = \{\mathcal{F}_k^{(p)}(z,u,v) \mid u_1 \le u \le u_2\,,\ v_1 \le v \le v_2\}\,. \tag{18}$$

Prior to providing the next definition, we introduce, in the $F_k^{(p)}$ — complex plane, a curve $F_k^{(p)}(z,.)$ determined by means of two functions $F_k^{\prime(p)}(z,.)$ and $F_k^{\prime\prime(p)}(z,.)$:

$$F^{(p)}(z,u) = \begin{cases} F_k^{\prime(1)}(z,u)\,, & \text{if} \quad -1 \leq u \leq 0\,, \\ F_k^{\prime\prime(1)}(z,u)\,, & \text{if} \quad 0 \leq u \leq 1\,. \end{cases} \tag{19}$$

Obviously z is a fixed complex number given by $(3)_{2,3}$.

Definition 2. $F_k^{(p)}(z)$ is said to be the bound for $zf_k(z)$, while $F_k^{(p)}(z,u)$ the bounding function, if

$$F^{(p)}(z,u) = \partial\mathcal{F}_k^{(p)}(z) = \{F_k^{(p)}(z,u) \mid -1 \leq u \leq 1\}\,. \tag{20}$$

Here $\partial\mathcal{F}_k^{(p)}(z)$ denotes the boundary of the inclusion region $\mathcal{F}_k^{(p)}(z)$.

Definition 3. If $k=1$ ($k=p$, $p>1$) then $\mathcal{F}_k^{(p)}(z)$ and $F_k^{(p)}(z)$ are called basic inclusion regions and basic bounds (auxiliary inclusion regions and auxiliary bounds), respectively.

Let us assume that the auxiliary including function $\mathcal{F}_p^{(p)}(z,u,v)$ for $zf_p(z)$ is known. On account of (16), the including functions $\mathcal{F}_p^{(p)}(z,u,v)$ for $zf_p(z)$ has, in the $\mathcal{F}_k^{(p)}$ — complex plane, the following S–continued fraction representation

$$\mathcal{F}_k^{(p)}(z,u,v) = \frac{g_k z}{1} \, {}_{+} \, \frac{g_{k+1} z}{1} \, {}_{+} \cdots {}_{+} \, \frac{g_{p-1} z}{1} \, {}_{+} \, \frac{\mathcal{F}_p^{(p)}(z,u,v)}{1}\,, \quad 1 \leq k \leq p\,. \tag{21}$$

Similarly, also on the basis of (16), we can write the S–continued fraction representation for the bounding function $F_k^{(p)}(z,u)$, determined in the $F_k^{(p)}$, — complex plane

$$F_k^{(p)}(z,u) = \frac{g_k z}{1} \, {}_{+} \, \frac{g_{k+1} z}{1} \, {}_{+} \cdots {}_{+} \, \frac{g_{p-1} z}{1} \, {}_{+} \, \frac{F_p^{(p)}(z,u)}{1}\,, \quad 1 \leq k \leq p\,. \tag{22}$$

To understand better the last formula, we rewrite (22) explicitly

$$F_k^{\prime(p)}(z,u) = \frac{g_k z}{1} \, {}_{+} \, \frac{g_{k+1} z}{1} \, {}_{+} \cdots {}_{+} \, \frac{g_{p-1} z}{1} \, {}_{+} \, \frac{F_p^{\prime(p)}(z,u)}{1}\,, \quad 1 \leq k \leq p\,, \ -1 \leq u \leq 0\,. \tag{22a}$$

$$F_k^{\prime\prime(p)}(z,u) = \frac{g_k z}{1} \, {}_{+} \, \frac{g_{k+1} z}{1} \, {}_{+} \cdots {}_{+} \, \frac{g_{p-1} z}{1} \, {}_{+} \, \frac{F_p^{\prime\prime(p)}(z,u)}{1}\,, \quad 1 \leq k \leq p\,, \quad 0 \leq u \leq 1\,. \tag{22b}$$

For a fixed z, the following recurrence formulae

$$\mathcal{F}_{p-1-j}^{(p)}(z,u,v) = \frac{g_{p-1} z}{1} \, {}_{+} \, \frac{\mathcal{F}_{p-j}^{(p)}(z,u,v)}{1}\,, \quad j = 0,\ldots,p-1-k\,, \tag{23}$$

$$F_{p-1-j}^{(p)}(z,u,v) = \frac{g_{p-1} z}{1} \, {}_{+} \, \frac{F_{p-j}^{(p)}(z,u,v)}{1}\,, \quad j = 0,\ldots,p-1-k \tag{24}$$

allow us to calculate the values of the including functions $\mathcal{F}_k^{(p)}(z,u,v)$ represented by (23) (bounding function $F_k^{(p)}(z,u)$ represented by (24)) from the values of the auxiliary including functions $\mathcal{F}_p^{(p)}(z,u,v)$ (bounding functions $F_p^{(p)}(z,u)$).

On the basis of (2)–(3) and (23)–(24), the including functions $\mathcal{B}_p(z,u,v)$ and the bounding functions $B_p(z,u)$ for the effective, dielectric constants $\varepsilon_e(z)/\varepsilon_1$ are given by:

$$\mathcal{B}_f^{(p)}(z,u,v) = 1 + \mathcal{F}_1^{(p)}(z,u,v)\,, \quad B_f^{(p)}(z,u,v) = 1 + F_1^{(p)}(z,u,v)\,. \tag{25}$$

According to (23)–(25), the determination of the inclusion regions (bounds) for $\varepsilon_e(z)/\varepsilon_1$ by the method of the S–continued fractions requires the knowledge of only the auxiliary regions $\mathcal{F}_p^{(p)}(z)$ (bounds $F_p^{(p)}(z)$) for the Stieltjes function $zf_p(z)$, see (10). In the next Section, we shall derive analytic expressions for both the inclusion regions $\mathcal{F}_p^{(p)}(z)$ and bounds $F_p^{(p)}(z)$, $p = 1,2,\ldots$.

4. Auxiliary inclusion regions and bounds

To derive an explicit formula for the auxiliary bounds $F_p^{(p)}(z)$ we use, as the available data concerning a microstructure of a composite, the $(p-1)$ coefficients of the power series (4) and the inequality [6]

$$\varepsilon_e \leq \varepsilon_1 \quad \text{for} \quad \varepsilon_2 = 0 < \varepsilon_1 \,. \tag{26}$$

On account of (26) and (2)–(3) we obtain

$$f_1(-1) \leq 1 \,. \tag{27}$$

Note that for real z, the inequalities

$$f_p(z) > 0\,, \quad \frac{\partial f_p(z)}{\partial z} < 0\,, \quad \frac{\partial\,[z f_p(z)]}{\partial z} > 0\,, \quad z \in (-1,\infty)\,, \quad p = 1,2,\ldots \tag{28}$$

are a direct consequence of (7). By using (10) we can rewrite (27) in the form of the sequence of the following continued fractions:

$$\frac{g_1}{1-f_2(-1)} \leq 1\,, \qquad \frac{g_1}{1-\dfrac{g_2}{1-f_3(-1)}} \leq 1\,, \qquad \frac{g_1}{1-\dfrac{g_2}{1-\dfrac{g_3}{1-f_4(-1)}}} \leq 1\,, \quad \ldots\,. \tag{29}$$

Due to (28), relations (29) yield

$$f_p(-1) \leq V_p\,, \quad p = 1,2,\ldots\,, \tag{30}$$

where

$$V_1 = 1\,, \quad V_{p-1} = \frac{g_{p-1}}{1-V_p}\,, \quad p = 2,3,\ldots\,. \tag{31}$$

The relations $g_p = \int_0^1 \mathrm{d}\gamma_p(u) > 0$, $f_p(-1) = \int_0^1 (\mathrm{d}\gamma_p(u)/(1-u))$ result in

$$f_p(-1) > g_p\,, \quad g_p > 0\,. \tag{32}$$

Hence, on account of (30)–(32) we obtain

$$V_1 = 1\,, \quad 0 < V_p < 1\,, \quad p = 2,3,\ldots\,. \tag{33}$$

We now pass to the problem of finding the range of functions $f_p(z)$ satisfying (30). Relations (7) and (30) give

$$f_p(-1) = \int_0^1 \frac{\mathrm{d}\gamma_p(u)}{1-u} \leq V_p\,, \quad p = 1,2,3,\ldots\,. \tag{34}$$

Consequently,

$$\mathrm{d}\omega_p(u) = \frac{\mathrm{d}\gamma_p(u)}{1-u} \leq V_p\,, \quad p = 1,2,3,\ldots \tag{35}$$

are also Stieltjes measures satisfying

$$\int_0^1 \mathrm{d}\omega_p(u) \leq V_p\,. \tag{36}$$

Hence we can write

$$f_p(z) = \int_0^1 \frac{1-u}{1+zu}\mathrm{d}\omega_p(u)\,, \tag{37}$$

where $d\omega_p(u)$ is an arbitrary, nonnegative measure obeying the inequality (36). The range of admissible values of $f_p(z)$, resulting from (36)–(37), forms a convex region obtainable for measures written formally in the following manner:

$$d\omega_p(u) = V_p\delta(u-u_0)du \quad \text{and} \quad d\omega_p(u) = V_p\left[(1-\alpha)\delta(u)+\alpha\delta(u-1)\right]du\,, \tag{38}$$

where $u \in [0,1]$ and $0 \le \alpha \le 1$, *cf.* [1, Chap. 17A]. By substituting (38) into (37), we obtain the bounding functions $F_p^{(p)}(z,u)$:

$$F_p^{(p)}(z,u) = V_p F(z,u)\,, \tag{39}$$

where

$$F(z,u) = \begin{cases} (1+u)z\,, & \text{if} \quad -1 \le u \le 0\,, \\ \dfrac{z(1-u)}{1+zu}\,, & \text{if} \quad 0 \le u \le 1\,. \end{cases} \tag{40}$$

Note that $F_p^{(p)}(z,u)$ describes, in the cut $(-\infty,-1)$ $F_p^{(p)}$ — complex plane, the boundary of a convex region $\mathcal{F}_p^{(p)}(z)$ of admissible values of the Stieltjes function $zf_p(z)$ appearing in (10). The curve $F(z,u)$ given by (40) consists of the straight line $F'(u,z)$ and of the circular arc $F''(u,z)$:

$$F'(z,u) = (1+u)z\,, \quad -1 \le u \le 0\,; \qquad F''(z,u) = \frac{z(1-u)}{1+zu}\,, \quad 0 \le u \le 1\,. \tag{41}$$

The curve $F(z,u)$ determined by (40) is called the elementary bounding function. Relations (41) were originally derived in a different manner by Bergman [14]. For $p=1$ we have $V_p = 1$. Hence the bounding function (40)–(41) forms in the cut $(-\infty,-1)$ complex plane the convex, lens-shaped region

$$\mathcal{F}_1^{(1)}(z) = \left\{\frac{zv(1-u)}{1+zu} \mid 0 \le u,v \le 1\right\}\,, \tag{42}$$

of admissible values of the family of Stieltjes functions $zf_1(z)$ satisfying the inequality (34). Sometimes it is useful to characterize the straight line $(41)_1$ and the whole circle $(41)_2$ by triplets of points $\{0,z,1\}$ and $\{0,z,\infty\}$ obtainable from $(41)_1$ for $u=-1,0,\infty$ and from $(41)_2$ for $u=1,0,\infty$.

5. Regular arrays of cylinders

To illustrate our theoretical developments, let us consider the problem of determination of the bounds on the effective dielectric constants ε_e of square and hexagonal arrays of circular cylinders. Each cylinder in the arrays has the dielectric constant h, where both ε_e and h are so normalized that the dielectric constant of the matrix material may be taken to be equal to unity. The bulk dielectric constant ε_e is defined by

$$\langle \mathbf{D} \rangle = \varepsilon_e \langle \nabla\Phi \rangle\,, \tag{43}$$

where Φ denotes the electric potential; from (1), it follows $\mathbf{U} = \nabla\Phi$. The averaging $\langle\,.\,\rangle$ is performed over the unit square or hexagonal cell. The electric potential Φ appearing in (43) have to fulfill:

1) The Maxwell equation of the form

$$\nabla\cdot(1+z\theta)\nabla\Phi = 0\,, \quad z = h-1 \tag{44}$$

2) The continuity condition for the normal component of the electric displacement $\mathbf{D} = (1+z\theta)\nabla\Phi$ at the surfaces of the cylinders

$$\mathbf{m}\cdot\mathbf{D}_- = \mathbf{m}\cdot\mathbf{D}_+\,, \tag{45}$$

where $\mathbf{m}$ is the unit vector normal to the surface of a cylinder, while $\mathbf{D}_-$, $\mathbf{D}_+$ denote the electric displacement on the inside and on the outside of a cylinder surface. The function θ appearing in (44) stands for the characteristic function of the cylinder; i.e. $\theta(x) = 1$ ($\theta(x) = 0$), if x belongs (does not belong) to the domain occupied by the cylinders.

As an input for calculation of the S–continued fractions, the coefficients of the expansions of $\varepsilon_e(z)$ in powers of $z = h - 1$ have been obtained by solving the system of equations (43)–(45), *cf.* [18,27]. The results are summarized in Tables 1 and 2.

Table 1. Coefficients of a power expansion of the effective dielectric constant $\varepsilon_2/\varepsilon_1$ for a square array of cylinders

φ_2	c_1	c_2	c_3	c_4	c_5	c_6	c_7
0.10	0.10	0.04500	0.020250	0.009113	0.004101	0.001846	0.000831
0.20	0.20	0.08800	0.032024	0.012830	0.005148	0.002068	0.000832
0.30	0.30	0.10500	0.036936	0.013086	0.004682	0.001698	0.000626
0.40	0.40	0.12000	0.036784	0.011662	0.003884	0.001381	0.000530
0.50	0.50	0.12500	0.033646	0.010208	0.003615	0.001488	0.000685
0.60	0.60	0.12000	0.029979	0.010181	0.004465	0.002255	0.001215
0.70	0.70	0.10500	0.028735	0.012751	0.006975	0.004169	0.002617
0.75	0.75	0.09375	0.030114	0.015261	0.009200	0.006077	0.004242

Table 2. Coefficients of a power expansion of the effective dielectric constant $\varepsilon_2/\varepsilon_1$ for a hexagonal array of cylinders

φ_2	c_1	c_2	c_3	c_4	c_5	c_6	c_7
0.10	0.10	0.04500	0.020250	0.009113	0.004101	0.001846	0.000831
0.20	0.20	0.08000	0.032000	0.012800	0.005120	0.002048	0.000819
0.30	0.30	0.10500	0.036754	0.012867	0.004506	0.001578	0.000553
0.40	0.40	0.12000	0.036031	0.010834	0.003265	0.000988	0.000301
0.50	0.50	0.12500	0.031397	0.007960	0.002054	0.000548	0.000155
0.60	0.60	0.12000	0.024528	0.005275	0.001262	0.000361	0.000128
0.70	0.70	0.10500	0.017303	0.003605	0.001089	0.000452	0.000220
0.80	0.80	0.08000	0.011955	0.003569	0.001676	0.000926	0.000546
0.88	0.88	0.05280	0.010877	0.004970	0.003014	0.002072	0.001515
0.90	0.90	0.04500	0.011273	0.005526	0.003584	0.002637	0.002061

By employing the algorithm (14) to power series coefficients presented in Tables 1 and 2, the parameters g_n of S–continued fractions (22) have been evaluated and shown in Tables 3 and 4 and in Figs. 1 and 2.

Table 3. Coefficients of S–continued fraction to the effective dielectric constant $\varepsilon_2/\varepsilon_1$ of a square array of cylinders

φ_2	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
0.10	0.10	0.450						
0.20	0.20	0.400	0.0003	0.4997				
0.30	0.30	0.350	0.0018	0.4982				
0.40	0.40	0.300	0.0065	0.4935	0.0005	0.4995		
0.50	0.50	0.250	0.0192	0.4808	0.0030	0.4970	0.0002	0.4998
0.60	0.60	0.200	0.0498	0.4502	0.0142	0.4858	0.0025	0.4975
0.70	0.70	0.150	0.1237	0.3763	0.0632	0.4368	0.0220	0.4780
0.75	0.75	0.125	0.1962	0.3038	0.1441	0.3559	0.0741	0.4259

Table 4. Coefficients of S–continued fraction to the effective dielectric constant $\varepsilon_2/\varepsilon_1$ of a hexagonal array of cylinders

φ_2	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
0.10	0.10	0.450						
0.20	0.20	0.400						
0.30	0.30	0.350						
0.40	0.40	0.300	0.0003	0.4997				
0.50	0.50	0.250	0.0012	0.4988	0.0001	0.4999		
0.60	0.60	0.200	0.0044	0.4956	0.0012	0.4988		
0.70	0.70	0.150	0.0148	0.4852	0.0076	0.4924	0.0006	0.4994
0.80	0.80	0.100	0.0494	0.4506	0.0415	0.4585	0.0081	0.4919
0.88	0.88	0.060	0.1460	0.3540	0.1693	0.3307	0.0746	0.4254
0.90	0.90	0.050	0.2005	0.2995	0.2640	0.2360	0.1667	0.3333

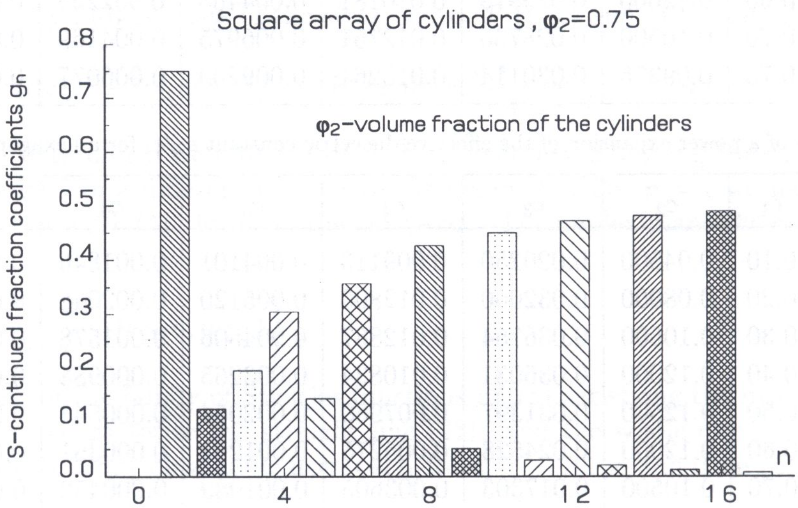

Fig. 1. The sequence of coefficients g_n of S–continued fraction bounds for a square array of cylinders

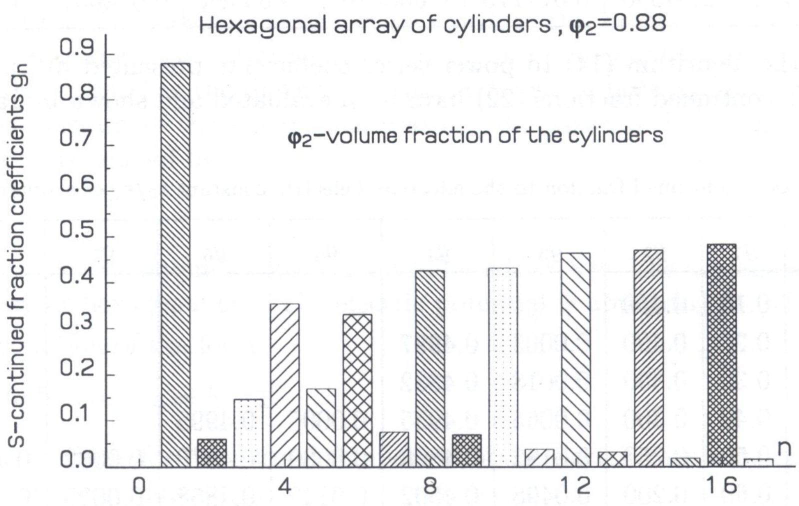

Fig. 2. The sequence of coefficients g_n of S–continued fraction bounds for a hexagonal array of cylinders

Finally, by using the recurrence formula (24) we computed the sequences of lens-shaped complex bounds on ε_e presented in Figs. 3, 4, 5, 6.

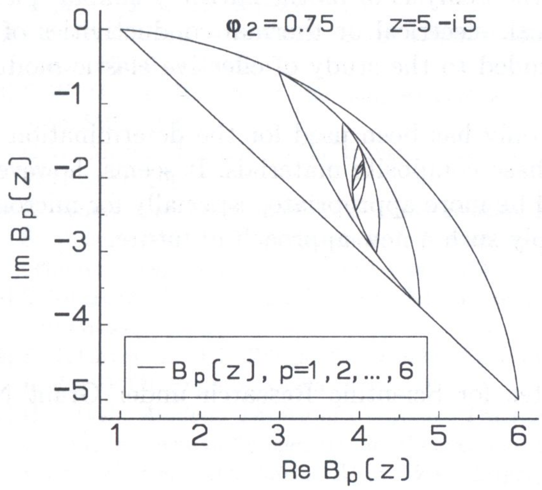

Fig. 3. The sequence of lens-shaped narrowing bounds on $\varepsilon_e(z)$ for square array of cylinders; $z = 5 - i5$

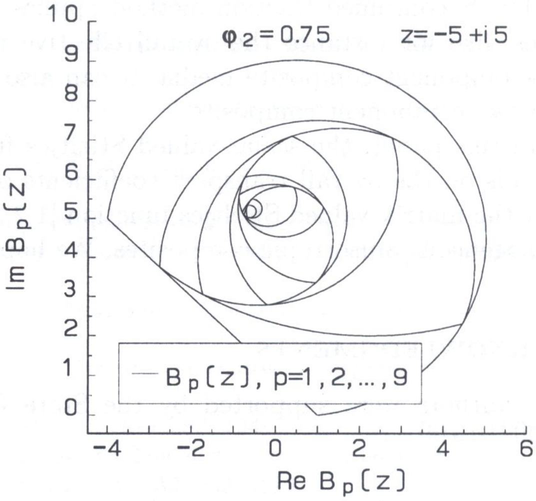

Fig. 4. The sequence of lens–shaped narrowing bounds on $\varepsilon_e(z)$ for square array of cylinders; $z = -5 + i5$

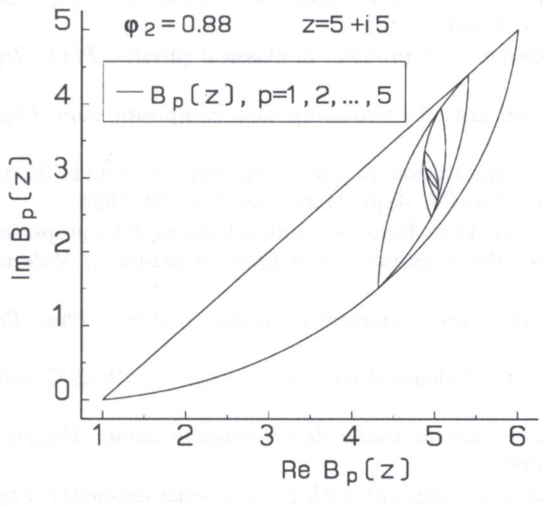

Fig. 5. The sequence of lens-shaped narrowing bounds on $\varepsilon_e(z)$ for hexagonal array of cylinders; $z = 5 + i5$

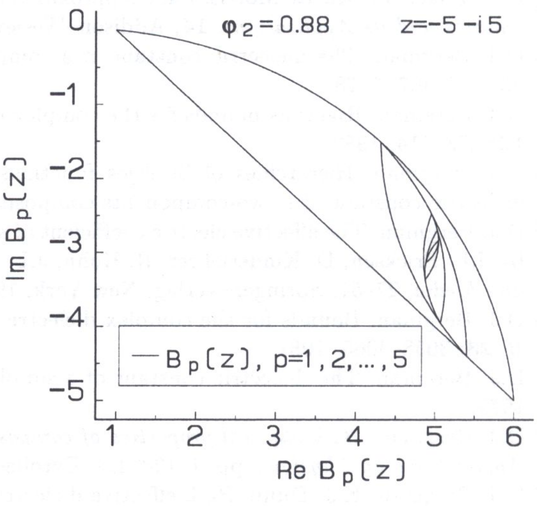

Fig. 6. The sequence of lens-shaped narrowing bounds on $\varepsilon_e(z)$ for hexagonal array of cylinders; $z = -5 - i5$

6. Discussion and Summary

Starting from a partial information about power expansions of the geometrical functions $zf_1(z)$, we have derived and numerically investigated the S–continued fraction method of determination of complex bounds on ε_e (z) for two-phase composite media.

The proposed S–continued fraction approach to the transport coefficients of two-phase composite media is based mainly on the algorithms given by (14), (24) and (31), which are simply recursive and do not require solving of a large number of coupled equations.

It is worth noting that the derived S–fractions bounds agree with the corresponding bounds derived earlier in a different manner by Milton [24–25], and independently by Bergman [5].

As an example of practical calculations, the sequences of lens-shaped narrowing bounds on $\varepsilon_e(z)$, for square and hexagonal arrays of circular cylinders, have been found and depicted in Figs. 3, 4, 5 and 6.

The S–continued fraction method applies also to the analysis of mathematically similar quantities, like for instance the overall effective magnetical, electrical or thermal conductivities of a two-component composite media. It can also be extended to the study of effective elastic moduli of a two-component composite.

In this paper, the scalar-valued Stieltjes function only has been used for the determination of bounds on the overall transport coefficients of two-phase composite materials. It seems, however, that the matrix-valued Stieltjes function [1,2,3] would be more appropriate, especially for microinhomogeneous anisotropic composites. We hope to apply such a new approach in future.

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