

# Solving thermoelasticity problems by means of Trefftz functions

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The paper presents a new method of approximate solving of the two- and three-dimensional thermoelasticity problems in a finite body. The method presented here can be used for solving direct and inverse problems as well. System of thermoelasticity equations is reduced to the system of wave equations where the temperature occurs as inhomogeneity in one of them. The thermal field is approximated by linear combination of heat polynomials (Trefftz functions for heat conduction equation). The system of wave equations is solved by means of wave polynomials (Trefftz functions for wave equation). Convergence of the T-functions method is proved. The procedure of solving direct and inverse thermoelasticity problems by means of Trefftz functions is tested on an example. Sensitiveness of the method according to data disturbance was checked.

## 1. INTRODUCTION

The approach presented in this paper belongs to the Trefftz methods [21]. The main idea of the method is to determine functions (polynomials) satisfying a given differential equation (T-functions) and to fit the linear combination of them to the governing initial and boundary conditions. The method is suitable for linear differential equations in finite domain. Obtained solution satisfies the equation identically. In the paper we consider boundary direct and inverse problems. In this case one or more boundary conditions are not known. Instead of that, values of solution in discrete points inside the domain (internal responses) are known.

The Trefftz functions method for linear partial differential equations has been developed by Herrera, Jirousek, Kupradze, Leon, Sabina, Zieliński and Zienkiewicz [11–14, 23]. These authors considered equations without time variable or the time was discretized. A little bit different approach towards using Trefftz functions (the solution depends in continuous way on time) was first described in the paper [20], where it was applied to solving one-dimensional heat conduction equation. Although his approach has been developed only recently, it has rich literature. In the papers [7–10, 22] the authors described heat functions (Trefftz functions for heat conduction equation) in different coordinate systems and for direct and inverse heat conduction problems. Using T-functions as a new type of finite-element base function is shown in [2, 6]. The paper [4] deals with numerous cases, involving other differential equations, such as the Laplace, Poisson, Helmholtz and one dimensional wave equation ones. Solving functions for the wave equation are presented in [3, 5, 15, 16, 18, 19]. Applications of wave polynomials for elasticity problems are described in [17]. Solving polynomials for beam vibration problems are described in [1].

The approach proposed here has a few remarkable advantages in comparison with other methods. Firstly, the approximate solution (a linear combination of the solving functions) satisfies the equation identically and depends continuously on all space variables and time. In standard Finite Element Method (FEM) (probably the most popular numerical method of solving wide class of engineering

problems) base functions (usually polynomials) fulfil given equation but only if the degree of them is lower than the degree of equation. Therefore we have to divide the domain into a lot of elements. For direct problem FEM gives then a good approximation. When using FEM for solving these problems the internal responses should be given on the first layer of the elements from the border. The location of the internal responses on further layers leads to large propagation of the error. In general more elements (dense mesh) means better results but only for direct problems. In inverse problem more elements can give a worse approximation of boundary condition. Secondly, the method is flexible in terms of given boundary and initial conditions (discrete, missing) and the shape of the body can be relatively more complicated [19]. Most analytical methods require regular shape of the domain and each initial and boundary condition should be given. Thirdly, the solving functions can be used as base functions for several variants of the Finite Element Method with time-space elements. It means that the approach is perfectly suitable for inverse problems.

What is very useful in the solving functions' method is the properties of the Taylor series:

$$f(x + \Delta x, y + \Delta y, t + \Delta t) = f(x, y, t) + \frac{df}{1!} + \frac{d^2f}{2!} + \dots + \frac{d^N f}{N!} + R_{N+1} \quad (1)$$

where  $d^n f = (\frac{\partial}{\partial x} \Delta x + \frac{\partial}{\partial y} \Delta y + \frac{\partial}{\partial t} \Delta t)^n f$ , and  $R_{N+1}$  is the remainder term.

The problem is described in Section 2. Section 3 presents wave polynomials. Trefftz functions' method for thermoelasticity problems and its convergence is presented in section 4. In Section 5 the test example is discussed. Concluding remarks are given in last section.

## 2. FORMULATION OF THE PROBLEM

Let us consider thermoelasticity equations

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu) \text{grad div } \mathbf{u} = \rho \ddot{\mathbf{u}} + \gamma \text{grad } T, \quad (\mathbf{x}, t) \in \Omega \times (0, t), \quad (2)$$

where  $\Omega \subset \mathbf{R}^n$ ,  $\mathbf{u}$  – displacement vector,  $\nabla$  – nabla operator,  $\mu, \lambda$  – Lamé constants,  $\rho$  – mass density,  $\gamma = \frac{E}{1-2\nu} \alpha$ ,  $E$  – Young's modulus,  $\nu$  – Poisson's ratio,  $\alpha$  – coefficient of thermal expansion and  $t$  denotes time. The temperature field is described by equation

$$\frac{1}{\kappa} \frac{\partial T}{\partial t} = \nabla^2 T, \quad (3)$$

where  $\kappa$  – coefficient of thermal diffusivity. Equations (2) and (3) are completed by initial and boundary conditions for displacements (and/or stresses) and for temperature. The relationship between strains and stresses describes formula  $\sigma_{ij} = 2\mu \varepsilon_{ij} + \lambda \varepsilon_{kk} \delta_{ij} - \gamma T \delta_{ij}$ , where  $\varepsilon_{ij} = \frac{1}{2} (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$  – strain tensor and  $\delta_{ij}$  – Kronecker delta. The system of equations (2) can be simplified by substitution:

$$\mathbf{u} = \text{grad } \phi + \text{rot } \Psi, \quad (4)$$

then we obtain

$$(\nabla^2 - \frac{1}{v_1^2} \frac{\partial^2}{\partial t^2}) \phi = mT, \quad (5)$$

$$(\nabla^2 - \frac{1}{v_2^2} \frac{\partial^2}{\partial t^2}) \psi_i = 0, \quad i = 1, 2, 3, \quad (6)$$

where  $v_1^2 = \frac{\lambda+2\mu}{\rho}$ ,  $v_2^2 = \frac{\mu}{\rho}$ ,  $m = \frac{\gamma}{c_1 \rho}$ . In a finite domain the wave equations (5) and (6) are still coupled by initial and boundary conditions. The main purpose of this work is to solve the system of equations (5)–(6) with given necessary initial and boundary conditions by means of Trefftz functions for wave equation (wave polynomials).

### 3. WAVE POLYNOMIALS

#### 3.1. Recurrent formulas for wave polynomials

The way to obtain two- and three-dimensional wave polynomials and their properties was described in papers [15, 18, 19]. Let us consider wave equation

$$\nabla^2 u = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2}. \quad (7)$$

For two space variables the function

$$g = e^{i(ax+by+ct)} \quad (8)$$

satisfying equation (7) when  $c^2 = a^2 + b^2$  is called a generating function for wave polynomials. The power series expansion for (8) is

$$e^{i(ax+by+ct)} = \sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{l=0}^{n-k} S_{(n-k-l)kl}(x, y, t) a^{n-k-l} b^k c^l, \quad (9)$$

where  $S_{(n-k-l)kl}(x, y, t)$  are polynomials of variables  $x, y, t$ . Substituting  $c^2 = a^2 + b^2$  in (9), we obtain

$$e^{i(ax+by+ct)} = \sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{\substack{l=0 \\ l < 2}}^{n-k} R_{(n-k-l)kl}(x, y, t) a^{n-k-l} b^k c^l. \quad (10)$$

The real and imaginary parts of  $R$  satisfy equation (7) and are called wave polynomials:

$$P_{(n-k-l)kl}(x, y, t) = \Re(R_{(n-k-l)kl}(x, y, t)), \quad Q_{(n-k-l)kl}(x, y, t) = \Im(R_{(n-k-l)kl}(x, y, t)),$$

e.g.,

$$\begin{aligned} P_{000}(x, y, t) &= 1, & Q_{000}(x, y, t) &= 0, & Q_{100}(x, y, t) &= x, & Q_{010}(x, y, t) &= y, & Q_{001}(x, y, t) &= t, \\ P_{200}(x, y, t) &= -\frac{x^2}{2} - \frac{t^2}{2}, & P_{110}(x, y, t) &= -xy, & P_{101}(x, y, t) &= -xt, & P_{011}(x, y, t) &= -yt, \\ P_{020}(x, y, t) &= -\frac{y^2}{2} - \frac{t^2}{2}, \dots, & P_{100} &= P_{010} = P_{001} = Q_{200} = Q_{110} = Q_{101} = Q_{011} = 0, \dots \end{aligned}$$

Note that there is no  $R_{002}$ , because  $l < 2$  (see formula (10)). For two-dimensional wave equation we take  $P_{000} = 1$ ,  $Q_{000} = 0$ . Then the recurrent formulas for wave polynomials are

$$P_{(n-k)k0} = \frac{1}{n}(-xQ_{(n-k-1)k0} - yQ_{(n-k)(k-1)0} - vtQ_{(n-k-2)k1} - vtQ_{(n-k)(k-2)1}), \quad (11)$$

$$P_{(n-k-1)k1} = \frac{1}{n}(-xQ_{(n-k-2)k1} - yQ_{(n-k-1)(k-1)1} - vtQ_{(n-k-1)k0}), \quad (12)$$

$$Q_{(n-k)k0} = \frac{1}{n}(xP_{(n-k-1)k0} + yP_{(n-k)(k-1)0} + vtP_{(n-k-2)k1} + vtP_{(n-k)(k-2)1}), \quad (13)$$

$$Q_{(n-k-1)k1} = \frac{1}{n}(xP_{(n-k-2)k1} + yP_{(n-k-1)(k-1)1} + vtP_{(n-k-1)k0}). \quad (14)$$

We put  $P_{(n-k-l)kl} = Q_{(n-k-l)kl} = 0$  when any subscript is negative. For three-dimensional wave polynomials we take  $P_{0000} = 1$ ,  $Q_{0000} = 0$  and have recurrent formulas

$$\begin{aligned} P_{(n-k-l)kl0} &= -\frac{1}{n}(xQ_{(n-k-l-1)kl0} + yQ_{(n-k-l)(k-1)l0} + zQ_{(n-k-l)k(l-1)0} \\ &\quad + vtQ_{(n-k-l-2)kl1} + vtQ_{(n-k-l)(k-2)l1} + vtQ_{(n-k-l)k(l-2)1}), \end{aligned} \quad (15)$$

$$P_{(n-k-l-1)kl1} = -\frac{1}{n}(xQ_{(n-k-l-2)kl1} + yQ_{(n-k-l-1)(k-1)l1} + zQ_{(n-k-l-1)k(l-1)1} + vtQ_{(n-k-l-1)kl0}), \quad (16)$$

$$Q_{(n-k-l)kl0} = \frac{1}{n}(xP_{(n-k-l-1)kl0} + yP_{(n-k-l)(k-1)l0} + zP_{(n-k-l)k(l-1)0} + vtP_{(n-k-l-2)kl1} + vtP_{(n-k-l)(k-2)l1} + vtP_{(n-k-l)k(l-2)1}), \quad (17)$$

$$Q_{(n-k-l-1)kl1} = \frac{1}{n}(xP_{(n-k-l-2)kl1} + yP_{(n-k-l-1)(k-1)l1} + zP_{(n-k-l-1)k(l-1)1} + vtP_{(n-k-l-1)kl0}). \quad (18)$$

We put  $P_{nklm} = Q_{nklm} = 0$  when any subscript is negative.

### 3.2. The second procedure to obtain wave polynomials

The second way to obtain wave polynomials is using wave equation in Taylor series (1) for the function  $u$ . For example for two space variables we assume that the function  $u(x, y, t)$  satisfies the wave equation (7). Let  $u \in C^{N+1}$  in the neighbourhood of  $(x_0, y_0, t_0)$  and  $\hat{x} = x - x_0$ ,  $\hat{y} = y - y_0$ ,  $\hat{t} = t - t_0$ . Then, the Taylor series for function  $u$  and for  $N = 2$  is

$$u(x, y, t) = u(x_0, y_0, t_0) + \frac{\partial u}{\partial x}\hat{x} + \frac{\partial u}{\partial y}\hat{y} + \frac{\partial u}{\partial t}\hat{t} + \frac{\partial^2 u}{\partial x^2}\frac{\hat{x}^2}{2} + \frac{\partial^2 u}{\partial y^2}\frac{\hat{y}^2}{2} + \frac{\partial^2 u}{\partial t^2}\frac{\hat{t}^2}{2} + \frac{\partial^2 u}{\partial x\partial y}\hat{x}\hat{y} + \frac{\partial^2 u}{\partial x\partial t}\hat{x}\hat{t} + \frac{\partial^2 u}{\partial y\partial t}\hat{y}\hat{t} + R_3.$$

By eliminating the derivative  $\frac{\partial^2 u}{\partial t^2}$  by equation (7) we get

$$u(x, y, t) = u(x_0, y_0, t_0) + \frac{\partial u}{\partial x}\hat{x} + \frac{\partial u}{\partial y}\hat{y} + \frac{\partial u}{\partial t}\hat{t} + \frac{\partial^2 u}{\partial x^2}\left(\frac{\hat{x}^2}{2} + \frac{v^2\hat{t}^2}{2}\right) + \frac{\partial^2 u}{\partial y^2}\left(\frac{\hat{y}^2}{2} + \frac{v^2\hat{t}^2}{2}\right) + \frac{\partial^2 u}{\partial x\partial y}\hat{x}\hat{y} + \frac{\partial^2 u}{\partial x\partial t}\hat{x}\hat{t} + \frac{\partial^2 u}{\partial y\partial t}\hat{y}\hat{t} + R_3.$$

The coefficients succeeding the derivation terms represent the wave polynomials obtained by using formulas (11-14). For  $N = 3, 4, \dots$  we follow similarly. For three space variables we get wave polynomials analogously. This way of obtaining the wave polynomials will be used to prove the convergence of the T-functions method.

### 4. TREFFTZ FUNCTIONS METHOD

We approximate the solution of wave equations (6) by

$$\psi_i \approx \hat{\psi}_i = \sum_{n=1}^N c_n^i V_n^i, \quad i = 1, 2, 3 \quad (19)$$

and for equation (5) we take

$$\phi \approx \hat{\phi} = \sum_{n=1}^N c_n^0 V_n^0 + \phi_p. \quad (20)$$

Here  $V_n^i$ ,  $i = 0, \dots, 3$ , are wave polynomials satisfying the corresponding wave equation and  $\phi_p$  is the particular solution of (5). Coefficients  $c_n^i$  in formulas (19) and (20) are determined by initial and boundary conditions for displacement and/or stresses (see examples).

#### 4.1. The convergence of wave-polynomial method

Each approximate method should be convergent. Let us consider two dimensional homogeneous wave equation (7). It is easy to prove the convergence of the method when in formula

$$u \approx w = \sum_{n=1}^N c_n V_n \quad (21)$$

all wave polynomials from order zero to  $K$  are taken, (*e.g.*, for  $K = 0$ ,  $N = 1$ , for  $K = 1$ ,  $N = 1 + 3 = 4$ , for  $K = 2$ ,  $N = 1 + 3 + 5 = 9$  and so on). Let us consider the wave polynomials obtained as in section 3.2. When in approximation (21) all wave polynomials from order zero to  $K$  are taken, then in this approximation full differentials of order from 0 to  $K$  are taken into account. Let us denote  $w_K$  the approximation (21) when all wave polynomials from order zero to  $K$  are considered. Taking all the points above into consideration we have

**Theorem 1.** *Let the function  $u$  satisfy the equation(7). When in formula (1) in certain point*

$$\lim_{N \rightarrow \infty} R_N = 0 \quad (22)$$

for function  $u$  satisfying equation (7) with given initial and boundary condition then

$$\lim_{K \rightarrow \infty} w_K = u.$$

*Proof.* For  $w_K$  we have  $|w_K - u| = |R_{K+1}|$ . But

$$\lim_{K \rightarrow \infty} R_{K+1} = 0,$$

hence

$$\lim_{K \rightarrow \infty} |w_K - u| = 0.$$

It means that

$$\lim_{K \rightarrow \infty} w_K = u.$$

□

The condition (22) is satisfied for example when all derivatives are commonly restricted. If function  $u \notin C^\infty$  the convergence is not provided but we can estimate the error of approximation by the remainder term in Taylor series of function  $u$ . The theorem presented above can give a criterion of choosing the number of polynomials which are used in approximations (19) and (20). It is easy to choose the number of polynomials if we can estimate the remainder term  $R_{N+1}$ . Unfortunately, it can be problematic. Other method of determining the order of used polynomials is based on the usage of adaptive algorithm. We assume the order of polynomials and check the quality of approximation. If the error is to big then we increase the number of wave polynomials. This procedure can be very effective but probably time-consuming.

#### 4.2. Particular solution

The particular solution  $\phi_p$  in (20) is calculated as  $L^{-1}(mT)$ , where  $L = \nabla^2 - \frac{1}{v_1^2} \frac{\partial^2}{\partial t^2}$ . The temperature distribution can be expressed as a sum of monomials. The first procedure to get such form is expanding  $T$  into Taylor series. The second is the usage of approximation by linear combination

of heat polynomials [4]. Therefore, we have to know how to calculate the inverse operator  $L^{-1}$  for monomials. Denote  $Z(x^k y^l t^m) = L^{-1}(x^k y^l t^m)$ . It is easy to prove that for two space variables we have three forms of  $Z(x^k y^l t^m)$

$$Z_1(x^k y^l t^m) = \frac{1}{(k+2)(k+1)}(x^{k+2} y^l t^m - l(l-1)Z_{(k+2)(l-2)m} + \frac{m(m-1)}{v_1^2}Z_{(k+2)l(m-2)}), \quad (23)$$

$$Z_2(x^k y^l t^m) = \frac{1}{(l+2)(l+1)}(x^k y^{l+2} t^m - k(k-1)Z_{(k-2)(l+2)m} + \frac{m(m-1)}{v_1^2}Z_{k(l+2)(m-2)}), \quad (24)$$

$$Z_3(x^k y^l t^m) = \frac{v_1^2}{(m+2)(m+1)}(-x^k y^l t^{m+2} + k(k-1)Z_{(k-2)l(m+2)} + l(l-1)Z_{k(l-2)(m+2)}). \quad (25)$$

For three space variables we have four forms of  $Z(x^n y^k z^l t^m)$

$$Z_1(x^n y^k z^l t^m) = \frac{1}{(n+2)(n+1)}(x^{n+2} y^k z^l t^m + \frac{m(m-1)}{v_1^2}Z_{(n+2)kl(m-2)} - k(k-1)Z_{(n+2)(k-2)lm} - l(l-1)Z_{(n+2)k(l-2)m}), \quad (26)$$

$$Z_2(x^n y^k z^l t^m) = \frac{1}{(k+2)(k+1)}(x^n y^{k+2} z^l t^m + \frac{m(m-1)}{v_1^2}Z_{n(k+2)l(m-2)} - n(n-1)Z_{(n-2)(k+2)lm} - l(l-1)Z_{n(k+2)(l-2)m}), \quad (27)$$

$$Z_3(x^n y^k z^l t^m) = \frac{1}{(l+2)(l+1)}(x^n y^k z^{l+2} t^m + \frac{m(m-1)}{v_1^2}Z_{nk(l+2)(m-2)} - n(n-1)Z_{(n-2)k(l+2)m} - k(k-1)Z_{n(k-2)(l+2)m}), \quad (28)$$

$$Z_4(x^n y^k z^l t^m) = \frac{v_1^2}{(m+2)(m+1)}(-x^n y^k z^l t^{m+2} + n(n-1)Z_{(n-2)kl(m+2)} + k(k-1)Z_{n(k-2)l(m+2)} + l(l-1)Z_{nk(l-2)(m+2)}). \quad (29)$$

In formulas (23)–(29) we put zero instead of a polynomial when any subscript is negative.

## 5. EXAMPLE

The method will be tested on an example when the exact solution is known. We consider the plane state of strain in a square  $(x, y) \in (-1, 1) \times (-1, 1)$  when the strain tensor depends on time and two variables  $\varepsilon_{ij} = \varepsilon_{ij}(x, y, t)$ ,  $(i, j = 1, 2)$  and  $\varepsilon_{i3} = 0$ ,  $(i = 1, 2, 3)$ . Then the system of equations (5), (6) has the following form

$$(\nabla^2 - \frac{1}{v_1^2} \frac{\partial^2}{\partial t^2})\phi(x, y, t) = mT(x, y, t) \quad (30)$$

$$(\nabla^2 - \frac{1}{v_2^2} \frac{\partial^2}{\partial t^2})\psi(x, y, t) = 0. \quad (31)$$

Displacements and stresses are given as

$$\begin{aligned} \mathbf{u} &= [u_x(x, y, t), u_y(x, y, t)] = \left[ \frac{\partial \phi(x, y, t)}{\partial x} + \frac{\partial \psi(x, y, t)}{\partial y}, \frac{\partial \phi(x, y, t)}{\partial y} - \frac{\partial \psi(x, y, t)}{\partial x} \right], \\ \sigma_{xx} &= (2\mu + \lambda) \frac{\partial u_x}{\partial x} + \lambda \frac{\partial u_y}{\partial y} - \gamma T, \sigma_{xy} = \mu \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right), \\ \sigma_{yy} &= \lambda \frac{\partial u_x}{\partial x} + (2\mu + \lambda) \frac{\partial u_y}{\partial y} - \gamma T, \sigma_{zz} = \lambda \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \right) - \gamma T. \end{aligned} \quad (32)$$

The direct and inverse problem in the case of the plane state of strain will be solved by means of Trefftz functions.

### 5.1. Direct problem

We assume the temperature distribution given by

$$T(x, y, t) = x^2/2 + y^2/2 + 2t, \quad (33)$$

and conditions for displacements

$$u_x(x, y, 0) = \frac{mxy^2}{3}, \quad u_y(x, y, 0) = \frac{mx^2y}{3}, \quad (34)$$

$$\dot{u}_x(x, y, 0) = \frac{2mx}{3} + \frac{\sqrt{2}c_2}{50000} \sin(x) \cos(y), \quad \dot{u}_y(x, y, 0) = \frac{2my}{3} - \frac{\sqrt{2}c_2}{50000} \cos(x) \sin(y), \quad (35)$$

$$u_x(\pm 1, y, t) = \pm \frac{m}{3} (2t + y^2 - \frac{c_1^2 t^2}{2}) \pm \frac{\sin(1)}{50000} \cos(y) \sin(\sqrt{2}c_2 t), \quad (36)$$

$$u_x(x, -1, t) = u_x(x, 1, t) = \frac{m}{3} (2xt + x - \frac{c_1^2 xt^2}{2}) + \frac{\cos(1)}{50000} \sin(x) \sin(\sqrt{2}c_2 t), \quad (37)$$

$$u_y(-1, y, t) = u_y(1, y, t) = \frac{m}{3} (2yt + y - \frac{c_1^2 yt^2}{2}) - \frac{\cos(1)}{50000} \sin(y) \sin(\sqrt{2}c_2 t), \quad (38)$$

$$u_y(x, \pm 1, t) = \pm \frac{m}{3} (2t + x^2 - \frac{c_1^2 t^2}{2}) \mp \frac{\sin(1)}{50000} \cos(x) \sin(\sqrt{2}c_2 t). \quad (39)$$

Then we have the exact solution

$$u_x = \frac{m}{3} (2xt + xy^2 - \frac{c_1^2 xt^2}{2}) + \frac{1}{50000} \sin(x) \cos(y) \sin(\sqrt{2}c_2 t), \quad (40)$$

$$u_y = \frac{m}{3} (x^2 y + 2yt - \frac{c_1^2 yt^2}{2}) - \frac{1}{50000} \cos(x) \sin(y) \sin(\sqrt{2}c_2 t). \quad (41)$$

We solve the problem described by a system of equations (3) and conditions (34–39) by means of wave polynomials. We reduce equations (3) to a system of wave equations (30–31). We assume the approximate solution for (31) in the form

$$\psi \approx \hat{\psi} = \sum_{n=1}^N c_n^1 V_n^1, \quad (42)$$

and for equation (30) we take

$$\phi \approx \hat{\phi} = \sum_{n=1}^N c_n^0 V_n^0 + \phi_p, \quad (43)$$

where  $V_n^i$ ,  $i = 0, 1$ , are wave polynomials satisfying corresponding wave equation. Then

$$\mathbf{u} \approx \hat{\mathbf{u}} = \text{grad } \hat{\phi} + \text{rot } \hat{\Psi}. \quad (44)$$

As a particular solution we take  $L^{-1}(T) = (Z_1(T) + Z_2(T) + Z_3(T))/3$  where  $Z_i(T)$  for  $i = 1, 2, 3$  are described by formulas (23–25). The notation  $Z_1(T)$  means that we take inverse operator for

each component (monomial) of the temperature  $T$ . The coefficients  $c_n^i$  are chosen so that the error of fulfilling given boundary and initial conditions is minimized. We build a functional  $I$

$$\begin{aligned}
I = & \int_{-1}^1 \int_{-1}^1 \underbrace{\{[\widehat{u}_x(x, y, 0) - u_x(x, y, 0)]^2 + [\widehat{u}_y(x, y, 0) - u_y(x, y, 0)]^2\}}_{\text{cond.(34)}} dy dx \\
& + \int_{-1}^1 \int_{-1}^1 \underbrace{\{[\widehat{\dot{u}}_x(x, y, 0) - \dot{u}_x(x, y, 0)]^2 + [\widehat{\dot{u}}_y(x, y, 0) - \dot{u}_y(x, y, 0)]^2\}}_{\text{cond.(35)}} dy dx \\
& + \int_{-1}^1 \int_0^{\Delta t} \underbrace{\{[\widehat{u}_x(-1, y, t) - u_x(-1, y, t)]^2 + [\widehat{u}_x(1, y, t) - u_x(1, y, t)]^2\}}_{\text{cond.(36)}} dt dy \\
& + \int_{-1}^1 \int_0^{\Delta t} \underbrace{\{[\widehat{u}_x(x, -1, t) - u_x(x, -1, t)]^2 + [\widehat{u}_x(x, 1, t) - u_x(x, 1, t)]^2\}}_{\text{cond.(37)}} dt dx \\
& + \int_{-1}^1 \int_0^{\Delta t} \underbrace{\{[\widehat{u}_y(-1, y, t) - u_y(-1, y, t)]^2 + [\widehat{u}_y(1, y, t) - u_y(1, y, t)]^2\}}_{\text{cond.(38)}} dt dy \\
& + \int_{-1}^1 \int_0^{\Delta t} \underbrace{\{[\widehat{u}_y(x, -1, t) - u_y(x, -1, t)]^2 + [\widehat{u}_y(x, 1, t) - u_y(x, 1, t)]^2\}}_{\text{cond.(39)}} dt dx.
\end{aligned} \tag{45}$$

In the time intervals  $(\Delta t, 2\Delta t)$ ,  $(2\Delta t, 3\Delta t)$ ,  $\dots$ , we proceed analogously. Here, the initial condition for time interval  $((m-1)\Delta t, m\Delta t)$  is the value of function  $u$  at the end of interval  $((m-2)\Delta t, (m-1)\Delta t)$ . Unfortunately, taking more time intervals we observe a propagation of the error. Each step the approximation is worse. The numerical calculation for the direct problems shows that we get acceptable results for about ten time steps. Here we have a general result that small error in the first time interval allows as to get a better approximation in the next time intervals. Unfortunately, we get a better approximation when we take more polynomials. This causes a longer time of calculations. It is a very interesting problem and should be studied wider.

A necessary condition to minimize the functional  $I$  is

$$\frac{\partial I}{\partial c_1^0} = \dots = \frac{\partial I}{\partial c_N^0} = \frac{\partial I}{\partial c_1^1} = \dots = \frac{\partial I}{\partial c_N^1} = 0. \tag{46}$$

The linear system of equations (46) can be written as

$$AC = B \tag{47}$$

where  $C = [c_1^0, \dots, c_N^0, c_1^1, \dots, c_N^1]^T$  and

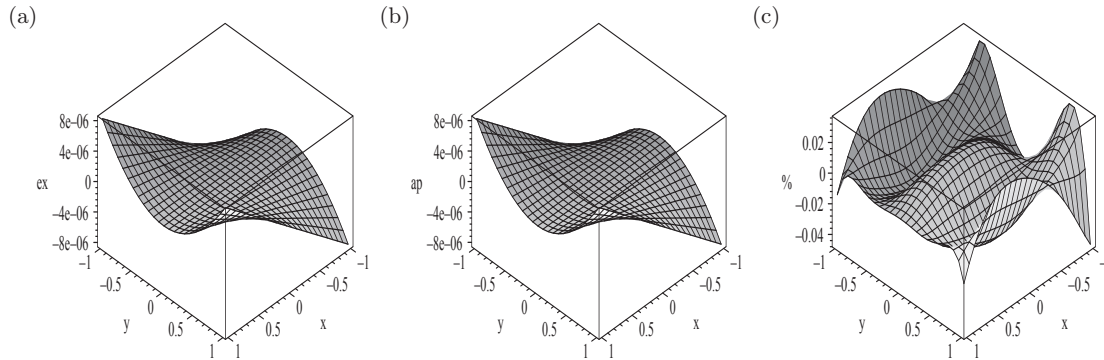
$$A = \begin{bmatrix} A^1 & A^2 \\ A^3 & A^4 \end{bmatrix} \begin{array}{l} \} \frac{\partial I}{\partial c_j^0} \\ \} \frac{\partial I}{\partial c_i^1} \end{array}.$$

$\underbrace{\hspace{1.5cm}}_{c_j^0} \quad \underbrace{\hspace{1.5cm}}_{c_i^1}$

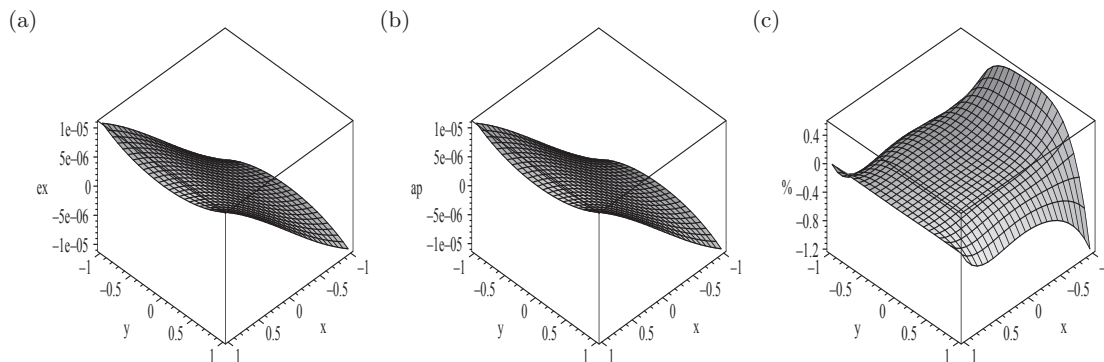
From Eq. (47) we obtain the coefficients  $c_n^i$ . In practice, this system of linear equations is indeterminate. Nevertheless, for different values of parameters we get the same solution. All results below were obtained for:  $\lambda = 10^{11}$  [Pa],  $\mu = 8 \cdot 10^{10}$  [Pa],  $\rho = 8000$  [kg/m<sup>3</sup>],  $\alpha = 14 \cdot 10^{-6}$  [1/deg] and  $\Delta t = 16/100000$  [s].



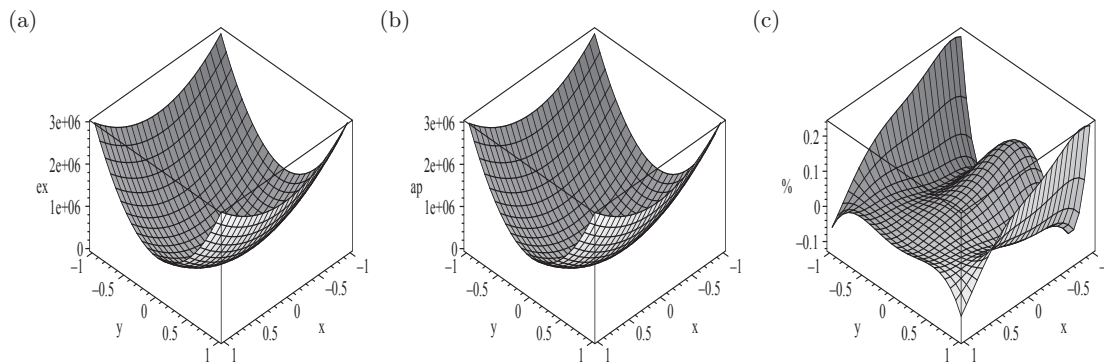
Figure 1 shows  $u_x$  for time  $t = 0$ : (a) the exact solution, (b) an approximation by polynomials from order 0 to 7, and (c) the relative error [%] between (a) and (b) related to the maximum value of the function for  $(x, y, t) \in (-1, 1) \times (-1, 1) \times (0, \Delta t)$ . It is visible that the approximation of the initial condition for  $u_x$  is very good – the relative error does not exceed 0.05%. The Trefftz functions' method gives the biggest error on the border of considered space-time domain. Therefore, Fig. 2 shows  $u_x$  for time  $t = \Delta t$ : (a) the exact solution, (b) an approximation by polynomials from order 0 to 7, and (c) the relative error specified above. Here, the error is bigger but still stays smaller than 1.3%. We can get very similar results for displacement  $u_y$ . Having displacements we can calculate stresses by using formulas (32).



**Fig. 1.** (a) The exact solution of  $u_x$  for  $t = 0$ , (b) an approximation by polynomials from order 0 to 7, (c) the relative error between (a) and (b)



**Fig. 2.** (a) The exact solution of  $u_x$  for  $t = \Delta t$ , (b) an approximation by polynomials from order 0 to 7, (c) the relative error between (a) and (b)



**Fig. 3.** (a) The exact solution of  $\sigma_{xx}$  for  $t = 0$ , (b) an approximation by polynomials from order 0 to 7, (c) relative error between (a) and (b)

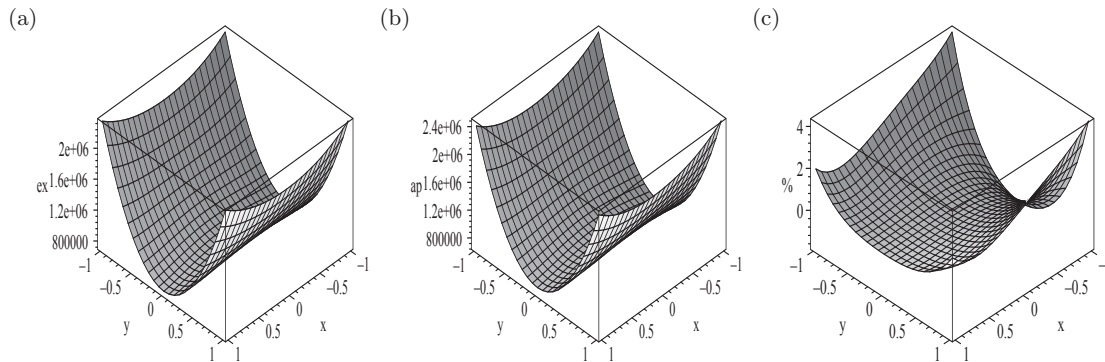
Figure 3 shows  $\sigma_{xx}$  for time  $t = 0$ : (a) the exact solution, (b) an approximation by polynomials from order 0 to 7, and (c) the relative error [%] between (a) and (b) related to the maximum value of the function for  $(x, y, t) \in (-1, 1) \times (-1, 1) \times (0, \Delta t)$ . The approximation of the initial condition for  $\sigma_{xx}$  is very good – the relative error does not exceed 0.2%. We calculate the stresses by differentiation of displacements. Therefore, the error for stresses is bigger than for displacement.

Figure 4 shows  $\sigma_{xx}$  for time  $t = \Delta t$ : (a) the exact solution, (b) an approximation by polynomials from order 0 to 7, and (c) the relative error between (a) and (b) specified above. Similarly as before, the error for  $t = \Delta t$  is bigger but still stays smaller than 4%.

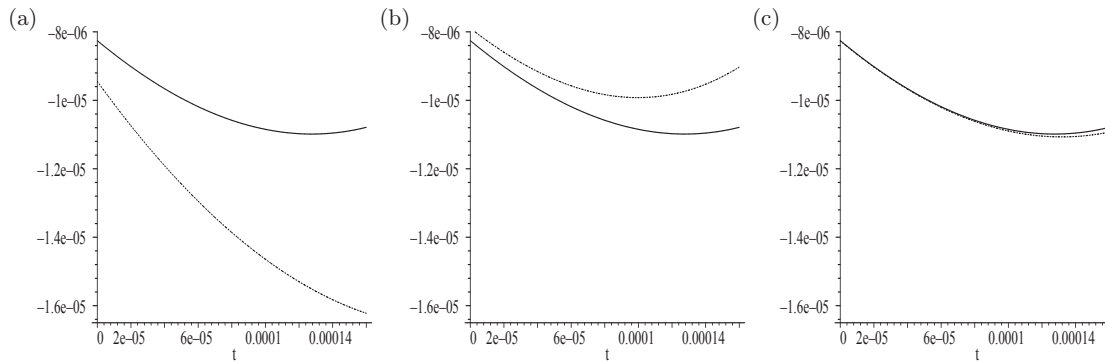
In Sec. 4 we proved that Trefftz functions’ method is convergent. It means that the approximation is better when we take more wave polynomials in approximate solution. Figures 1 and 2 show that the approximation is the worst near the point  $x = -1, y = 1$ . Figure 5 shows the exact result (solid line) for the  $u_x$  as a function of time for the location  $x = -1, y = 1$  and the approximation (dash line) by polynomials from order 0 to: (a) 4, (b) 5, and (c) 7. The results presented in Fig. 5 are not a proof of convergence, but in this case we get a better approximation when the approximate solution contains more wave polynomials. We get a similar result for stresses. Figure 6 shows the exact result (solid line) for the  $\sigma_{xx}$  as a function of time for the location  $x = -1, y = 1$  and the approximation (dash line) by polynomials from order 0 to: (a) 4, (b) 5, and (c) 7.

We can calculate the mean relative error of approximation of  $u_x$  in the norm  $H^1(\Omega)$ , where  $\Omega = (0, \Delta t)$  for point  $x = -1, y = 1$ . The error is defined as:

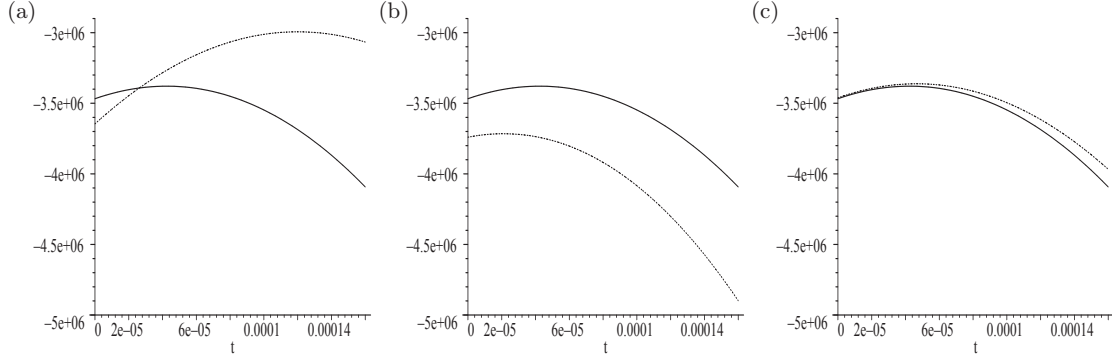
$$E = \sqrt{\frac{\int_0^{\Delta t} [\hat{u}(-1; 1; t) - u(-1; 1; t)]^2 dt}{\int_0^{\Delta t} [u(1; 1; t)]^2 dt}} \cdot 100\%.$$



**Fig. 4.** (a) The exact solution of  $\sigma_{xx}$  for  $t = \Delta t$ , (b) an approximation by polynomials from order 0 to 7, (c) relative error between (a) and (b)



**Fig. 5.** (The exact solution (solid line) of  $u_x$  as a function of time for the location  $x = -1, y = 1$  and the approximation (dash line) by polynomials from order 0 to: (a) 4, (b) 5, (c) 7



**Fig. 6.** The exact solution (solid line) of  $\sigma_{xx}$  as a function of time for the location  $x = -1, y = 1$  and the approximation (dash line) by polynomials from order 0 to (a) 4, (b) 5, (c) 7

**Table 1.** Error  $E$  dependence of the polynomial order

Order $K$	4	5	7
E [%]	34.2	9.17	0.57

In approximation  $\hat{u}$  we take all wave polynomials from order 0 to  $K$ . Table 1 shows the error, which depends on the order  $K$ . The error  $E$  decreases when the number of the polynomials in approximation  $\hat{u}$  increases. For  $K = 7$  it does not exceed 0.6%, which is a very good result. The decreasing of mean relative error while taking more wave polynomials in approximation suggests that the wave-polynomials method is convergent.

## 5.2. Inverse problem

Similarly as for direct problem we assume the temperature distribution given by

$$T(x, y, t) = x^2/2 + y^2/2 + 2t, \quad (48)$$

and conditions for displacements

$$u_x(x, y, 0) = \frac{mxy^2}{3}, \quad u_y(x, y, 0) = \frac{mx^2y}{3}, \quad (49)$$

$$\dot{u}_x(x, y, 0) = \frac{2mx}{3} + \frac{\sqrt{2}c_2}{50000} \sin(x) \cos(y), \quad \dot{u}_y(x, y, 0) = \frac{2my}{3} - \frac{\sqrt{2}c_2}{50000} \cos(x) \sin(y), \quad (50)$$

$$u_x(-1, y, t) = -\frac{m}{3}(2t + y^2 - \frac{c_1^2 t^2}{2}) - \frac{\sin(1)}{50000} \cos(y) \sin(\sqrt{2}c_2 t), \quad (51)$$

$$u_x(x, -1, t) = u_x(x, 1, t) = \frac{m}{3}(2xt + x - \frac{c_1^2 xt^2}{2}) + \frac{\cos(1)}{50000} \sin(x) \sin(\sqrt{2}c_2 t), \quad (52)$$

$$u_y(-1, y, t) = \frac{m}{3}(2yt + y - \frac{c_1^2 yt^2}{2}) - \frac{\cos(1)}{50000} \sin(y) \sin(\sqrt{2}c_2 t), \quad (53)$$

$$u_y(x, \pm 1, t) = \pm \frac{m}{3}(2t + x^2 - \frac{c_1^2 t^2}{2}) \mp \frac{\sin(1)}{50000} \cos(x) \sin(\sqrt{2}c_2 t). \quad (54)$$

The continuous conditions  $u_x(1, y, t)$  and  $u_y(1, y, t)$  are not known. Instead of that we know the values of them in discrete points  $(x - \delta, -1 + \frac{k}{5}, \frac{l\Delta t}{10})$ ,  $k, l = 0, \dots, 10$  (internal responses). If  $\delta = 0$  then we have direct problem. For  $\delta > 0$  occur inverse problems. Internal responses are simulated

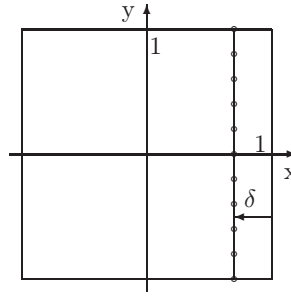


Fig. 7. Localization of internal responses

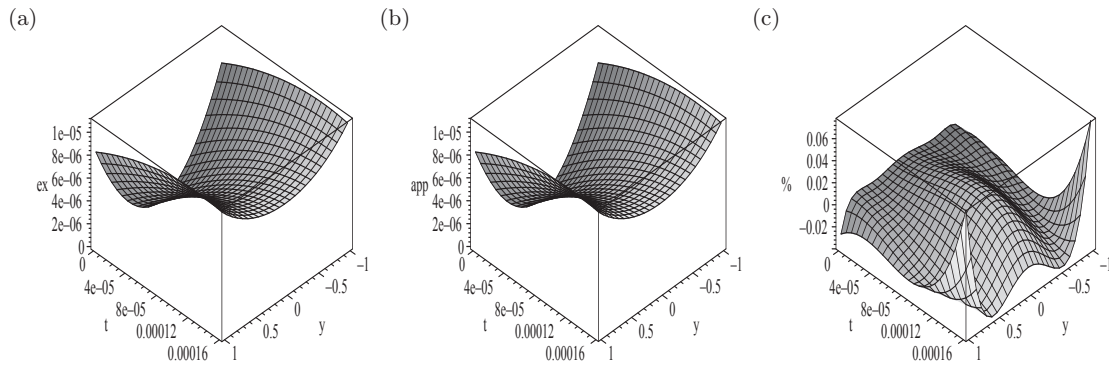
from the exact solution (40) and (41). Location of the internal responses for displacements are shown in Fig. 7.

Problem described by a system of equation (30)-(31) and conditions (49)-(54) with internal responses we solve by means of wave polynomials. We proceed analogously as for direct problem. The functional  $I$  has the following form

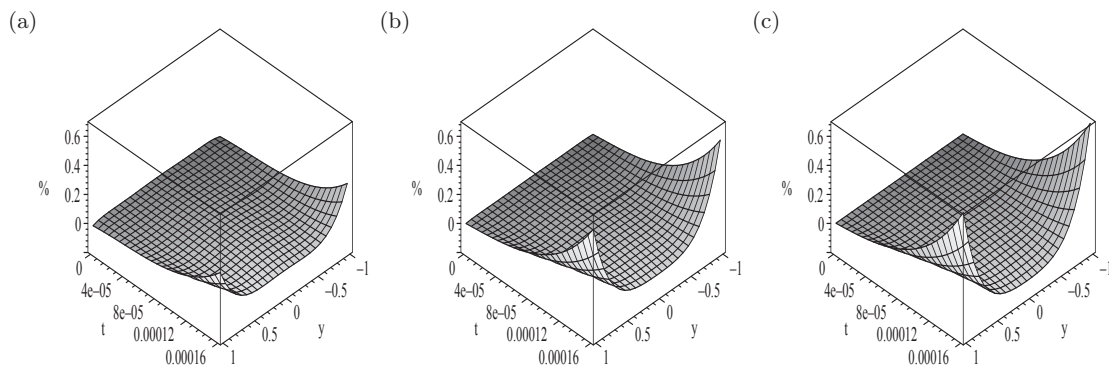
$$\begin{aligned}
I = & w \int_{-1}^1 \int_{-1}^1 \underbrace{\{[\widehat{u}_x(x, y, 0) - u_x(x, y, 0)]^2 + [\widehat{u}_y(x, y, 0) - u_y(x, y, 0)]^2\}}_{\text{cond.(49)}} dy dx \\
& + w \int_{-1}^1 \int_{-1}^1 \underbrace{\{[\widehat{u}_x(x, y, 0) - \dot{u}_x(x, y, 0)]^2 + [\widehat{u}_y(x, y, 0) - \dot{u}_y(x, y, 0)]^2\}}_{\text{cond.(50)}} dy dx \\
& + w \int_{-1}^1 \int_0^{\Delta t} \underbrace{\{[\widehat{u}_x(-1, y, t) - u_x(-1, y, t)]^2\}}_{\text{cond.(51)}} dt dy \\
& + w \int_{-1}^1 \int_0^{\Delta t} \underbrace{\{[\widehat{u}_x(x, -1, t) - u_x(x, -1, t)]^2 + [\widehat{u}_x(x, 1, t) - u_x(x, 1, t)]^2\}}_{\text{cond.(52)}} dt dx \\
& + w \int_{-1}^1 \int_0^{\Delta t} \underbrace{\{[\widehat{u}_y(-1, y, t) - u_y(-1, y, t)]^2\}}_{\text{cond.(53)}} dt dy \\
& + w \int_{-1}^1 \int_0^{\Delta t} \underbrace{\{[\widehat{u}_y(x, -1, t) - u_y(x, -1, t)]^2 + [\widehat{u}_y(x, 1, t) - u_y(x, 1, t)]^2\}}_{\text{cond.(54)}} dt dx \\
& + \Delta t \sum_{k=0}^{10} \sum_{l=0}^{10} \underbrace{\{[\widehat{u}_x(1 - \delta, -1 + \frac{k}{5}, \frac{l\Delta t}{10}) - u_x(1 - \delta, -1 + \frac{k}{5}, \frac{l\Delta t}{10})]^2\}}_{\text{internal responses for } u_x} \\
& + \Delta t \sum_{k=0}^{10} \sum_{l=0}^{10} \underbrace{\{[\widehat{u}_y(1 - \delta, -1 + \frac{k}{5}, \frac{l\Delta t}{10}) - u_y(1 - \delta, -1 + \frac{k}{5}, \frac{l\Delta t}{10})]^2\}}_{\text{internal responses for } u_y}.
\end{aligned} \tag{55}$$

Here, weights  $w = (1 - \Delta t)/4$  should occur because the time interval  $\Delta t$  is small. If we omit this weights the influence of internal responses on the solution is to big.

Figure 8 shows for  $\delta = 0$  (direct problem)  $u_x(1, y, t)$ : (a) the exact solution, (b) an approximation by polynomials from order 0 to 7, and (c) the relative error [%] between (a) and (b) related to the maximum value of the function for  $(x, y, t) \in (-1, 1) \times (-1, 1) \times (0, \Delta t)$ . In the whole time interval the relative error for direct problem does not exceed 0.08%.



**Fig. 8.** The solution for  $u_x(1, y, t)$ : (a) exact (b) an approximation by polynomials from order 0 to 7, (c) the relative error between (a) and (b)



**Fig. 9.** The relative error for identification of boundary condition  $u_x(1, y, t)$  for: (a)  $\delta = 0.1$ , (b)  $\delta = 0.3$ , (c)  $\delta = 0.5$

Figure 9 shows the relative error for identification of boundary condition  $u_x(1, y, t)$  (similarly as in Fig. 8(c)) for: (a)  $\delta = 0.1$ , (b)  $\delta = 0.3$ , and (c)  $\delta = 0.5$ . Even for  $\delta = 0.5$  (internal responses for displacement are located in distance 0.5 from the border  $x = 1$ ) the error of approximation the boundary condition not exceed 0.7% in the whole time interval. Of course for  $\delta < 0.5$  the error is lower. In this case we can calculate the mean relative error of approximation of  $u_x(1, y, t)$ . The error is defined as:

$$E = \sqrt{\frac{\int_{-1}^1 \int_0^{\Delta t} [\hat{u}(1; y; t) - u(1; y; t)]^2 dt dy}{\int_{-1}^1 \int_0^{\Delta t} [u(1; y; t)]^2 dt dy}} \cdot 100\%. \quad (56)$$

In approximation  $\hat{u}$  we take all wave polynomials from order 0 to 7. Table 2 shows the error (56), which depends on the distance  $\delta$ . The error  $E$  increases when the distance  $\delta$  increases. Still, even for  $\delta = 0.5$  it does not exceed 0.135%, which is a very good result. The increasing of mean relative error while taking bigger distance  $\delta$  is typical for inverse problems.

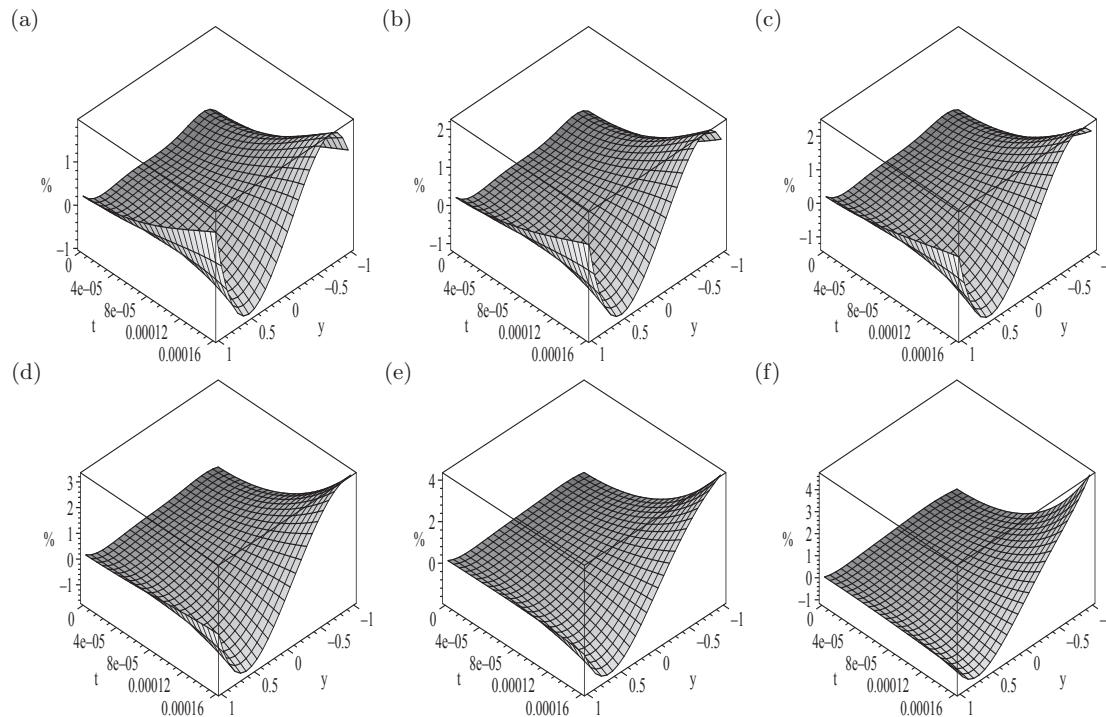
**Table 2.** Error  $E$  dependence of the distance  $\delta$

Distance $\delta$	0	0.1	0.3	0.5
$E$ [%]	0.022	0.044	0.108	0.133

### 5.2.1. Noisy data

We usually get internal responses from measurement and they are very often disturbed. In order to check the sensitivity of the presented method according to noisy data the internal responses were disturbed by using random number. For example for displacement  $u_x$  we get  $u_x^m = u_x \cdot (1 + \alpha)$  where  $\alpha$  is a random number generated by using normal distribution  $N(0, 0.04)$ . Upper subscript  $m$  denotes measurement. In this case we can say that the mean error of measurement equals 4%.

Figure 10 shows the relative error for identification of boundary condition  $u_x(1, y, t)$  (similarly as on figure 9) obtained by using noisy data  $u_x^m$  for a)  $\delta = 0$ , b)  $\delta = 0.05$ , c)  $\delta = 0.1$ , d)  $\delta = 0.2$ , e)  $\delta = 0.3$ , f)  $\delta = 0.5$ . Here all polynomials from order 0 to 7 are taken into consideration. Even for  $\delta = 0.5$  the relative error stays smaller than 5%. For inverse problem with noisy data it is very good result.



**Fig. 10.** The relative error for identification of boundary condition  $u_x(1, y, t)$  for: (a)  $\delta = 0$ , (b)  $\delta = 0.05$ , (c)  $\delta = 0.1$ , (d)  $\delta = 0.2$ , (e)  $\delta = 0.3$ , (f)  $\delta = 0.5$

Table 3 shows the mean relative error in this case described by formula (56), which depends on the distance  $\delta$ . The error  $E$  slowly increases when the distance  $\delta$  increases. Even for  $\delta = 0.5$  it does not exceed 2%. In fact, we are considering here an inverse problem with noisy data. Therefore the mean error smaller than 2% can be admitted as good result.

**Table 3.** Error  $E$  dependence of the distance  $\delta$

Distance $\delta$	0	0.05	0.1	0.2	0.3	0.5
$E[\%]$	0.896	0.976	1.073	1.303	1.503	1.394

Table 4 shows the comparison of the error (56) for original and noisy data, which depends on the distance  $\delta$ . It is visible that for noisy data the results are more than ten times worse but still stay low. The data was disturbed by using normal distribution with standard deviation equalling

**Table 4.** Error  $E$  dependence of the distance  $\delta$  for original and noisy data

Distance $\delta$	0	0.1	0.3	0.5
Original data	0.022	0.044	0.108	0.133
Noisy data	0.896	1.073	1.503	1.394

0.04. We can see that this disturbance causes the error lower than 4%. Therefore smoothing the disturbed data is not necessary in this case.

## CONCLUSIONS

As a rule, the direct and especially inverse thermoelasticity problems are difficult to solve. In this paper a new, relatively simple approach to solving these problems was proposed. Thanks to this method we obtain an analytical solution which depends continuously on all space variables and time in the whole domain. The example presented shows that we get a satisfactory approximation of displacement and stresses both for direct and inverse problem as well. The example considered suggests that the wave polynomials method is convergent, which means that more polynomials in approximate solution leads to better results. The method proposed is suitable for solving thermoelasticity inverse problems. Even for noisy data the internal responses can be distant from the corresponding border. In considered example it was not necessary to smooth noisy data. We got very satisfactory results without any kind of regularization. Application of the wave polynomials for solving direct and inverse thermoelasticity problems is very simple and the quality of the obtained approximation is good. The method presented here can be used also in more complicated geometry. In this case we can get global approximation or we can use a Finite Element Method with Trefftz base functions (FEMT). If we use FEMT the number of polynomials in approximations (19) and (20) can be obviously smaller than for global approximation. The criterion of choosing this number for most complicated domains is a good subject for next paper.

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## REFERENCES

- [1] M.J. Al-Khatib, K. Grysa, A. Maciąg. The method of solving polynomials in the beam vibration problems. *Journal of Theoretical and Applied Mechanics*, **46**(2): 347–366, 2008.
- [2] M.J. Ciałkowski. Solution of inverse heat conduction problem with use new type of finite element base functions. In: B.T. Maruszewski, W. Muschik, A. Radowicz (eds.), *Proceedings of the International Symposium on Trends in Continuum Physics*, pp. 64–78. World Scientific Publishing, Singapore, New Jersey, London, Hong Kong, 1999.
- [3] M.J. Ciałkowski. Thermal and related functions used in solving certain problems of mechanics. Part I: Solution of certain differential equations by means of inverse operations (in Polish). *Studia i materiały LIII. Technika 3*, pp. 7–70, Uniwersytet Zielonogórski, 2003.
- [4] M.J. Ciałkowski, A. Frąckowiak. *Heat Functions and Their Application for Solving Heat Transfer and Mechanical Problems* (in Polish). Poznań University of Technology Publishers, 2000.
- [5] M.J. Ciałkowski, A. Frąckowiak. Thermal and related functions used in solving certain problems of mechanics. Part II: Effective determination of inverse operations applied to harmonic functions (in Polish). *Studia i materiały LIII. Technika 3*, pp. 71–98, Uniwersytet Zielonogórski, 2003.
- [6] M.J. Ciałkowski, A. Frąckowiak, K. Grysa. Solution of a stationary inverse heat conduction problems by means of Trefftz non-continuous method. *Int. J. of Heat Mass Transfer*, **50**: 2170–2181, 2007.
- [7] M.J. Ciałkowski, S. Futakiewicz, L. Hożejowski. Method of heat polynomials in solving the inverse heat conduction problems. *ZAMM* **79**: 709–711, 1999.

- [8] M.J. Ciałkowski, S. Futakiewicz, L. Hożejowski. Heat polynomials applied to direct and inverse heat conduction problems. In: B.T. Maruszewski, W. Muschik, A. Radowicz (eds.), *Proceedings of the International Symposium on Trends in Continuum Physics*, pp. 79–88. World Scientific Publishing, Singapore, New Jersey, London, Hong Kong, 1999.
- [9] S. Futakiewicz, K. Grysa, L. Hożejowski. On a problem of boundary temperature identification in a cylindrical layer. In: B.T. Maruszewski, W. Muschik, A. Radowicz (eds.), *Proceedings of the International Symposium on Trends in Continuum Physics*, pp. 119–125. World Scientific Publishing, Singapore, New Jersey, London, Hong Kong, 1999.
- [10] S. Futakiewicz, L. Hożejowski. Heat polynomials in solving the direct and inverse heat conduction problems in a cylindrical system of coordinates. In: A.J. Nowak, C. A. Brebbia, R. Bialecki, M. Zerroukat (eds.), *Advanced Computational Method in Heat Transfer V*, pp. 71–80. Computational Mechanics Publications, Southampton, Boston, 1998.
- [11] I. Herrera, F. Sabina. Connectivity as an alternative to boundary integral equations: Construction of bases. *Appl. Math. Phys. Sc.*, **75**(5): 2059–2063, 1978.
- [12] J. Jirousek. Basis for development of large finite elements locally satisfying all fields equations. *Comp. Meth. Appl. Mech. Eng.*, **14**: 65–92, 1978.
- [13] J. Jirousek, A.P. Zieliński, H. Rabemantantsoa, A. Venkatesh. Survey of Trefftz-type element formulations. *Computers and Structures*, **63**(2): 225–242, 1997.
- [14] V.D. Kupradze. *Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*. North-Holland Publ. Cmp, Amsterdam, 1989.
- [15] A. Maciąg. Solution of the Three-dimensional Wave Equation by Using Wave Polynomials. *PAMM – Proc. Math. Mech.*, **4**: 706–707, 2004.
- [16] A. Maciąg. Three-dimensional Wave Polynomials. *Mathematical Problems in Engineering*, **5**: 583–598, 2005.
- [17] A. Maciąg. Wave polynomials in elasticity problems. *Engineering Transactions*, **55**(2): 129–153, 2007.
- [18] A. Maciąg, J. Wauer. Solution of the two-dimensional wave equation by using wave polynomials. *Journal of Engineering Mathematics*, **51**(4): 339–350, 2005.
- [19] A. Maciąg, J. Wauer. Wave Polynomials for Solving Different Types of Two-Dimensional Wave Equations. *Computer Assisted Mechanics and Engineering Sciences*, **12**: 87–102, 2005.
- [20] P.C. Rosenbloom, D.V. Widder. Expansion in terms of heat polynomials and associated functions. *Trans. Am. Math. Soc.*, **92**: 220–266, 1956.
- [21] E. Trefftz. Ein Gegenstück zum Ritz’schen Verfahren. *Proceedings 2nd International Congress of Applied Mechanics*, (Zurich), pp. 131–137, 1926.
- [22] H. Yano, S. Fukutani, A. Kieda. A boundary residual method with heat polynomials for solving unsteady heat conduction problems. *Franklin Inst.*, **316**: 291–298, 1983.
- [23] A.P. Zieliński, O.C. Zienkiewicz. Generalized finite element analysis with T-complete boundary solution functions. *Int. J. Numer. Meth. Eng.*, **21**: 509–528, 1985.