Computer Assisted Mechanics and Engineering Sciences, **3**: 9–21, 1996.

Homogenization in elastic random media

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(Received January 30, 1995)

The theoretical foundation and a numerical procedure of deriving stochastic effective properties of linear-elastic periodic fibre composite are presented. Using Monte-Carlo method, a Fortran program based on the deterministic rectangular plane strain element of the Finite Element Method has been worked out to evaluate probabilistic density functions of these properties. The expected values of elastic effective characteristics thus obtained are compared with deterministic results of COSAN modelling.

1. Introduction

The homogenization theory [11, 18] is one of the main mathematical methods of modelling materials and multi-component (composite) structures. It consists in replacing a real medium or a structure made of many materials with an effective (homogenized) medium with properly averaged characteristics. Considering the character of material properties variations it is possible to treat them as deterministic or random media.

In the 1980's scientists finally derived, with appropriate numerical models, the methods of homogenization of composites with various deterministic properties: linear-elastic [7, 30, 42], elasto-plastic continuous [4, 44] and with cavities [33] as well as visco-plastic [14, 20]. Dynamical problems were also considered in [13]. A comprehensive review of literature dealing with these problems, and especially with the application of the finite element method (FEM), can be found in [15, 36].

The idea of a composite material with random elastic properties appeared for the first time over 25 years ago, see Beran [9], Hashin and Shtrikman [21], for instance. Among the present probabilistic models of composite materials, it is possible to distinguish, for example, methods of upper-lower bounds of effective properties [3], direct estimation of these properties with the use of definition of limit density [2], stochastic dynamical systems [8, 43] or planar Delaunay networks [37]. These problems are usually connected with methods of composite optimization [29, 38]; literature dealing with stochastic methods in the homogenization theory can be found in [28, 41] and [2, 3, 8]. However, the probabilistic models obtained there are independent of the geometry of the considered composite and too complicated for computational implementation.

In this paper, a theoretical-numerical realization of the idea of randomizing elastic properties of component materials of a fibre-reinforced composite is presented. The mathematical basis has been derived by extending the deterministic model [7, 30, 42, 44] to the stochastic case. The necessary elements of the probability theory and statistical methods have been taken from [6, 35, 40]. The numerical code MCCEFF based on the POLSAP program [5, 22, 27] makes it possible to compute the probabilistic distribution of effective properties and their upper and lower bounds. In this case, the classical structure of the deterministic plane strain problem in the FEM context has been used with a specially adapted random number generator realizing the Gaussian distribution [39]. The expectations of elastic effective characteristics thus obtained are compared with the results of deterministic modelling with the COSAN code [16]; additional tests have been made with the use of ABAQUS [1]. The applied method of Monte-Carlo simulation has been used in other static and dynamic problems [10,32,41].

2. Mathematical model

Let us suppose that $Y \subset \mathcal{R}^2$ is occupied by a periodic random two-phase linear-elastic composite structure in the undeformed and the unstressed state [47]; Ω is a periodicity cell of Y, $\partial\Omega$ is its external boundary, Ω_1 is a fibre region, Ω_2 is a matrix region and $\partial\Omega_{12}$ is a boundary between these regions (see Fig. 1). We assume that Ω is a bounded coherent region uniplanar with $x_3 = 0$ plane (square with centre placed fibre with a round section). Let Ω_1 and Ω_2 be disjoint coherent regions such as $\Omega = \Omega_1 \cup \Omega_2$ and let them contain transversely isotropic homogeneous media.

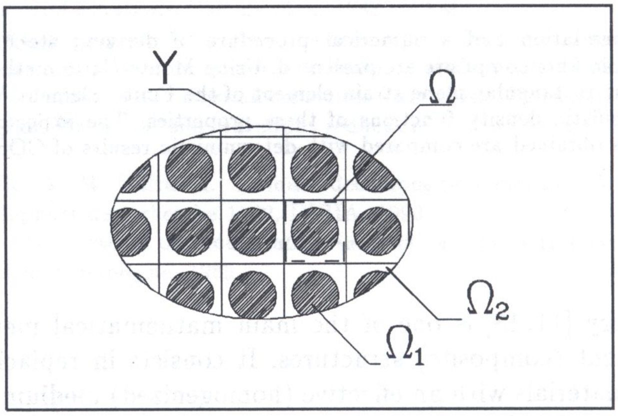

Fig. 1. Geometry of periodic composite structure and periodicity cell

Let the ratio ε relate the "microscopic" length scale connected with the periodicity cell Ω to the "macroscopic" one, connected with the composite structure Y.

We can observe that:

(i) if Ω is a stationary stochastic variable, then Y is a random composite;

(ii) Y is periodic in the stochastic sense if, for an additional ω belonging to a suitable probability space, there exists a translation of Ω which covers the whole region occupied by Y (this translation is assumed to be ergodic, so that ensemble averaging is equivalent to spatial averaging);

(iii) the tensor of effective moduli $C_{ijkl}^{(\mathrm{eff})}$ is a constitutive tensor which represents the behaviour of the composite as $\varepsilon \to 0$, which means that for any ε-independent load $\mathbf{Q}$ the displacement field $\mathbf{u}^{(\varepsilon)}$ satisfying the fundamental elasticity theory equations

$$\sigma_{ij}^{(\varepsilon)} = C_{ijkl}^{(\varepsilon)} \varepsilon_{kl}^{(\varepsilon)} ,$$

$$\varepsilon_{ij}^{(\varepsilon)} = \frac{1}{2}\left(u_{i,j}^{(\varepsilon)} + u_{j,i}^{(\varepsilon)}\right)$$

$$\sigma_{ij,j}^{(\varepsilon)} = 0 ,$$

$$\sigma_{ij}^{(\varepsilon)} n_j = \bar{t}_i ; \qquad \mathbf{x} \in \partial\Omega_{12} ,$$

$$u_i^{(\varepsilon)} = 0 ; \qquad \mathbf{x} \in \partial\Omega ,$$

converges to $\mathbf{u}^0$ — the solution of the corresponding system with $C_{ijkl}^{(\varepsilon)}$ replaced by $C_{ijkl}^{(\mathrm{eff})}$ — as $\varepsilon \to 0$.

We shall find the first two moments of effective elasticity tensors $C_{ijkl}^{(\mathrm{eff})}$ stated by (iii) by means of homogenization of the periodicity cell Ω.

For this purpose, we assume that the Young modulus function is a Gaussian field $\mathbf{e}(\mathbf{x}) = \mathbf{e}(\mathbf{x}, \omega)$, where $\mathbf{x} \in \Omega$ and ω runs over some probability space. We also assume that random variables $e_1(\mathbf{x}, \omega)$, $e_2(\mathbf{x}, \omega)$, $\mathbf{x} \in \Omega$, are uncorrelated. Next, let $E[\mathbf{e}(\mathbf{x}, \omega)]$ and $\mathrm{Var}\,(\mathbf{e}(\mathbf{x}; \omega))$ denote the vector of expected values and the variances, respectively, defined as

$$E(\mathbf{e}(\mathbf{x})) = \begin{bmatrix} E(e_1); & \mathbf{x} \in \Omega_1 \\ E(e_2); & \mathbf{x} \in \Omega_2 \end{bmatrix} \tag{1}$$

and

$$\mathrm{Var}\,(\mathbf{e}(\mathbf{x}; \omega)) = \begin{bmatrix} \mathrm{Var}\,(e_1) \\ \mathrm{Var}\,(e_2) \end{bmatrix}. \tag{2}$$

It is well known that these vectors determine uniquely the Young modulus random field.

We assume that Poisson ratios are deterministic, i.e.

$$\boldsymbol{\nu}(\mathbf{x}) = \begin{bmatrix} \nu_1\,; & \mathbf{x} \in \Omega_1 \\ \nu_2\,; & \mathbf{x} \in \Omega_2 \end{bmatrix}. \tag{3}$$

We shall consider the random elasticity tensor field $C_{ijkl}(\mathbf{x}; \omega)$ by extending the deterministic case to the random one given by

$$C_{ijkl}(\mathbf{x}; \omega) = \mathbf{e}(\mathbf{x}; \omega) \left[\delta_{ij}\delta_{kl} \frac{\boldsymbol{\nu}(\mathbf{x})}{(1+\boldsymbol{\nu}(\mathbf{x}))(1-2\boldsymbol{\nu}(\mathbf{x}))} + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \frac{1}{2(1+\boldsymbol{\nu}(\mathbf{x}))} \right], \qquad i,j,k,l = 1,2\,. \tag{4}$$

Because of linear dependence between elasticity tensor components and the Young modulus (4), these components have Gaussian distribution so they can be derived uniquely from their first two moments.

From the definition of expected values and the variance of a random variable [40], and with the notation

$$A_{ijkl}(\mathbf{x}) = \delta_{ij}\delta_{kl} \frac{\boldsymbol{\nu}(\mathbf{x})}{(1+\boldsymbol{\nu}(\mathbf{x}))(1-2\boldsymbol{\nu}(\mathbf{x}))} + (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \frac{1}{2(1+\boldsymbol{\nu}(\mathbf{x}))} \tag{5}$$

we obtain

$$E(C_{ijkl}(\mathbf{x}; \omega)) = A_{ijkl}(\mathbf{x})\, E(\mathbf{e}(\mathbf{x}; \omega)) \tag{6}$$

and

$$\mathrm{Var}\,(C_{ijkl}(\mathbf{x}; \omega)) = (A_{ijkl}(\mathbf{x}))^2\, \mathrm{Var}\,(\mathbf{e}(\mathbf{x}; \omega)) \qquad \text{(no sum on } i,j,k,l). \tag{7}$$

In order to derive the first two moments of the effective elasticity tensors we use the classical definition [30, 44]

$$C_{ijkl}^{(\mathrm{eff})}(\omega) = \frac{1}{|\Omega|} \int_\Omega \left[\sigma_{kl}(\chi^{ij}(\mathbf{x}, \omega)) + C_{ijkl}(\mathbf{x}, \omega) \right] \mathrm{d}\Omega\,, \tag{8}$$

where $\chi^{ij}(\mathbf{x}, \omega)$ are displacement fields with periodic boundary displacement terms on the cell boundaries and random boundary forces on the phase boundary. Expectations and variations of the boundary forces can be expressed by the following Eqs. [16, 31]

$$E(F_q^{ij}(\omega)) \;=\; E\left\{ -[\lambda(\omega)]\delta_{ij}n_q - [\mu(\omega)](n_i\delta_{jq} + n_j\delta_{iq}) \right\}, \tag{9}$$

$$\mathrm{Var}\,(F_q^{ij}(\omega)) \;=\; \mathrm{Var}\,\left\{ -[\lambda(\omega)]\delta_{ij}n_q - [\mu(\omega)](n_i\delta_{jq} + n_j\delta_{iq}) \right\}, \tag{10}$$

where operator $[\cdot]$ denotes the difference of function values on the fibre-matrix boundary

$$[F(\mathbf{x})] = F^{\Omega_2} - F^{\Omega_1} \tag{11}$$

and

$$\lambda(\mathbf{x};\omega) = \frac{\nu(\mathbf{x})}{(1+\nu(\mathbf{x}))(1-2\nu(\mathbf{x}))}\,\mathrm{e}(\mathbf{x};\omega)\,, \tag{12}$$

$$\mu(\mathbf{x};\omega) = \frac{1}{2(1+\nu(\mathbf{x}))}\,\mathrm{e}(\mathbf{x};\omega)\,. \tag{13}$$

Finally we obtain expected values of boundary forces by splitting the r.h.s. of Eq. (9) into fibre and matrix parts and via certain algebraic operations

$$E(F_q^{ij}(\omega)) = B_{ijq}(\nu_1)\,E(e_1) + B_{ijq}(\nu_2)\,E(e_2)\,, \tag{14}$$

where operator $B_{ijq}(\nu(\mathbf{x}))$ [44] is defined by

$$B_{ijq}(\nu(\mathbf{x})) = -\delta_{ij} n_q \frac{\nu(\mathbf{x})}{(1+\nu(\mathbf{x}))(1-2\nu(\mathbf{x}))} - (n_i\delta_{jq} + n_j\delta_{iq})\frac{1}{2(1+\nu(\mathbf{x}))} \tag{15}$$

and the variances

$$\mathrm{Var}\,(F_q^{ij}(\omega)) = (B_{ijq}(\nu_1))^2\,\mathrm{Var}\,(e_1) + (B_{ijq}(\nu_2))^2\,\mathrm{Var}\,(e_2)\,, \qquad \text{(no sum on } i,j,k,l\text{)}. \tag{16}$$

By using the definition of the averaging function $F(\mathbf{x})$ on the region Ω

$$\langle F(\mathbf{x})\rangle_\Omega = \frac{1}{|\Omega|}\int_\Omega F(\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{17}$$

we can derive the expectations of the effective characteristics from (8), i.e.

$$E(C_{ijkl}^{(\mathrm{eff})}(\omega)) = E\Big(\langle\sigma_{kl}(\chi^{ij}(\mathbf{x};\omega))\rangle_\Omega\Big) + E\left(\langle C_{ijkl}(\mathbf{x};\omega)\rangle_\Omega\right)\,. \tag{18}$$

It should be noted that the expressions for the variances have a more complicated form than the expectations given by (18) because the averaged stresses and elasticity tensor are correlated variables [40, 45]. We obtain

$$\begin{aligned}\mathrm{Var}\,(C_{ijkl}^{(\mathrm{eff})}(\omega)) = {} & \mathrm{Var}\,\Big(\langle\sigma_{kl}(\chi^{ij}(\mathbf{x};\omega))\rangle_\Omega\Big) + \mathrm{Cov}\,\Big(\langle\sigma_{kl}(\chi^{ij}(\mathbf{x};\omega))\rangle_\Omega\,,\,\langle C_{ijkl}(\mathbf{x};\omega)\rangle_\Omega\Big) \\ & + \mathrm{Var}\,\left(\langle C_{ijkl}(\mathbf{x};\omega)\rangle_\Omega\right)\,.\end{aligned} \tag{19}$$

As we know, the displacement fields $\chi^{ij}(\mathbf{x};\omega)$, similarly to the deterministic ones [16, 30], are calculated numerically. Therefore, considering the constitutive relation from (iii) and Eq. (19), the variance of the effective property tensor may be obtained as a result of computer simulation. It will be done by means of statistical estimation methods [6, 10, 23], according to which expected values and the respective variances (the unbiased estimator [6]) of effective elasticity tensor components are calculated as

$$E(C_{ijkl}^{(\mathrm{eff})}(\omega)) = \frac{1}{M}\sum_{m=1}^{M} C_{ijkl}^{(\mathrm{eff})m}\,, \tag{20}$$

$$\mathrm{Var}\,(C_{ijkl}^{(\mathrm{eff})}(\omega)) = \frac{1}{M-1}\sum_{m=1}^{M}\left(C_{ijkl}^{(\mathrm{eff})m} - E(C_{ijkl}^{(\mathrm{eff})})\right)\,, \tag{21}$$

where $C_{ijkl}^{(\mathrm{eff})M}$ are given random series of desired tensor components obtained as a result of numerical random values generation.

3. COMPUTATIONAL IMPLEMENTATION

3.1. Homogenization finite element model

Let us introduce the following approximation of homogenization displacement functions χ_i^{pq} at any point of the considered continuum Ω by the finite number of generalized coordinates q_α^{pq} and shape functions $\varphi_{i\alpha}$

$$\chi_i^{pq} = \varphi_{i\alpha} q_\alpha^{pq}\,, \qquad i,p,q = 1,2; \quad \alpha = 1,\ldots,N\,. \tag{22}$$

The strain $\varepsilon_{ij}(\chi^{pq})$ and stress tensors $\sigma_{ij}(\chi^{pq})$ can be rewritten in the same way

$$\varepsilon_{ij}(\chi^{pq}) = B_{ij\alpha} q_\alpha^{pq}\,, \tag{23}$$

$$\sigma_{ij}(\chi^{pq}) = C_{ijkl}\varepsilon_{kl}(\chi^{pq}) = C_{ijkl}B_{kl\alpha} q_\alpha^{pq}\,, \tag{24}$$

where $B_{ij\alpha}$ is a typical FEM strain-displacement operator. Introducing Eqs. (23–24) into the virtual work equation we obtain

$$\delta\chi_{i,j}^{pq} \int_\Omega C_{ijkl}\chi_{k,l}^{pq}\,\mathrm{d}\Omega = \delta\chi_i^{pq} \int_{\partial\Omega} [F_i^{pq}]\,\mathrm{d}(\partial\Omega)\,, \tag{25}$$

where $[F_i^{pq}]$ are the boundary forces given by Eqs. (9–10). Further, let us define the global stiffness matrix

$$K_{\alpha\beta} = \sum_{e=1}^{E} K_{\alpha\beta}^{(e)} = \sum_{e=1}^{E} \int_{\Omega_e} C_{ijkl}B_{ij\alpha}B_{kl\beta}\,\mathrm{d}\Omega\,. \tag{26}$$

By introducing this matrix into Eq. (25) and minimizing it we obtain

$$K_{\alpha\beta} q_\beta^{pq} = Q_\alpha^{pq}\,, \tag{27}$$

where Q_α^{pq} is the external load vector including $[F_i^{pq}]$, which is defined by Eqs. (14–16). In the three numerical tests we obtain the homogenization function χ_i^{pq} for $p,q = 1,2$. In order to provide symmetry conditions on the periodicity cell quarter we fix vertical displacements on its external boundaries and rotations for every nodal point belonging to these boundaries. For the resulting functions χ_i^{pq} we compute the stresses $\sigma_{ij}(\chi_i^{pq})$ and average the tensor numerically on Ω. Finally, we obtain the effective properties from Eqs. (18–19) using the Monte-Carlo method described below.

3.2. Monte-Carlo simulation

The Monte-Carlo simulation is, in fact, a numerical method based on random sampling via a random number generator. In our problem of homogenization we define the Young modulus as an input Gaussian variable. In order to obtain random sequences of this variable we have to produce a numerically uniform distribution first. There is a sequuence of random numbers which lie within a specified range ([0,1] typically), and each number is as likely to occur as any other in the range. Generating uniform distributions is performed by means of a FORTRAN library routine which is a linear congruential generator. It generates a sequence of integer numbers $I_1, I_2, \ldots$, each between 0 and $m-1$, by the recurrence relation

$$I_{j+1} = aI_j + c \qquad (\mathrm{mod}\ m)\,, \tag{28}$$

where m is called the modulus and a, c are positive integers called the multiplier and the increment, respectively. The recurrence (28) will eventually repeat, with a period that is obviously no greater than m. If m, a and c are properly chosen, then the period of recurrence (28) is of the maximum length m. The sequence of real numbers between 0 and 1 is returned by dividing I_{j+1} by m, so that it is strictly less than 1, but occasionally (once in m calls) exactly equal to 0. The linear congruential method is very fast, it requires only a few operations per call, but it is not free of sequential correlation on successive calls. A special shuffling routine has to be added to eliminate this disadvantage [39]. Next, we use the Box-Muller method for generating random variables with a Gaussian distribution

$$p(y)\,\mathrm{d}y = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) \mathrm{d}y\,. \tag{29}$$

Let us consider the transformation between two uniform deviates on (0,1), x_1, x_2, and two quantities y_1, y_2 given as follows

$$\begin{aligned} y_1 &= \sqrt{-2\ln x_1}\,\cos 2\pi x_2\,, \\ y_2 &= \sqrt{-2\ln x_1}\,\sin 2\pi x_2\,. \end{aligned} \tag{30}$$

We can write equivalently

$$\begin{aligned} x_1 &= \exp\left[-\frac{1}{2}\left(y_1^2 + y_2^2\right)\right], \\ x_2 &= \frac{1}{2\pi}\arctan\frac{y_2}{y_1}\,, \end{aligned} \tag{31}$$

and the Jacobian determinant has the form

$$\frac{\partial(x_1,x_2)}{\partial(y_1,y_2)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = -\frac{1}{2\pi}\exp\left[-\frac{1}{2}\left(y_1^2 + y_2^2\right)\right] \tag{32}$$

since this is a product of functions of y_1 and y_2 separately. Finally, each y is independently distributed according to the Gaussian distribution.

4. Numerical results

Computer simulation has been done by means of the MCCEFF numerical procedure based on a 4-node 2D rectangular element of the POLSAP system [5, 22] (stress–strain analysis with membrane elements) written in FORTRAN 77. On the basis of a given cell geometry and a random number sequence of the Young modulus, this code calculates the component sequences of the effective property tensor according to the formulas obtained before. Generation of necessary random variables with sample volume of 10 000 has been done with the use of the D7R6 program; its code can be found in [39] (generation of input data as well as statistical processing of the results can be done with the use of Microsoft Excel Spreadsheet).

Numerical example of searching effective properties of a composite has been tested earlier for the deterministic case in [16, 31]. with the assumed expected values of the Young modulus $e_1 = 86.0$ GPa for the fibre and $e_2 = 4.0$ GPa for the matrix (Poisson ratios $\nu_1 = 0.22$, $\nu_2 = 0.34$, respectively), four numerical tests have been conducted for different combinations of variation coefficients of the moduli for both materials (Table 1).

In each cell the upper number represents the coefficient of variation α given by

$$\alpha[X] = \sqrt{\frac{\mathrm{Var}\,[X]}{E^2[X]}} \tag{33}$$

and the lower — standard deviation (in GPa) by

$$\sigma[X] = \sqrt{\mathrm{Var}\,[X]}\,. \tag{34}$$

The purpose of those tests was to state which of the random Young moduli influenced the components of the effective properties tensor $C_{ijkl}^{(\mathrm{eff})}(\omega)$ of the composite and how it influenced those

Table 1. Coefficients of variation (and standard deviations) for numerical tests

test number	e_1	e_2
1	0.1 8.6	0.1 0.4
2	0.1 8.6	0.05 0.2
3	0.05 4.3	0.1 0.4
4	0.05 4.3	0.05 0.2

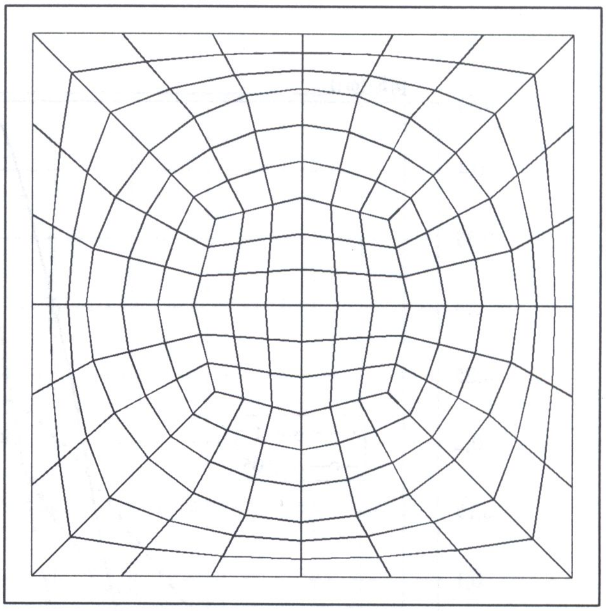

Fig. 2. FEM discretization of periodicity cell

components. The assumed values of variance coefficients were considered, in numerical analysis, to be their lower and upper bounds, respectively [22, 27]. In all the cases, calculations were done on a PC-486/50 computer and one test took approximately 31 hours. Discretization of a composite periodicity cell subjected to the tests is shown in Fig. 2 (180 finite elements and 527 degrees of freedom).

In Figs. 3 to 7, the probability density function (PDF) of the occurence of a given value in a population is marked on the vertical axis, while values obtained as a result of generation are marked on the horizontal axis (in GPa). Figures 3 and 4 show input random distributions of the Young modulus in a fibre region and in a matrix, respectively. On these charts, series 1 is equivalent to random distribution with greater variance ($\alpha = 0.1$), and series 2 — with smaller variance ($\alpha = 0.05$).

A graphic illustration of the effective properties PDFs has been shown in Figs. 5–7, and random distribution parameters can be found in Table 2.

Table 2. Expectations, variances and variation coefficients in individual tests

Test number	$C^{(\mathrm{eff})}_{1111}(\omega)$	$C^{(\mathrm{eff})}_{1122}(\omega)$	$C^{(\mathrm{eff})}_{1212}(\omega)$
1	14.77 1.29 0.087	4.94 0.47 0.094	18.07 1.74 0.096
2	14.78 0.66 0.045	4.95 0.23 0.047	18.07 1.74 0.096
3	14.78 1.27 0.086	4.95 0.46 0.094	18.06 0.87 0.048
4	14.79 0.64 0.043	4.95 0.23 0.047	18.06 0.87 0.048
COSAN	14.66	4.95	18.38

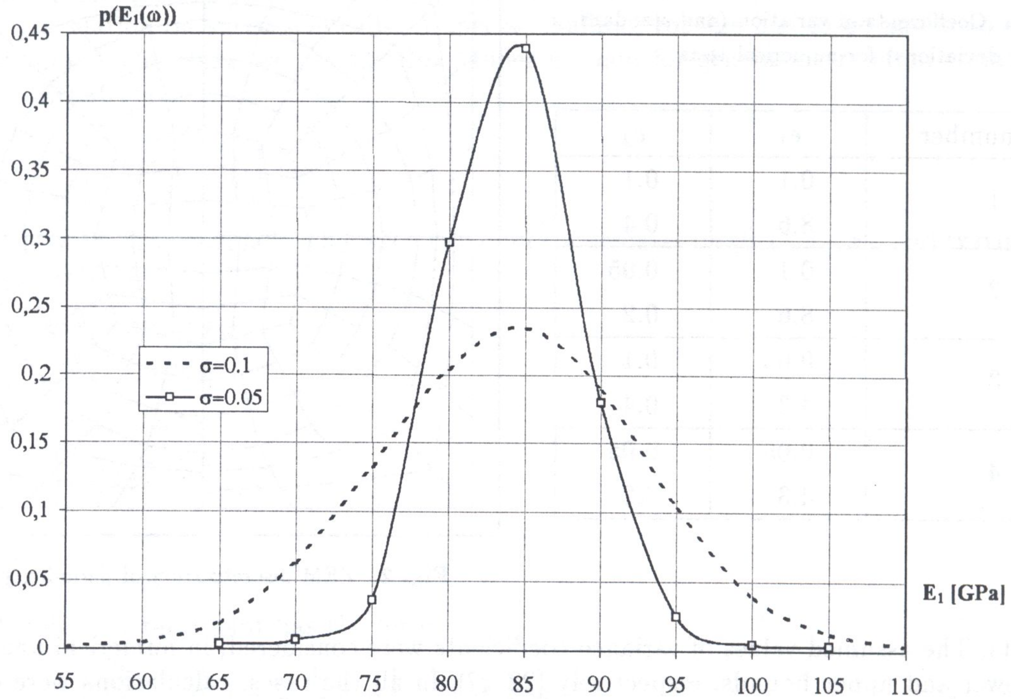

Fig. 3. Input probability density function of Young moduli in fibre

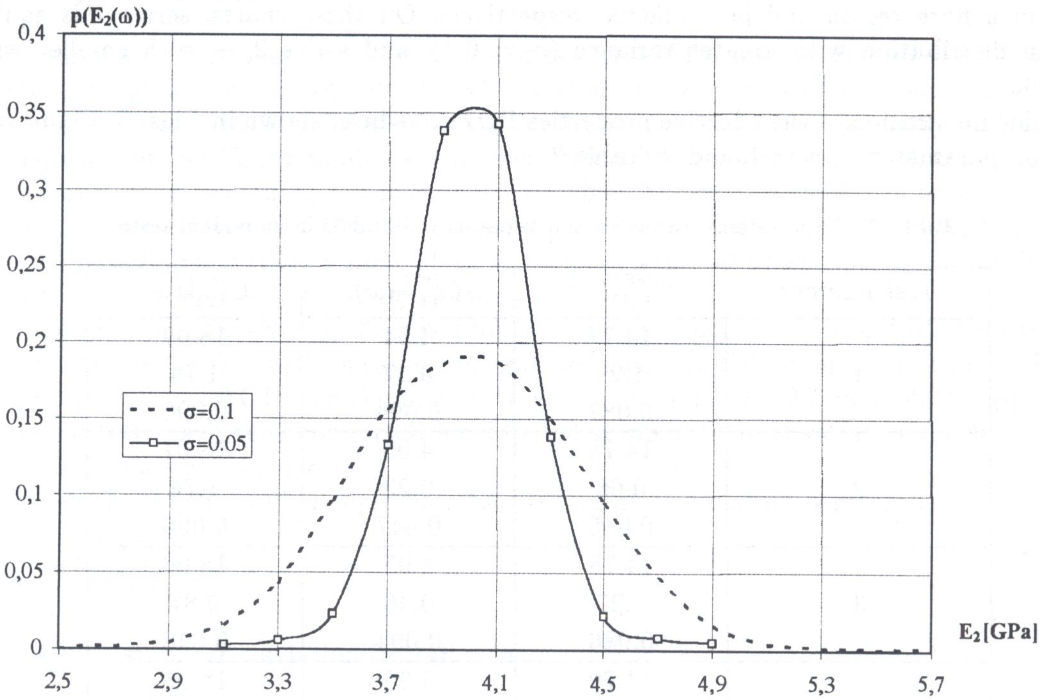

Fig. 4. Input probability density function of Young moduli in matrix

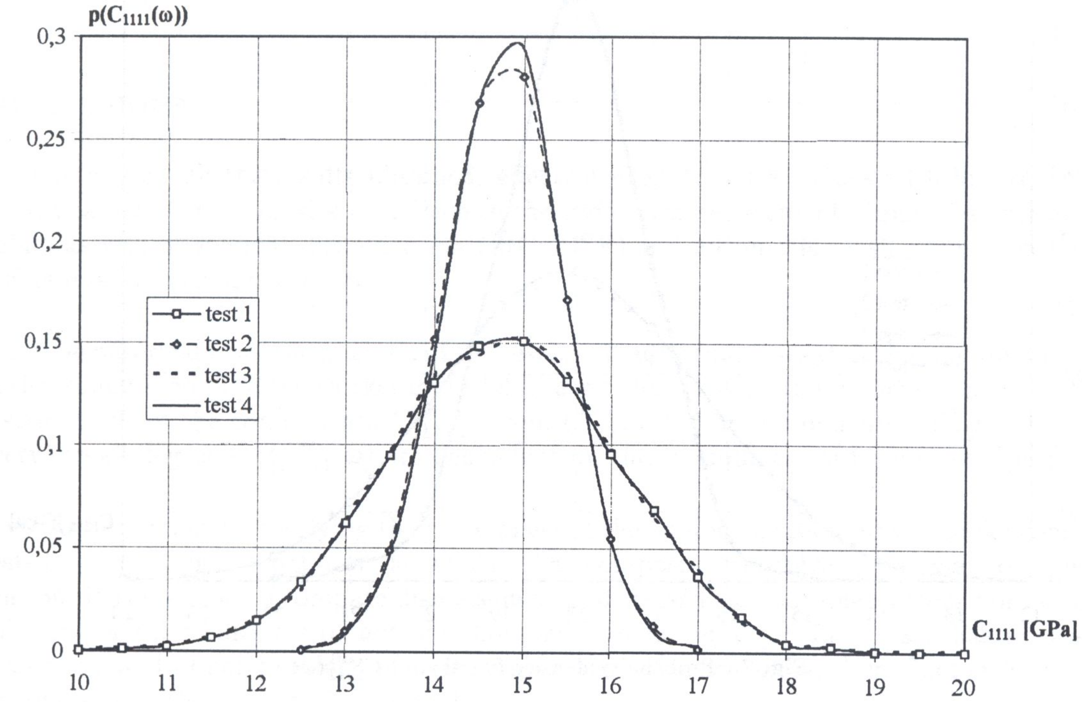

Fig. 5. Probability density function of $C_{1111}^{(\mathrm{eff})}(\omega)$

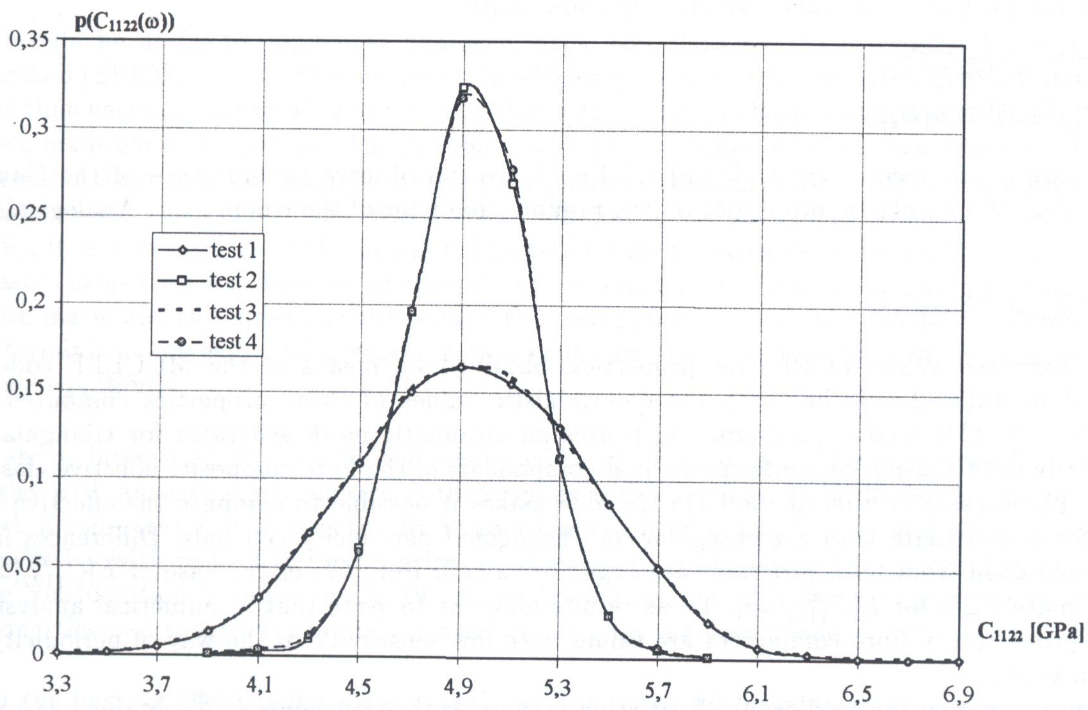

Fig. 6. Probability density function of $C_{1122}^{(\mathrm{eff})}(\omega)$

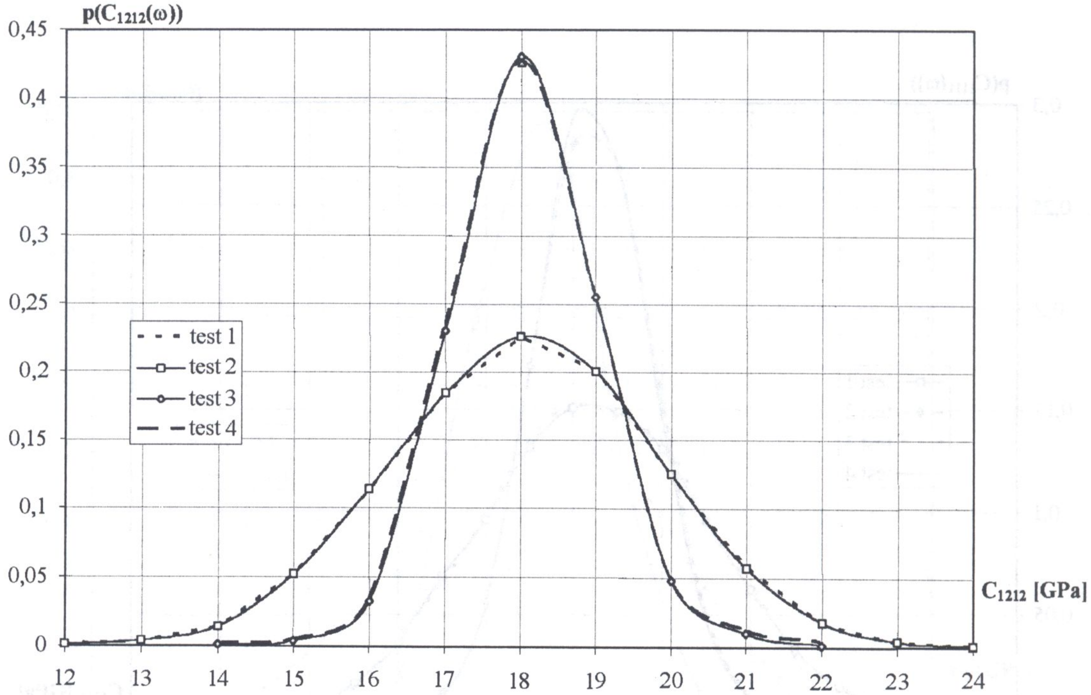

Fig. 7. Probability density function of $C_{1212}^{(\text{eff})}(\omega)$

Probabilistic distributions of the components $C_{1111}^{(\text{eff})}(\omega)$ and $C_{1122}^{(\text{eff})}(\omega)$, shown in Figs. 5 and 6, respectively, prove that random character of Young modulus changes in the matrix has a dominant influence on these distributions. The bell-shaped curve for test 1 agrees with the one for test 3 (analogically for tests 2 and 4). Variation coefficients of the component $C_{1111}^{(\text{eff})}(\omega)$, specified in Table 2, show that a random character of fibre elastic properties tends to influence the component very weakly. For both of these values we may therefore write

$$\alpha[C_{1111}^{(\text{eff})}(\omega)] \cong \alpha[E_2] \tag{35}$$

$$\alpha[C_{1122}^{(\text{eff})}(\omega)] \cong \alpha[E_2]\,. \tag{36}$$

In the case of the component $C_{1212}^{(\text{eff})}(\omega)$, cf. Fig. 7, we can observe the influence of the Gaussian distribution of fibre elastic properties on the random character of the component. Analogically, we have

$$\alpha[C_{1212}^{(\text{eff})}(\omega)] \cong \alpha[E_1]\,. \tag{37}$$

The expected values of effective properties, obtained by means of the MCCEFF code, are specified in Table 2 together with the deterministic values of these properties computed with COSAN [16]. This second program, which uses an automatic mesh generator for triangular elements only (COMESH), computes individual components of the fibre composite effective elasticity tensor. The present version of the COSAN code makes it possible to compute the effective properties for a composite with a rectangular and hexagonal periodicity cell only. Differences in the results obtained from both programs are negligibly small, from 0% in the case of $E[C_{1122}^{(\text{eff})}(\omega)]$ to approximately 2% for $E[C_{1111}^{(\text{eff})}(\omega)]$. These results allow us to state that in numerical analysis, effective properties of fibre composites are values with low sensitivity to the way of periodicity cell discretization.

When comparing the coefficients of variation of individual components $C_{ijkl}^{(\text{eff})}$ obtained as a result of simulation with the coefficients of variation of given random functions, the biggest difference can be seen in the case of the component $C_{1111}^{(\text{eff})}(\omega)$ (10%). For the other components, it is half as big. Therefore, we can find out that the random character of the Young modulus in the component materials of the considered composite has the smallest influence on the random character of the component $C_{1111}^{(\text{eff})}(\omega)$ of the effective properties tensor.

5. Conclusions

1. As it is proved by the results obtained, effective properties are values with low sensitivity to discretization of a composite cell. Despite the use of various elements (quadrilateral and triangular) as well as manual discretization (MCCEFF) and automatic mesh generation (COSAN) differences are negligibly small.

2. It can be observed from the analysis of the results of individual tests that the random distribution of the Young moduli of the weaker material (matrix) has an essential influence on the probability distributions of the components $C_{1111}^{(\text{eff})}(\omega)$ and $C_{1122}^{(\text{eff})}(\omega)$. The components $C_{1212}^{(\text{eff})}(\omega)$ and — to a (very) small degree — $C_{1111}^{(\text{eff})}(\omega)$ are dependent on the randomness of fibre elastic properties.

3. It seems important for verifying the correctness of the homogenization idea to make comparative tests by subjecting a quarter or the whole of the composite structure to uniaxial compression or tension. It is possible to compare displacement and stress fields obtained before homogenization with analogical results for orthotropic homogenized material [1]. In the stochastic case, there is also an additional problem of determining the influence of random fluctuations of input parameters on the output of such analysis.

4. In order to increase the precision of computing the distribution of composite effective values, the composite should be homogenized with assumed periodicity conditions for a structure of more than one cell. For such a large scale system, larger computers are required because of too many degrees of freedom in such a system.

5. It would be useful to implement homogenization equations for the Stochastic Finite Element Method (SFEM) [22, 25–27] to increase the efficiency of computations (most of all by shortening the time necessary for making iteration). Attempts at deriving basic mathematical formulas have been made earlier in [24]. The basic problem which occurs when SFEM is used is the complicated formulas describing covariances between stress tensor and constitutive tensor components. In order to find the value of the covariance in relation to the remaining components of equation (19), it is necessary to make numerical tests for various expected values of fibre and matrix elastic properties for different shapes of the periodicity cell and various volume fractions of both materials. Depending on the results, the covariance can be either avoided or replaced with deterministic correctors of averaged stress variances and an averaged elasticity tensor in the same equation.

6. It seems interesting to compare the computed random parameters of composite effective properties with analogical parameters of their upper and lower bounds [3, 34, 46]. In the context of such a comparison, it is essential to make a numerical analysis of composite effective property sensitivity with respect to the fibre shape. The algorithm of such an analysis could be based on the consideration included in [12, 19]. As we know from [34], the relationship between effective properties and their lower bounds depends mainly on the fibre shape.

7. On the basis of the existing deterministic models, it would be interesting to create numerical procedures for analysing the problems of composites with anisotropic as well as non-linear materials. In the anisotropic case, it would be interesting to state the influence of random distribution changes of elastic properties in individual directions on effective characteristics.

ACKNOWLEDGEMENT

This paper has been supported by the State Committee for Scientific Research, project number 8 T11F 010 09.

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