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Identification of bilinear material models of nonlinear beams on nonlinear foundation

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In the analysis of the problem the beam is modelled by hinge elements connected together by rigid bars while the foundation is replaced by spring elements supporting the hinges. The nonlinear behaviour of the beam and the foundation is described by specially formulated bilinear material models. The characteristics of these models are considered to be unknown but, on the other hand, the deflections of certain points of the beam are given. The goal of the investigation is to determine the best values of the material characteristics by the use of a mixed variational principle based on the bilinear material model and by the application of the identification methods.

The problem is stated in the form of constrained, nonsmooth, nonlinear mathematical programming problem, and the identification is equivalent to finding the minima of a non-linear, multivariable functional. The application is illustrated by the solution of an example.

1. Introduction

For analysis of nonlinear behaviour of structures, a great number of solution techniques have been developed. The most commonly used approaches are incremental load methods which apply small load increments, locally linearize the nonlinear force–deformation characteristics of the elements and after each step update the stiffness matrix according to the changes in the states of stresses, strains and geometry of the structure.

Significant savings of computation time can be achieved by approximating the nonlinear material behaviour with bilinear relationships. An important special case of this approximation is the use of linearly elastic-perfectly plastic material which is the basis of the elasto-plastic and limit analysis of structures. For the analysis of elastic structures with nonlinear material characteristics Lógó and Taylor proposed a special bilinear material model [1]. In this model an idealized material with bilinear stress–strain relation is replaced by a composite material in which one component is linearly elastic with the material law

$$\hat{\sigma} = E\varepsilon \,, \tag{1}$$

while the other one is linearly elastic-"pseudo-plastic" following the same relations during loading and unloading,

$$\bar{\sigma} = \bar{E}\varepsilon \,, \qquad |\bar{\sigma}| - \bar{\sigma}_0 \le 0 \,. \tag{2}$$

By parallel connection of these two components a bilinear material model can be obtained,

$$\sigma = \bar{\sigma} + E\varepsilon \,, \qquad |\bar{\sigma}| - \bar{\sigma}_0 \le 0 \,, \qquad \bar{\sigma} = \bar{E}\varepsilon \,. \tag{3}$$

The details of the corresponding stress–strain diagram are shown in Fig. 1.

Using this model, mixed extremum principles have been developed for the analysis and optimal design of trusses and structures composed of one-dimensional elements and solution techniques have been elaborated based on a constrained nonsmooth, nonlinear programming algorithm [1–6].

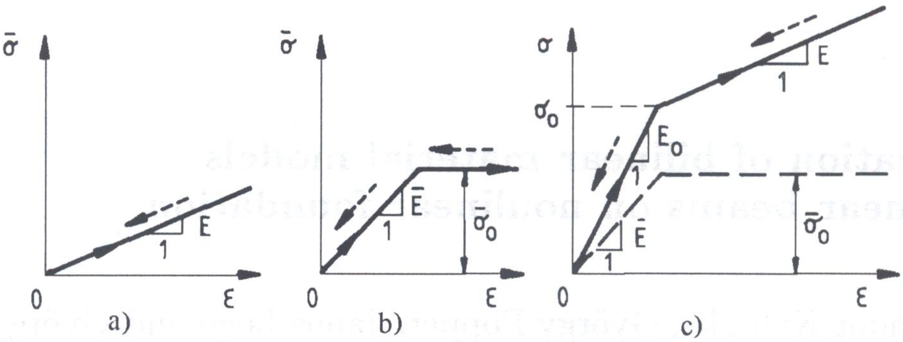

Fig. 1. Bilinear stress-strain diagram

The bilinear model and the mixed extremum principles have also been applied to the analysis of nonlinearly elastic beams resting on nonlinearly elastic foundation [7]. In the solution the beam was subdivided into a series of rigid bars interconnected by hinges and the foundation was replaced by a series of individual springs supporting the hinges. Then, after some modifications, the bilinear material model and the mixed extremum principles have been used to the analysis of this discretized structure.

Due to the nonlinear behaviour and a number of uncertain factors in the analysis of beams on soil, the reliable approximation of the material characteristics is a very difficult problem. On the basis of results of experiments with samples taken from the beam and the soil, only some very complicated methods (e.g., the three dimensional finite element method) can provide reliable information about the actual behaviour of the beam and the soil. In the large-scale in situ experiments, on the other hand, one can generally measure only the deflections of the beam, from which the material characteristics cannot be directly calculated. Applying, however, the parametric identification methods, the most reliable values of the material characteristics can be calculated and then applied to the analysis of various similar problems. The identification can be also based on more accurate solutions; in this case the results can be used for determination of stiffness characteristics for simpler approximate methods, like the method described above.

Following the above ideas, this paper presents a method where the stiffness characteristics of the beam and the foundation are unknown, but the deflections and the internal forces at certain points of the beam are known, as obtained from experiments. The goal of the investigation is to determine the estimated values of the stiffness characteristics that best approximate the real ones, by the use of a mixed variational principle based on complementary potential energy and the application of parametric identification [8, 9].

The first part of the paper shortly describes the fundamental equations and mixed extremum principle of the discretized structure. Then the basic idea of the method, the solution techniques and the application of the parametric identification to the present problem are presented. Finally, the application is illustrated by the solution of an example.

2. Models of the Beam and the Foundation

Consider a beam on a foundation subjected to a proportional loading with load parameter m (Fig. 2a). We assume that the nonlinear behaviour of the beam and the foundation are approximated by bilinear models described above.

First we discretize the structure (Fig. 2b). The beam is modelled by n hinge elements connected together by rigid bars while the foundation is modelled by n spring elements which are supporting the hinges. We assume that the beam can lift up from the foundation and therefore the hinges can separate from the springs. In this model all the deformations are concentrated in the hinge and spring elements and the external loads are also reduced to the hinges.

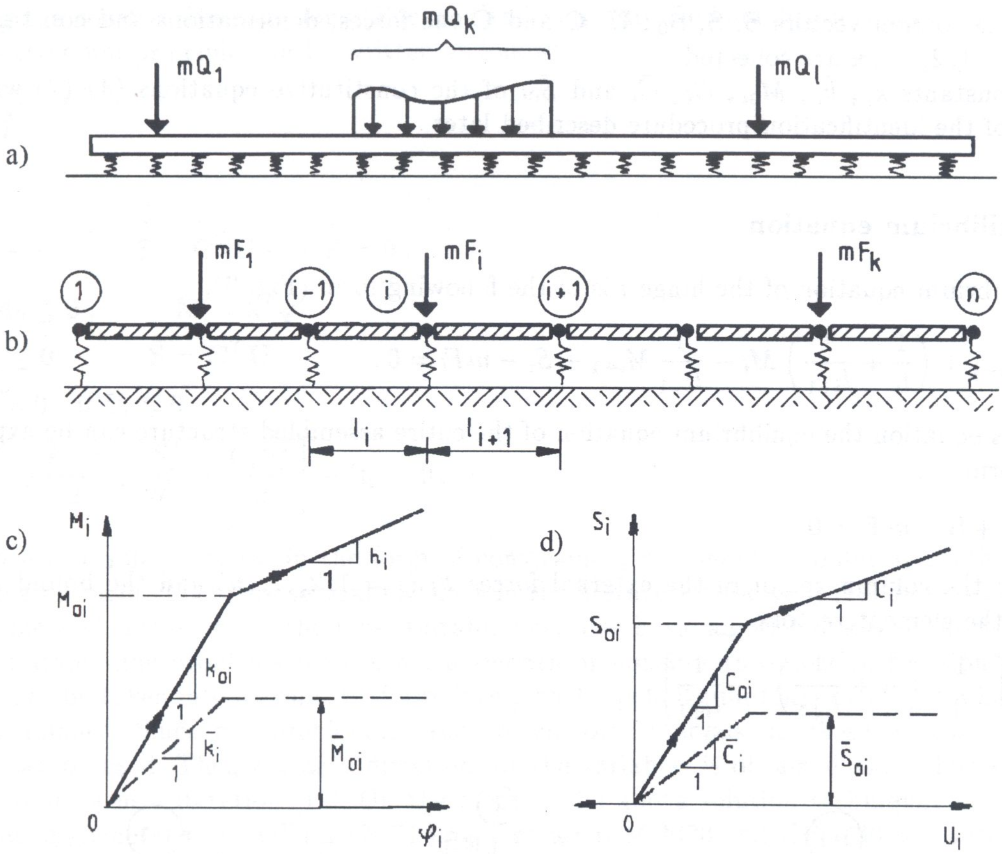

Fig. 2. a) Beam on foundation, b) model of the beam and foundation, c) bilinear moment–rotation relationship, d) bilinear force–deformation relationship

3. Fundamental equations

3.1. Constitutive equations

Generalizing the idea of the bilinear model defined above one can describe the relationships between the moment M_i and the rotation φ_i of the hinge element i as follows (Fig. 2c)

$$M_i = \bar{M}_i + k_i\varphi_i\,, \qquad |\bar{M}_i| - \bar{M}_{0i} \le 0\,, \qquad \bar{M}_i = \bar{k}_i\varphi_i\,. \tag{4}$$

For the whole assembly of hinges these equations can be written in vector form

$$\mathbf{M} = \bar{\mathbf{M}} + \mathbf{k}^T\boldsymbol{\varphi}\,, \qquad |\bar{\mathbf{M}}| - \bar{\mathbf{M}}_0 \le \mathbf{0}\,, \qquad \bar{\mathbf{M}} = \bar{\mathbf{k}}^T\boldsymbol{\varphi}\,, \tag{5}$$

where in the vectors $\mathbf{M}$, $\bar{\mathbf{M}}$, $\bar{\mathbf{M}}_0$, $\boldsymbol{\varphi}$, $\mathbf{k}$ and $\bar{\mathbf{k}}$ the moments, rotations and constants of the hinges $i = 1, 2, \ldots, n$ are collected.

Considering the spring element i with the force S_i and deformation U_i, and making again use of the bilinear model, we get the following relationships (Fig. 2d)

$$S_i = \bar{S}_i + C_iU_i\,, \qquad |\bar{S}_i| - \bar{S}_{0i} \le 0\,, \qquad \bar{S}_i = \bar{C}_iU_i \qquad \text{and if } U_i \le 0,\ S_i = 0. \tag{6}$$

Here compression and shortening of the springs are assumed to be positive and the unilateral contact between the beam and foundation is modelled by specifying zero tension capacity.

For the whole assembly of springs, Eqs. (6) can be expressed in vector form

$$\mathbf{S} = \bar{\mathbf{S}} + \mathbf{C}^T\mathbf{U}\,, \qquad |\bar{\mathbf{S}}| - \bar{\mathbf{S}}_0 \le \mathbf{0}\,, \qquad \bar{\mathbf{S}} = \bar{\mathbf{C}}^T\mathbf{U} \qquad \text{and if } U_i \le 0,\ S_i = 0\ (i = 1, 2, \ldots, n). \tag{7}$$

Here in the column vectors $\mathbf{S}$, $\bar{\mathbf{S}}$, $\bar{\mathbf{S}}_0$, $\mathbf{U}$, $\mathbf{C}$ and $\bar{\mathbf{C}}$ the forces, deformations and constants of the springs $i = 1, 2, \ldots, n$ are collected.

The constants k_i, $\bar{k}_i$, $\bar{M}_{0i}$, C_i, $\bar{C}_i$ and $\bar{S}_{0i}$ of the constitutive equations (4)–(7) will be the subjects of the identification procedure described later.

3.2. Equilibrium equation

The equilibrium equation of the hinge i is of the following form (Fig. 3),

$$-\frac{1}{l_i}M_{i-1} + \left(\frac{1}{l_i} + \frac{1}{l_{i+1}}\right) M_i - \frac{1}{l_{i+1}}M_{i+1} + S_i - mF_i = 0\,. \tag{8}$$

Using this equation the equilibrium equation of the entire assembled structure can be expressed in matrix form

$$\mathbf{G} \cdot \mathbf{M} + \mathbf{S} - m\,\mathbf{F} = \mathbf{0}\,. \tag{9}$$

Here $\mathbf{F}$ is the column vector of the external forces F_i $(i = 1, 2, \ldots, n)$ and the bound matrix $\mathbf{G}$ contains the element vectors

$$G_i = \left[-\frac{1}{l_i}\,,\ \left(\frac{1}{l_i} + \frac{1}{l_{i+1}}\right),\ -\frac{1}{l_{i+1}}\right]. \tag{10}$$

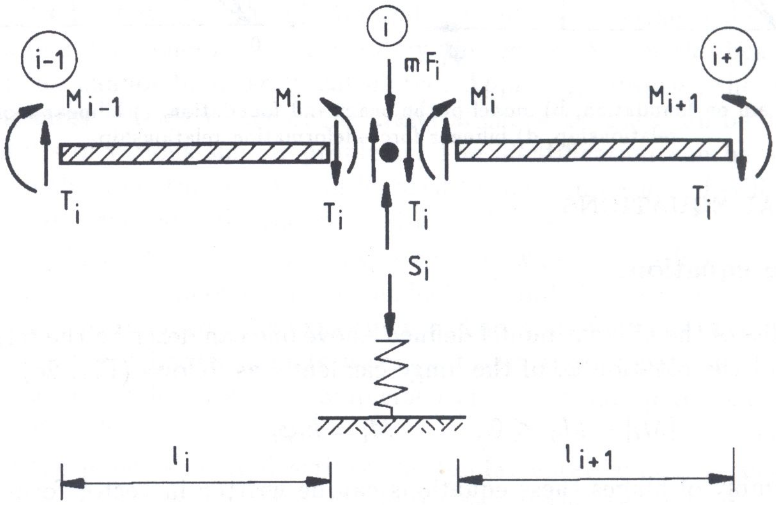

Fig. 3. Free-body diagrams of the beam elements

4. Formulation of the Extremum Principle

In the papers [2–5] two mixed extremum principles based on the bilinear material model and on the potential energy and the complementary potential energy, respectively, have been presented and applied to the analysis of trusses. Recently, the principles have been used to the analysis of nonlinear beams on nonlinear foundation [7]. In the following the principle based on the complementary potential energy will be presented. The proofs and detailed description of this principle can be found elsewhere [2, 5].

Among all states of the beam and foundation which satisfy the equilibrium and constitutive equations and correspond to a prescribed level $\bar{\Pi}_0$ of the complementary potential energy, the one at which the load multiplier m assumes its maximum value becomes the actual state. Hence, substituting the constitutive equations (Eqs. (5) and (7)) into the equilibrium equation (Eq. (9)) the mixed extremum principle can be written as follows:

$$m = \max_{(m,\bar{M},\bar{S},\varphi,U)}! \tag{11}$$

subject to

$$\mathbf{G}\cdot\left(\bar{\mathbf{M}}+\mathbf{k}^T\varphi\right)+\bar{\mathbf{S}}+\mathbf{C}^T\mathbf{U}-m\,\mathbf{F}=\mathbf{0}\,, \tag{12}$$

$$|\bar{\mathbf{M}}|-\bar{\mathbf{M}}_0\leq\mathbf{0}\,,\qquad \bar{\mathbf{M}}=\bar{\mathbf{k}}^T\varphi \tag{13}$$

$$|\bar{\mathbf{S}}|-\bar{\mathbf{S}}_0\leq\mathbf{0}\,,\qquad \bar{\mathbf{S}}=\bar{\mathbf{C}}^T\mathbf{U}\,, \tag{14}$$

$$\bar{S}_i=C_i=0\quad\text{if } U_i\leq 0, \tag{15}$$

$$\sum_{i=1}^{n}\left[\frac{\bar{M}_i^2}{2\bar{k}_i}+\frac{k_i\varphi_i^2}{2}+\frac{\bar{S}_i^2}{2\bar{C}_i}+\frac{C_iU_i^2}{2}\right]-\bar{\Pi}_0\leq 0\,. \tag{16}$$

The above principle is stated in the form of constrained, nonsmooth, nonlinear mathematical programming problem. There are several methods in the literature to solve it [10] and among them the bundle method [11] is one of the most suitable one. The basic algorithm solves unconstrained nonlinear programming problems with either a smooth or nonsmooth objective function and the constraints can be taken into account by formulating an L1-penalty function or using some barrier function techniques. Since our problem consists of smooth intervals, in these intervals smooth algorithms can be used. Then a search direction for the variables is obtained and a line search is performed to get a new iteration [12]. On the basis of the above solution techniques a computer program was implemented in FORTRAN 77 language for IBM 3090 and HP 9730 computers [13].

If the stiffness characteristics $\mathbf{k}$, $\bar{\mathbf{k}}$, $\bar{\mathbf{M}}_0$ and $\mathbf{C}$, $\bar{\mathbf{C}}$, $\bar{\mathbf{S}}_0$ of the beam and the foundation, respectively, are known, then during the analysis an appropriate level $\bar{\Pi}_0$ of the complementary potential energy has to be assumed rather than the intensity of the load. Then, using the principle and the solution techniques described above one can determine the load intensity m and all the state variables corresponding to the assumed energy level of the structure. Repeating this procedure, a load history analysis can be conducted or the specific state of the structure corresponding to a requested load intensity can be easily found.

5. Identification of the stiffness characteristics

In our former paper [7] using the bilinear material model and the corresponding mixed extremum principle, we presented the nonlinear analysis of beams on foundation. That analysis made possible the determination of internal forces and deflections of the structure. The subject of our present investigation is the inverse problem. Now stiffness characteristics of the beam and the foundation are unknown, and the deflections and internal forces at certain points at the beam as well as the load parameter are known from experiments. The goal of the present investigation is to determine the estimate values of the stiffness characteristics best approximating the real ones by the use of the mixed variational principle described above and by the application of the identification method to be presented below.

5.1. Mathematical formulation of identification

Assume a vector valued function $\mathbf{f}(\mathbf{x})$ with p real elements, where $\mathbf{x}=(x_1,\ldots,x_q)$ is a real vector of q unknown parameters. The function $\mathbf{f}:\mathcal{R}^q\to\mathcal{R}^p$ is generally given by a computer subroutine, sometimes requiring a rather time-consuming calculation, and the derivatives of $\mathbf{f}(\mathbf{x})$ may not be conveniently available.

The function $\mathbf{f}(\mathbf{x})$ is the solution computed from a mathematical model of the investigated mechanical structure. This solution depends on the unknown parameters $\mathbf{x} = (x_1, \ldots, x_q)$. Thus, $\mathbf{f}(\mathbf{x})$ represents a class of functions or a class of models depending on vector $\mathbf{x}$. The actual response of the mechanical structure known from measurements or from more exact analysis of the structure is given by a real vector $\tilde{\mathbf{f}}$ with p real elements.

We are looking for the "best" model of the structure within the mentioned class of models: we want to find the vector $\mathbf{x}^* = (x_1^*, \ldots, x_q^*)$, minimizing the norm of an error function

$$\mathbf{D}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - \tilde{\mathbf{f}} \tag{17}$$

over $\mathcal{R}^q$. Since the error (17) is a vector with p real elements, we may use the Euclidean norm, or more precisely, its square:

$$\|\mathbf{f}(\mathbf{x}) - \tilde{\mathbf{f}}\|^2 = [\mathbf{f}(\mathbf{x}) - \tilde{\mathbf{f}}]^T[\mathbf{f}(\mathbf{x}) - \tilde{\mathbf{f}}]\,. \tag{18}$$

We note that the minimization and consequently the identification of vector $\mathbf{x} = (x_1, \ldots, x_q)$ is equivalent to finding the least squares solution to the nonlinear over-determined system of equations

$$\mathbf{D}(\mathbf{x}) = \mathbf{0}\,, \qquad \mathbf{D} : \mathcal{R}^q \to \mathcal{R}^p, \quad p \geq q. \tag{19}$$

5.2. Application to the present problem

In the bilinear material model described above the behaviour of hinges and springs is characterised by three independent parameters. For the sake of simplicity assume that all the hinges and all the springs, respectively, have the same material characteristics and therefore there are only six independent parameters: k, $\bar{k}$, $\bar{M}_0$, C, $\bar{C}$ and $\bar{S}_0$, as defined in Section 3.1. The parameters are the elements of the unknown vector $\mathbf{x}$ introduced in Section 5.1.

Assume that deflections and internal forces at certain points of the beam are known from experiments. These quantities constitute the p elements of the vector $\tilde{\mathbf{f}}$. Then for any vector $\mathbf{x} = (x_1, \ldots, x_6) = (k, \bar{k}, \bar{M}_0, C, \bar{C}, \bar{S}_0)$ the deflections and internal forces of the beam can be computed using the method described in Section 4. These quantities form the vector–vector function $\mathbf{f}(\mathbf{x})$. Constructing the error function $\mathbf{D}(\mathbf{x})$ defined by Eq. (17) the minimizer $\mathbf{x}^* = (x_1^*, \ldots, x_6^*)$ of the scalar valued vector function (18) is to be found. In other words, the least squares solution to the nonlinear, over-determined system of Eqs. (19) is to be found.

5.3. Numerical techniques

There are several methods for finding the minima of the non-linear, multivariable functional of Eq. (18). In our investigations, a simple non-derivative method, namely the downhill simplex method of Nelder and Mead has been used [14]. This method seems to be very efficient and reliable for problems where the number of unknowns is not greater, say, than six. This solution technique was implemented in FORTRAN 77 programming language on HP 9730 computer.

6. Numerical example

Consider a nonlinear beam with uniform cross-section subjected to proportional loading and supported by a nonlinear foundation, as is shown in Fig. 4a. The nonlinear behaviour of the beam and the foundation are approximated by the bilinear models described above. The unknown material constants are related to the unknown stiffness characteristics k, $\bar{k}$, $\bar{M}_0$, C, $\bar{C}$ and $\bar{S}_0$ of the hinge and spring elements of the discretized structure (Fig. 4b).

Assume that the deflections and internal forces at the hinges of the beam are known from more accurate analysis of the structure. These results are shown in Table 1. The first column shows the

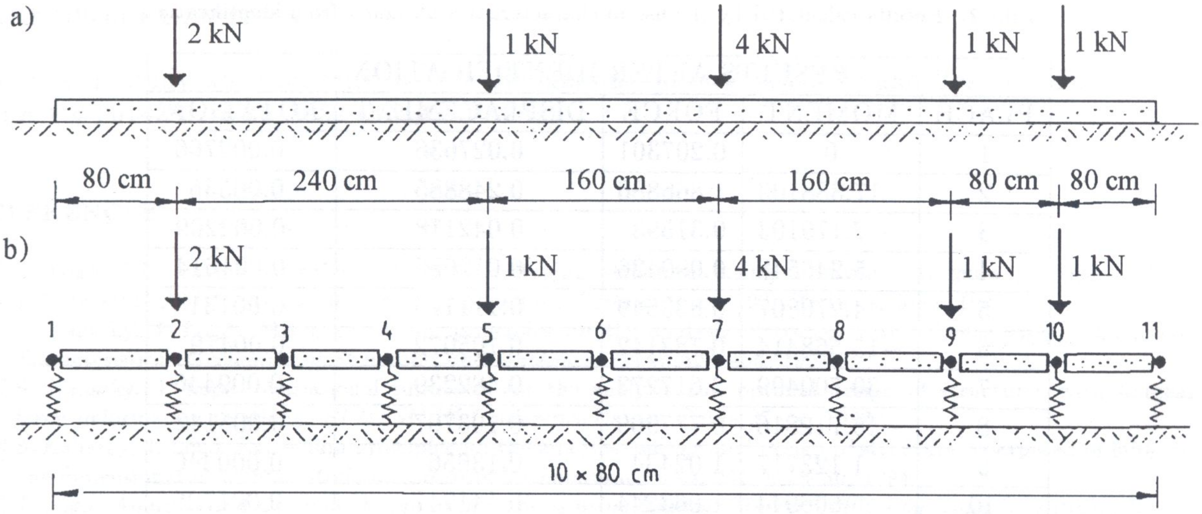

Fig. 4. a) Beam on foundation, b) model of the beam and foundation

Table 1. Results of the accurate analysis

ACCURATE RESULTS				
PLACE	MOMENT	FORCE	DISPLACEMENT	ROTATION
1	0	0.20731	0.027641	0.002766
2	17.387238	1.86685	0.248913	0.00535
3	−7.179371	0.315928	0.042124	−0.002209
4	−5.248762	0.090487	0.012065	−0.001615
5	4.27111	0.833478	0.11113	0.001314
6	15.568603	0.788246	0.105099	−0.00479
7	30.702633	3.6171	0.48228	0.009447
8	−16.713154	0.778037	0.103738	−0.005143
9	1.125814	1.024277	0.13657	0.000346
10	5.607756	1.063234	0.141765	0.001725
11	0	0.066862	0.008915	−0.001661

places where the stresses and strains are known. The numbers correspond to the discrete points 1...11 shown in Fig. 4b. The other columns contain the moments and rotations at the hinges and the forces and displacements of the supporting springs.

Applying the identification method described above, the unknown constants of the springs and hinges can be determined. They are supposed to best approximate the real values of the characteristics of the discretized structure. The values are as follows: $C = 1.501$, $\bar{C} = 3.000$, $\bar{S}_0 = 2.979$, $k = 470.0$, $\bar{k} = 1390$ and $\bar{M}_0 = 127.0$. Using these material characteristics and the solution method of Section 4, the deflections and internal forces of the discretized structure have been calculated and are shown in Table 2 in the same manner as they were given in Table 1. Comparing Tables 1 and 2 one can see that the results of the more accurate analysis and those obtained by the use of the presented solution method are very close to each other. The average deviation is of the order of 10^{-4}. It shows that the use of the discretized structure and of the bilinear material model provides approximate solutions and that the identification method described above is an efficient tool for the determination of the stiffness characteristics of the bilinear material model.

The necessary number of iterations in the downhill simplex method of Nelder and Mead [14] depends on the required accuracy of the stiffness characteristics. The accuracy was measured by the Euclidean norm of the error function, see Eq. (17). In our example this dependence is shown in Fig. 5.

Table 2. Results calculated by the use of characteristics obtained from identification

RESULTS AFTER IDENTIFICATION				
PLACE	MOMENT	FORCE	DISPLACEMENT	ROTATION
1	0	0.207301	0.027636	0.002766
2	17.388103	1.866886	0.248885	0.00535
3	−7.179103	0.31593	0.042118	−0.002209
4	−5.246548	0.090426	0.012055	−0.001614
5	4.270807	0.833549	0.111125	0.001314
6	−15.568414	0.788142	0.105072	−0.00479
7	30.700499	3.617273	0.482239	0.009446
8	−16.712816	0.777909	0.103707	−0.005142
9	1.123717	1.02434	0.13656	0.000346
10	5.606944	1.063244	0.141747	0.001725
11	0	0.066846	0.008912	−0.00166

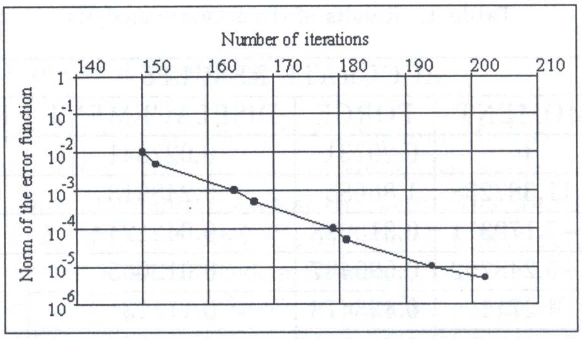

Fig. 5. Accuracy of the results in terms of number of iterations

The vertical axis shows the required maximum norm of the error function in logarithmic scale, while the horizontal axis gives the necessary number of iterations, i.e. the required computer time. According to Fig. 5, the number of iterations is approximately logarithmically proportional to the required accuracy, therefore a slight increase of the computer time improves the accuracy significantly. This rapid convergence allows the efficient application of the identification method to other similar problems.

7. Conclusion

The aim of this paper was to present a method and solution technique for the determination of unknown material characteristics of nonlinear beams on nonlinear foundation. In the analysis the beam and the foundation were discretized and special bilinear force–deformation relationships were applied. The solution based on a mixed variational principle was stated in the form of constrained, nonsmooth, nonlinear mathematical programming problem [7]. For the determination of the best values of the material characteristics a special identification technique formulated as a nonlinear minimization problem was used [8, 9].

The applicability of the bilinear material model for the analysis of nonlinear beams on nonlinear foundation has already been proved previously [7]. This paper shows that this approximate method can be successfully combined with a special identification techniques and applied to the determination of the unknown material characteristics. The results of the numerical example illustrate the efficiency and accuracy of the proposed solution procedure.

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