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Scattering on a spherical shell. Comparison of 3-D elasticity and Kirchhoff shell theory results

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The report is a continuation of [1]. The closed-form solutions of the scattering problem by the 3-D elasticity and Kirchhoff shell theories are investigated.

1. Introduction

The purpose of this paper is twofold:

- Scattering of a plane wave on a hollow, elastic sphere seems to be the only 3-D scattering problem admitting a closed form, exact solution in the whole range of wave numbers $0 < k < \infty$. Thus, as a such, it serves as an indispensable verification test for boundary and finite element approximations including the h-p BE/FE code presented in [2].
- For practical purposes the shell is usually approximated using the Kirchhoff theory (compare [3]). Obviously, for shells with ratio of thickness over the midsurface radius bigger than a critical value, the results based on the 3-D elasticity and shell theories must differ and it is that critical value we shall try to determine in this paper.

The compatibility conditions on the surface of an elastic scatterer reduce to the equality of normal components of acceleration vector $\ddot{u}_r$ for the body, and acceleration vector $\dot{v}_r$ for the fluid, expressed in terms of pressure p

$$-\omega^2 u_r = \ddot{u}_r = \dot{v}_r = -\frac{1}{\rho_w}\frac{\partial p}{\partial n} \tag{1}$$

and the equality of normal stress components

$$\sigma_{ij}(\mathbf{u})n_j n_i = -p \tag{2}$$

with the tangential stress components in the elastic body set to zero. As usual, ω denotes the frequency and ρ_w is the density of the water.

In the case of rigid scattering we have only one condition expressing vanishing of the velocity normal component on scatterer's boundary

$$0 = -\frac{1}{\rho_w}\frac{\partial p}{\partial n}\,. \tag{3}$$

We remind that in both cases p denotes the total pressure being the sum of incident pressure p^{inc} and scattered pressure p^s with the last one satisfying the Sommerfeld radiation boundary condition at infinity. It is convenient to represent the total pressure p for the elastic scattering in the form

$$p = p^{inc} + p^{s,\infty} + p^{s,r} \tag{4}$$

where p^{inc} is the incident pressure (same for both rigid and elastic scattering problems), $p^{s,\infty}$ is the scattered pressure corresponding to the rigid boundary (the ∞ symbol indicates the infinite impedance) and $p^{s,r}$ is a new variable. The compatibility conditions for displacement u and the new variable $p^{s,r}$ read now as follows

$$-\omega^2 u_r = -\frac{1}{\rho_w}\frac{\partial p^{s,r}}{\partial n} \tag{5}$$

and

$$\sigma_{ij} n_j n_i = -(p^{inc} + p^{s,\infty} + p^{s,r}) \tag{6}$$

with the additional Sommerfeld condition imposed on $p^{s,r}$.

Consequently, the *elastic body scattered pressure* p^s is equal to the *rigid body scattered pressure* $p^{s,\infty}$ and component $p^{s,r}$ which represents the pressure corresponding to *forced vibrations of the elastic body in fluid*, loaded with the sum of incident pressure p^{inc} and the rigid body scattered pressure $p^{s,\infty}$.

We notice finally that rigid-body scattered pressure $p^{s,\infty}$ coincides with the solution of the radiation problem with velocity v_r of the vibration boundary opposite to that corresponding to incident pressure p^{inc}.

These simple observations motivate the following order of presentation.

In Sections 2.1–2.3, we review first the shell theory and briefly summarize free and forced vibrations in vacuum. Radiation and free as well as forced vibrations in fluid problems are reviewed in Sections 2.4, 2.5 and 2.6, respectively, and the solutions of the rigid scattering problem and the final elastic scattering problem in terms of the Legendre's polynomials are discussed in Sections 2.7 and 2.8, respectively.

We use the same order when presenting the 3-D elasticity-based solution. Following essentials of the 3-D theory summarized in Section 3.1, we discuss first free and forced vibrations in vacuum and then turn again to radiation and free as well as forced vibrations in fluid problems. Solutions to the final scattering problems, obtained in terms of series expansion of the Bessel's functions and the Legendre's polynomials are given in Sections 3.7 and 3.8.

Numerical experiments and accompanying discussion in Section 4 conclude the paper.

2. Kirchhoff-Love shell theory solution

Following [3] we review the exact solution of the classical scattering problem for a hollow sphere. Based on the Kirchhoff-Love shell theory, the closed-formed solution is obtained in the form of series expansions in terms of the Legendre's polynomials of first kind of order n.

2.1. A review of the shell theory

Under the customary Kirchhoff-Love assumptions and restricting ourselves to axisymmetric modes only, steady-state equations of motion for a spherical shell in spherical coordinates r, θ, ϕ ($x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$) take on the following form:

$$\begin{aligned}&\left(1+\beta^2\right)\left[\frac{\partial^2 u_\theta}{\partial\theta^2} + \cot\theta\frac{\partial u_\theta}{\partial\theta} - \left(\nu + \cot^2\theta\right)u_\theta\right] \\ &\quad -\beta^2\frac{\partial^3 u_r}{\partial\theta^3} - \beta^2\cot\theta\frac{\partial^2 u_r}{\partial\theta^2} + \left[(1+\nu) + \beta^2\left(\nu+\cot^2\theta\right)\right]\frac{\partial u_r}{\partial\theta} - \frac{a^2\ddot{u}_\theta}{c_p^2} = 0,\end{aligned} \tag{7}$$

$$\beta^2\frac{\partial^3 u_\theta}{\partial\theta^3}+2\beta^2\cot\theta\frac{\partial^2 u_\theta}{\partial\theta^2}-\left[(1+\nu)\left(1+\beta^2\right)+\beta^2\cot^2\theta\right]\frac{\partial u_\theta}{\partial\theta}$$
$$+\cot\theta\left[\left(2-\nu+\cot^2\theta\right)\beta^2-(1+\nu)\right]u_\theta-\beta^2\frac{\partial^4 u_r}{\partial\theta^4}-2\beta^2\cot\theta\frac{\partial^3 u_r}{\partial\theta^3}$$
$$+\beta^2\left(1+\nu+\cot^2\theta\right)\frac{\partial^2 u_r}{\partial\theta^2}-\beta^2\cot\theta\left(2-\nu+\cot^2\theta\right)\frac{\partial u_r}{\partial\theta}-2(1+\nu)u_r-\frac{a^2\ddot{u}_r}{c_p^2}$$
$$=-p_a\frac{(1-\nu^2)a^2}{Eh}\,, \tag{8}$$

or, in a more compact form,

$$L_{\theta\theta}u_\theta+L_{\theta r}u_r+\Omega^2u_\theta=0\,, \tag{9}$$

$$L_{r\theta}u_\theta+L_{rr}u_r+\Omega^2u_r=-p_a\frac{(1-\nu^2)a^2}{Eh}\,, \tag{10}$$

where

$$L_{\theta\theta}=\left(1+\beta^2\right)\left\{\left(1-\eta^2\right)^{\frac{1}{2}}\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\left(1-\eta^2\right)^{\frac{1}{2}}+(1-\nu)\right\}, \tag{11}$$

$$L_{\theta r}=\left(1-\eta^2\right)^{\frac{1}{2}}\left\{\left[\beta^2(1-\nu)-(1+\nu)\right]\frac{\mathrm{d}}{\mathrm{d}\eta}+\beta^2\frac{\mathrm{d}}{\mathrm{d}\eta}\nabla_\eta^2\right\}, \tag{12}$$

$$L_{r\theta}=-\left\{\left[\beta^2(1-\nu)-(1+\nu)\right]\frac{\mathrm{d}}{\mathrm{d}\eta}\left(1-\eta^2\right)^{\frac{1}{2}}+\beta^2\nabla_\eta^2\frac{\mathrm{d}}{\mathrm{d}\eta}\left(1-\eta^2\right)^{\frac{1}{2}}\right\}, \tag{13}$$

$$L_{rr}=-\beta^2\nabla_\eta^4-\beta^2(1-\nu)\nabla_\eta^2-2(1+\nu) \tag{14}$$

and

$$\nabla_\eta^2=\frac{\mathrm{d}}{\mathrm{d}\eta}\left(1-\eta^2\right)\frac{\mathrm{d}}{\mathrm{d}\eta}\,. \tag{15}$$

The following notation has been used

a – the radius of the middle surface of the shell,
E,ν – the Young modulus and Poisson ratio,
h – the shell thickness,
$\eta=\cos\theta$,
Ω – the dimensionless frequency of the shell, $\Omega=\frac{wa}{c_p}=(\frac{c_w}{c_p})ka$,
c_w – the wave velocity in water,
c_p – the low frequency phase velocity of compressional waves in an elastic plate,
ω – the frequency,
k – the wave number,
$\beta=\sqrt{\frac{h^2}{12a^2}}$,
p_a – normal force per unit area applied to a shell.

2.2. Free vibrations in vacuum

For free vibrations in vacuum, the displacement field admits spectral representation in terms of the Legendre's polynomials

$$u_r(\eta)=\sum_{n=0}^{\infty}u_{rn}P_n(\eta)\,, \tag{16}$$

$$u_\theta(\eta)=\sum_{n=1}^{\infty}u_{\theta n}\left(1-\eta^2\right)^{\frac{1}{2}}\frac{dP_n}{\mathrm{d}\eta}\,, \tag{17}$$

where $u_{\theta n}$ and u_{rn} are displacements in the θ and r directions for the n-th mode, respectively, and $P_n(\eta)$ is the Legendre's polynomial of order n. Plugging (16)–(17) into (7)–(8), with the forcing function p_a set to zero we obtain the following characteristic equations:

– for $n = 0$,

$$\left[\Omega^2 - 2(1+\nu)\right] u_{rn} = 0\,, \tag{18}$$

– for $n > 0$,

$$\left[\Omega^2 - \left(1+\beta^2\right)(\nu+\lambda_n-1)\right] u_{\theta n} - \left[\beta^2(\nu+\lambda_n-1)+(1+\nu)\right] u_{rn} = 0\,, \tag{19}$$

$$-\lambda_n\left[\beta^2(\nu+\lambda_n-1)+(1+\nu)\right] u_{\theta n} + \left[\Omega^2 - 2(1+\nu) - \beta^2\lambda_n(\nu+\lambda_n-1)\right] u_{rn} = 0\,. \tag{20}$$

For non-trivial solutions of (18) and system (19)–(20), we end up with the following characteristic equation for natural frequency Ω:

– for $n = 0$,

$$\Omega^2 - 2(1+\nu) = 0\,, \tag{21}$$

– for $n > 0$,

$$\Omega^4 - \left[1 + 3\nu + \lambda_n - \beta^2\left(1 - \nu - \lambda_n^2 - \nu\lambda_n\right)\right]\Omega^2 + (\lambda_n - 2)(1-\nu^2) + \beta^2\left[\lambda_n^3 - 4\lambda_n^2 + \lambda_n\left(5-\nu^2\right) - 2\left(1-\nu^2\right)\right] = 0\,, \tag{22}$$

where $\lambda_n = n(n+1)$.

2.3. Forced vibrations in vacuum

Applying the method of spectral decomposition to the force function, we express p_a as $\sum_{n=0}^{\infty} f_n P_n(\eta)$ and substitute it into (7) and (8). We obtain the following equations:

– for $n = 0$,

$$\left[\Omega^2 - 2(1+\nu)\right] u_{r0} = -\frac{a^2(1-\nu^2)}{Eh} f_0\,, \tag{23}$$

– for $n > 0$,

$$\left[\Omega^2 - \left(1+\beta^2\right)(\nu+\lambda_n-1)\right] u_{\theta n} - \left[\beta^2(\nu+\lambda_n-1)+(1+\nu)\right] u_{rn} = 0\,, \tag{24}$$

$$-\lambda_n\left[\beta^2(\nu+\lambda_n-1)+(1+\nu)\right] u_{\theta n} + \left[\Omega^2 - 2(1+\nu) - \beta^2\lambda_n(\nu+\lambda_n-1)\right] u_{rn} = -\frac{a^2(1-\nu^2)}{Eh} f_n\,, \tag{25}$$

where f_n is component force for the n-th mode. Solving (23) and system (24)–(25), we obtain

– for $n = 0$,

$$u_{r0} = -\frac{a^2(1-\nu^2)}{Eh[\Omega^2 - 2(1+\nu)]} f_0\,, \tag{26}$$

– for $n > 0$,

$$u_{rn} = \frac{\begin{vmatrix} \Omega^2 - (1+\beta^2)(\nu+\lambda_n-1) & 0 \\ -\lambda_n[\beta^2(\nu+\lambda_n-1)+(1+\nu)] & -\frac{a^2(1-\nu^2)}{Eh} f_n \end{vmatrix}}{\begin{vmatrix} \Omega^2 - (1+\beta^2)(\nu+\lambda_n-1) & -[\beta^2(\nu+\lambda_n-1)+(1+\nu)] \\ -\lambda_n[\beta^2(\nu+\lambda_n-1)+(1+\nu)] & \Omega^2 - 2(1+\nu) - \beta^2\lambda_n(\nu+\lambda_n-1) \end{vmatrix}}\,, \tag{27}$$

$$u_{\theta n} = \frac{\begin{vmatrix} 0 & -[\beta^2(\nu+\lambda_n-1)+(1+\nu)] \\ -\frac{a^2(1-\nu^2)}{Eh} f_n & \Omega^2 - 2(1+\nu) - \beta^2\lambda_n(\nu+\lambda_n-1) \end{vmatrix}}{\begin{vmatrix} \Omega^2-(1+\beta^2)(\nu+\lambda_n-1) & -[\beta^2(\nu+\lambda_n-1)+(1+\nu)] \\ -\lambda_n[\beta^2(\nu+\lambda_n-1)+(1+\nu)] & \Omega^2-2(1+\nu)-\beta^2\lambda_n(\nu+\lambda_n-1) \end{vmatrix}} \tag{28}$$

It is useful to define a modal mechanical impedance in vacuum as follows,

$$Z_n = \frac{f_n}{-i\omega u_{rn}}\,; \tag{29}$$

– for $n = 0$,

$$Z_0 = \frac{Eh[\Omega^2 - 2(1+\nu)]}{i\omega a^2(1-\nu^2)}\,, \tag{30}$$

– for $n > 0$,

$$Z_n = \frac{\begin{vmatrix} \Omega^2-(1+\beta^2)(\nu+\lambda_n-1) & -[\beta^2(\nu+\lambda_n-1)+(1+\nu)] \\ -\lambda_n[\beta^2(\nu+\lambda_n-1)+(1+\nu)] & \Omega^2-2(1+\nu)-\beta^2\lambda_n(\nu+\lambda_n-1) \end{vmatrix}}{-i\omega\begin{vmatrix} \Omega^2-(1+\beta^2)(\nu+\lambda_n-1) & 0 \\ -\lambda_n[\beta^2(\nu+\lambda_n-1)+(1+\nu)] & -\frac{a^2(1-\nu^2)}{Eh} \end{vmatrix}}\,. \tag{31}$$

2.4. Radiation problem

Solution of the radiation problem for a sphere vibrating with normal acceleration,

$$\ddot{u}_r(\theta) = \sum_{n=0}^{\infty} \ddot{u}_{rn} P_n(\cos\theta)\,, \tag{32}$$

takes on the form [3]

$$p(r,\theta) = -\frac{\rho_w}{k}\sum_{n=0}^{\infty} \ddot{u}_{rn} P_n(\cos\theta)\frac{h_n(kr)}{h_n'(kr_o)}\,. \tag{33}$$

By setting $r = r_o$ in Eq. (33) and substituting $-i\omega\dot{u}_{rn}$ for $\ddot{u}_{rn}$, the surface pressure can be expressed as

$$p(r_o,\theta) = \sum_{n=0}^{\infty} z_n \dot{u}_{rn} P_n(\cos\theta) \tag{34}$$

where z_n is the *modal specific acoustic impedance*,

$$z_n = r_n - i\omega m_n = i\rho_w c_w \frac{h_n(kr_o)}{h_n'(kr_o)}\,, \tag{35}$$

with *modal resistance* r_n and *modal accession to inertia* m_n given by the formulas

$$r_n = \rho_w c_w\, \Re\left[\frac{ih_n(kr_o)}{h_n'(kr_o)}\right], \tag{36}$$

$$m_n = \frac{-\rho_w c_w}{\omega}\, \Im\left[\frac{ih_n(kr_o)}{h_n'(kr_o)}\right]. \tag{37}$$

As usual, ρ_w is the density of water, h_n and h_n' are the Hankel's functions and their derivatives, respectively.

2.5. Free vibrations in fluid

By treating radiating pressure (33) as a forcing term f_n in (23) and (25) we arrive at the following generalized eigenvalue problem (see [6] for a detailed mathematical analysis):

– for $n = 0$

$$-\Omega^2\left(1+\frac{m_n}{\rho_s h}\right) - i\Omega\frac{a}{h}\frac{r_n}{\rho_s c_p} + 2(1+\nu) = 0\,, \tag{38}$$

– for $n > 0$

$$\Omega^4\left(1+\frac{m_n}{\rho_s h}\right) - \Omega^2\left[1+3\nu+\left(1+\frac{m_n}{\rho_s h}\right)\left(n^2+n\right) - \frac{m_n}{\rho_s h}(1-\nu)\right] + \left(1-\nu^2\right)\left(n^2+n-2\right) = 0\,. \tag{39}$$

As usual, ρ_s is the density of solid.

Note that the natural frequencies in fluid may be complex, and the Bessel functions with complex argument need to be evaluated. A nonlinear root search, such as the Newton iterations, may be implemented in order to solve the corresponding system of two real-valued nonlinear equations.

2.6. Forced vibrations in fluid

Forced vibrations of a spherical shell can be considered as a particular case of forced vibrations in vacuum with the forcing term equal to the sum of the actual modal component force f_n and radiating negative pressure term expressed in terms of modal specific impedance z_n and the modal velocity component. Thus, the final modal equation takes on the form

$$Z_n \dot{u}_{rn} = f_n - z_n \dot{u}_{rn} \tag{40}$$

where Z_n is modal mechanical impedance given in (30) and (31). Solving for $\dot{u}_{rn}$ in (40), we get

$$\dot{u}_{rn} = \frac{f_n}{Z_n + z_n}\,. \tag{41}$$

2.7. Rigid scattering problem

Consider a spherical scatterer with midsurface radius a, and a distant source generating a train of sound waves which impinge on the outer boundary of the sphere. Let the rigid scattered pressure be denoted by $p^{s,\infty}$ and the total pressure by p

$$p = p^{inc} + p^{s,\infty}\,. \tag{42}$$

Since the boundary is rigid, the resultant particle acceleration must have zero component in the radial direction on the scatterer,

$$\ddot{u}_r^{s,\infty} + \ddot{u}_r^{inc} = 0 \qquad \text{at } r = r_o\,, \tag{43}$$

where $\ddot{u}_r^{inc}$ and $\ddot{u}_r^{s,\infty}$ are the normal acceleration components corresponding to incident and scattered waves, and r_o denotes the outer radius, respectively. Consequently,

$$\ddot{u}_r^{s,\infty} = -\ddot{u}_r^{inc} = \frac{1}{\rho_w}\frac{\partial p^{inc}(r,\theta)}{\partial r} \qquad \text{at } r = r_o\,. \tag{44}$$

Thus, the rigid scattering problem reduces simply to the radiation problem with the vibrating boundary acceleration $\ddot{u}_{rn}$ specified by (44). We restrict now ourselves to an incident *plane wave* only, representing it in terms of spherical harmonics

$$p^{inc}(r,\theta) = P_{inc}e^{ikr\cos\theta} = P_{inc}\sum_{n=0}^{\infty}(2n+1)i^n P_n(\cos\theta)j_n(kr) \tag{45}$$

where P_{inc} is a prescribed coefficient. Substituting (45) into (44) and comparing with Eqs. (32)–(33), we obtain finally

$$p^{s,\infty}(r,\theta) = -P_{inc}\sum_{n=0}^{\infty}(2n+1)i^n P_n(\cos\theta)\frac{j_n'(kr_o)}{h_n'(kr_o)}h_n(kr)\,. \tag{46}$$

2.8. Elastic scattering problem

We begin again by expanding the incident plane wave in terms of the spherical harmonics

$$p^{inc}(r,\theta) = P_{inc}e^{ikr\cos\theta} = P_{inc}\sum_{n=0}^{\infty}(2n+1)i^n P_n(\cos\theta)j_n(kr) \tag{47}$$

where P_{inc} is a prescribed coefficient. As indicated in Introduction, the total elastic-body scattered pressure $p^s(r,\theta)$ is equal to the sum of rigid-body scattered pressure $p^{s,\infty}(r,\theta)$,

$$p^{s,\infty}(r,\theta) = -P_{inc}\sum_{n=0}^{\infty}(2n+1)i^n P_n(\cos\theta)\frac{j_n'(kr_o)}{h_n'(kr_o)}h_n(kr)\,, \tag{48}$$

and pressure $p^{s,r}(r,\theta)$ radiated by the elastic shell in fluid loaded with forcing term, equal to $p^{s,\infty}(r,\theta)+p^{inc}(r,\theta)$,

$$\begin{aligned} p^{s,r}(r,\theta) &= i\rho_w c_w\sum_{n=0}^{\infty}\dot{u}_{rn}P_n(\cos\theta)\frac{h_n(kr)}{h_n'(kr_o)} \\ &= i^n\rho_w c_w\sum_{n=0}^{\infty}P_n(\cos\theta)\frac{(2n+1)P_{inc}h_n(kr)}{(kr_o)^2(Z_n+z_n)(h_n'(kr_o))^2}\,, \end{aligned} \tag{49}$$

where $\dot{u}_{rn}$ denotes velocity component for the n-th mode,

$$\dot{u}_{rn} = -\frac{p_n}{Z_n+z_n} = -\frac{i^{n+1}(2n+1)P_{inc}}{(kr_o)^2h_n'(kr_o)(Z_n+z_n)}\,, \tag{50}$$

with the forcing term equal to the sum of p^{inc} and $p^{s,\infty}$,

$$p_n = \frac{i^{n+1}(2n+1)P_{inc}}{(kr_o)^2h_n'(kr_o)}\,, \tag{51}$$

and modal mechanical impedance Z_n defined by (29).

3. 3-D ELASTICITY SOLUTION

We follow a similar order of presentation as for the shell model.

3.1. A review of the 3-D theory

By means of the Helmholtz potentials, solution of the Navier equations for a vibrating elastic hollow sphere problem can be reduced to the solution of equivalent, three decoupled wave equations for three Helmholtz potentials Φ, Ψ, and χ,

$$c_1^2\nabla^2\Phi = \ddot{\Phi}, \qquad c_2^2\nabla^2\Psi = \ddot{\Psi}, \qquad c_2^2\nabla^2\chi = \ddot{\chi} \tag{52}$$

Referring to [1] for details, we list here only the final formulas

$$\sigma_{rr} = \sum_{n=0}^{\infty} \frac{2\mu}{r^2}\left[A_n T_{11}^{(1)}(\alpha r) + B_n l T_{13}^{(1)}(\beta r) + C_n T_{11}^{(2)}(\alpha r) + D_n l T_{13}^{(2)}(\beta r)\right] P_n(\cos\theta)\exp(-iwt). \tag{53}$$

Following [4], we shall use a simplified notation of the form

$$\sigma_{rr} = \frac{2\mu}{r^2}\left[T_{11}^{(i)}(\alpha r) + l T_{13}^{(i)}(\beta r)\right] P_n(\cos\theta)\exp(-iwt) \tag{54}$$

with $T_{11}^{(i)}(\alpha r)$ replacing $A_n T_{11}^{(1)}(\alpha r) + C_n T_{11}^{(2)}(\alpha r)$, $T_{13}^{(i)}(\alpha r)$ replacing $B_n l T_{13}^{(1)}(\beta r) + D_n l T_{13}^{(2)}(\beta r)$ and the summation convention being used. Continuing formulas for the stresses, we have

$$\begin{aligned}\sigma_{\theta\theta} = \frac{2\mu}{r^2}\Big\{ & T_{21}^{(i)}(\alpha r) P_n(\cos\theta) + \hat{T}_{21}^{(i)}(\alpha r)\frac{1}{\sin^2\theta}[-n\cos^2\theta P_n(\cos\theta) + n\cos\theta P_{n-1}(\cos\theta)] \\ & + l T_{23}^{(i)}(\beta r) P_n(\cos\theta) + l\hat{T}_{23}^{(i)}(\beta r)\frac{1}{\sin^2\theta}\left[\left(-n\cos^2\theta\right) P_n(\cos\theta) + n\cos\theta P_{n-1}(\cos\theta)\right]\Big\} \\ & \times\exp(-iwt),\end{aligned} \tag{55}$$

$$\begin{aligned}\sigma_{\phi\phi} = \frac{2\mu}{r^2}\Big\{ & T_{31}^{(i)}(\alpha r) P_n(\cos\theta) + \hat{T}_{31}^{(i)}(\alpha r)\frac{1}{\sin^2\theta}[n\cos^2\theta P_n(\cos\theta) - n\cos\theta P_{n-1}(\cos\theta)] \\ & + l T_{33}^{(i)}(\beta r) P_n(\cos\theta) + l\hat{T}_{33}^{(i)}(\beta r)\frac{1}{\sin^2\theta}\left[\left(n\cos^2\theta\right) P_n(\cos\theta) - n\cos\theta P_{n-1}(\cos\theta)\right]\Big\} \\ & \times\exp(-iwt),\end{aligned} \tag{56}$$

$$\begin{aligned}\sigma_{r\theta} = \frac{2\mu}{r^2}\Big\{ & T_{41}^{(i)}(\alpha r)\left[n\cot\theta P_n(\cos\theta) - \frac{n}{\sin\theta}P_{n-1}(\cos\theta)\right] \\ & + l T_{43}^{(i)}(\beta r)\left[n\cot\theta P_n(\cos\theta) - \frac{n}{\sin\theta}P_{n-1}(\cos\theta)\right]\Big\}\exp(-iwt),\end{aligned} \tag{57}$$

$$\sigma_{r\phi} = \sigma_{\theta\phi} = 0, \tag{58}$$

where:

$$T_{11}^{(i)}(\alpha r) = \left(n^2 - n - \frac{1}{2}\beta^2 r^2\right) Z_n^{(i)}(\alpha r) + 2\alpha r Z_{n+1}^{(i)}(\alpha r), \tag{59}$$

$$T_{13}^{(i)}(\beta r) = n(n+1)\left[(n-1)Z_n^{(i)}(\beta r) - \beta r Z_{n+1}^{(i)}(\beta r)\right], \tag{60}$$

$$T_{21}^{(i)}(\alpha r) = \left(-n^2 - \frac{1}{2}\beta^2 r^2 + \alpha^2 r^2\right) Z_n^{(i)}(\alpha r) - \alpha r Z_{n+1}^{(i)}(\alpha r), \tag{61}$$

$$\hat{T}_{21}^{(i)}(\alpha r) = Z_n^{(i)}(\alpha r), \tag{62}$$

$$T_{23}^{(i)}(\beta r) = -(n^2+n)\left[n Z_n^{(i)}(\beta r) - \beta r Z_{n+1}^{(i)}(\beta r)\right], \tag{63}$$

$$\hat{T}_{23}^{(i)}(\beta r) = (n+1)Z_n^{(i)}(\beta r) - \beta r Z_{n+1}^{(i)}(\beta r), \tag{64}$$

$$T_{31}^{(i)}(\beta r) = \left(n - \frac{1}{2}\beta^2 r^2 + \alpha^2 r^2\right) Z_n^{(i)}(\alpha r) - \alpha r Z_{n+1}^{(i)}(\alpha r), \tag{65}$$

$$\hat{T}_{31}^{(i)}(\alpha r) = Z_n^{(i)}(\alpha r), \tag{66}$$

$$T_{33}^{(i)}(\beta r) = n(n+1)Z_n^{(i)}(\beta r), \tag{67}$$

$$\hat{T}_{33}^{(i)}(\beta r) = (n+1)Z_n^{(i)}(\beta r) - \beta r Z_{n+1}^{(i)}(\beta r), \tag{68}$$

$$T_{41}^{(i)}(\alpha r) = (n-1)Z_n^{(i)}(\alpha r) - \alpha r Z_{n+1}^{(i)}(\alpha r), \tag{69}$$

$$T_{43}^{(i)}(\beta r) = \left(n^2 - 1 - \frac{1}{2}\beta^2 r^2\right) Z_n^{(i)}(\beta r) + \beta r Z_{n+1}^{(i)}(\beta r). \tag{70}$$

$$\alpha = \frac{\omega}{c_1}, \qquad \beta = \frac{\omega}{c_2}, \qquad c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \qquad c_2 = \sqrt{\frac{\mu}{\rho}}, \tag{71}$$

$$Z_n^{(1)} \equiv j_n(kr) \equiv (\pi/2kr)^{\frac{1}{2}} J_{n+\frac{1}{2}}(kr), \tag{72}$$

$$Z_n^{(2)} \equiv y_n(kr) \equiv (\pi/2kr)^{\frac{1}{2}} Y_{n+\frac{1}{2}}(kr). \tag{73}$$

Here, $j_n(kr)$ and $y_n(kr)$ are the spherical Bessel's functions.

The corresponding displacement field takes the form

$$u_r = \frac{1}{r}\left[U_1^{(i)}(\alpha r) + lU_3^{(i)}(\beta r)\right] P_n(\cos\theta)\exp(-i\omega t), \tag{74}$$

$$u_\theta = \frac{1}{r}\left\{V_1^{(i)}(\alpha r)\left[n\cot\theta P_n(\cos\theta) - \frac{n}{\sin\theta}P_{n-1}(\cos\theta)\right]\right.$$
$$\left. + lV_3^{(i)}(\beta r)\left[n\cot\theta P_n(\cos\theta) - \frac{n}{\sin\theta}P_{n-1}(\cos\theta)\right]\right\}\exp(-i\omega t), \tag{75}$$

$$u_\phi = 0, \tag{76}$$

where

$$U_1^{(i)}(\alpha r) = nZ_n^{(i)}(\alpha r) - \alpha r Z_{n+1}^{(i)}(\alpha r), \qquad U_3^{(i)}(\beta r) = n(n+1)Z_n^{(i)}(\beta r), \tag{77}$$

$$V_1^{(i)}(\alpha r) = Z_n^{(i)}(\alpha r), \qquad V_2^{(i)}(\beta r) = Z_n^{(i)}(\beta r), \tag{78}$$

$$V_3^{(i)}(\beta r) = (n+1)Z_n^{(i)}(\beta r) - \beta r Z_{n+1}^{(i)}(\beta r). \tag{79}$$

3.2. Free vibrations in vacuum

Imposing the traction-free boundary conditions, we obtain the characteristic modal equations for natural frequencies ω:

$$\triangle_n = \begin{vmatrix} T_{11}^{(1)}(\alpha r_o) & T_{11}^{(2)}(\alpha r_o) \\ T_{11}^{(1)}(\alpha r_i) & T_{11}^{(2)}(\alpha r_i) \end{vmatrix} = 0 \qquad \text{for } n = 0, \tag{80}$$

$$\triangle_n = \begin{vmatrix} T_{11}^{(1)}(\alpha r_o) & T_{13}^{(1)}(\beta r_o) & T_{11}^{(2)}(\alpha r_o) & T_{13}^{(2)}(\beta r_o) \\ T_{11}^{(1)}(\alpha r_i) & T_{13}^{(1)}(\beta r_i) & T_{11}^{(2)}(\alpha r_i) & T_{13}^{(2)}(\beta r_i) \\ T_{41}^{(1)}(\alpha r_o) & T_{43}^{(1)}(\beta r_o) & T_{41}^{(2)}(\alpha r_o) & T_{43}^{(2)}(\beta r_o) \\ T_{41}^{(1)}(\alpha r_i) & T_{43}^{(1)}(\beta r_i) & T_{41}^{(2)}(\alpha r_i) & T_{43}^{(2)}(\beta r_i) \end{vmatrix} = 0 \qquad \text{for } n > 0. \tag{81}$$

Here, r_i and r_o denote inner and outer radiuses, respectively.

3.3. Forced vibrations in vacuum

Expanding the excitation pressure in terms of the Legendre polynomials we reduce again the whole problem, as for the free vibrations, to separate modal systems of equations. The only different equation compared with the free vibrations is as follows:

$$\sigma_{rr}\,|_{r=r_0} = \frac{2\mu}{r_o^2}\left[A_n T_{11}^{(1)}(\alpha r_o) + B_n l T_{13}^{(1)}(\beta r_o) + C_n T_{11}^{(2)}(\alpha r_o) + D_n l T_{13}^{(2)}(\beta r_o)\right] P_n(\cos\theta)e^{-i\omega t}$$
$$= f_n P_n(\cos\theta)e^{-i\omega t}. \tag{82}$$

For n=0, we obtain

$$\begin{aligned} A_0 T_{11}^{(1)}(\alpha r_o) + C_0 T_{11}^{(2)}(\alpha r_o) &= \frac{r_o^2}{2\mu} f_0 \,, \\ A_0 T_{11}^{(1)}(\alpha r_i) + C_0 T_{11}^{(2)}(\alpha r_i) &= 0 \,. \end{aligned} \tag{83}$$

Solving for A_0 and C_0 in (83), we obtain

$$A_0 = \frac{\begin{vmatrix} \frac{r_o^2}{2\mu} f_0 & T_{11}^{(2)}(\alpha r_o) \\ 0 & T_{11}^{(2)}(\alpha r_i) \end{vmatrix}}{\begin{vmatrix} T_{11}^{(1)}(\alpha r_o) & T_{11}^{(2)}(\alpha r_o) \\ T_{11}^{(1)}(\alpha r_i) & T_{11}^{(2)}(\alpha r_i) \end{vmatrix}} \,, \qquad C_0 = \frac{\begin{vmatrix} T_{11}^{(1)}(\alpha r_o) & \frac{r_o^2}{2\mu} f_0 \\ T_{11}^{(1)}(\alpha r_i) & 0 \end{vmatrix}}{\begin{vmatrix} T_{11}^{(1)}(\alpha r_o) & T_{11}^{(2)}(\alpha r_o) \\ T_{11}^{(1)}(\alpha r_i) & T_{11}^{(2)}(\alpha r_i) \end{vmatrix}} \,. \tag{84}$$

For $n > 0$, we have

$$\begin{aligned} A_n T_{11}^{(1)}(\alpha r_o) + B_n l T_{13}^{(1)}(\beta r_o) + C_n T_{11}^{(2)}(\alpha r_o) + D_n l T_{13}^{(2)}(\beta r_o) &= \frac{r_o^2}{2\mu} f_n \,. \\ A_n T_{11}^{(1)}(\alpha r_i) + B_n l T_{13}^{(1)}(\beta r_i) + C_n T_{11}^{(2)}(\alpha r_i) + D_n l T_{13}^{(2)}(\beta r_i) &= 0 \,, \\ A_n T_{41}^{(1)}(\alpha r_o) + B_n l T_{43}^{(1)}(\beta r_o) + C_n T_{41}^{(2)}(\alpha r_o) + D_n l T_{43}^{(2)}(\beta r_o) &= 0 \,, \\ A_n T_{41}^{(1)}(\alpha r_i) + B_n l T_{43}^{(1)}(\beta r_i) + C_n T_{41}^{(2)}(\alpha r_i) + D_n l T_{43}^{(2)}(\beta r_i) &= 0 \,. \end{aligned} \tag{85}$$

Solving for coefficients A_n, B_n, C_n and D_n, we obtain

$$A_n = \frac{\triangle_{n1}}{\triangle_n} \,, \qquad B_n = \frac{\triangle_{n2}}{\triangle_n} \,, \qquad C_n = \frac{\triangle_{n3}}{\triangle_n} \,, \qquad D_n = \frac{\triangle_{n4}}{\triangle_n} \,, \tag{86}$$

where

$$\triangle_n = \begin{vmatrix} T_{11}^{(1)}(\alpha r_o) & T_{13}^{(1)}(\beta r_o) & T_{11}^{(2)}(\alpha r_o) & T_{13}^{(2)}(\beta r_o) \\ T_{11}^{(1)}(\alpha r_i) & T_{13}^{(1)}(\beta r_i) & T_{11}^{(2)}(\alpha r_i) & T_{13}^{(2)}(\beta r_i) \\ T_{41}^{(1)}(\alpha r_o) & T_{43}^{(1)}(\beta r_o) & T_{41}^{(2)}(\alpha r_o) & T_{43}^{(2)}(\beta r_o) \\ T_{41}^{(1)}(\alpha r_i) & T_{43}^{(1)}(\beta r_i) & T_{41}^{(2)}(\alpha r_i) & T_{43}^{(2)}(\beta r_i) \end{vmatrix} \,, \tag{87}$$

$$\triangle_{n1} = \begin{vmatrix} \frac{r_o^2}{2\mu} f_n & T_{13}^{(1)}(\beta r_o) & T_{11}^{(2)}(\alpha r_o) & T_{13}^{(2)}(\beta r_o) \\ 0 & T_{13}^{(1)}(\beta r_i) & T_{11}^{(2)}(\alpha r_i) & T_{13}^{(2)}(\beta r_i) \\ 0 & T_{43}^{(1)}(\beta r_o) & T_{41}^{(2)}(\alpha r_o) & T_{43}^{(2)}(\beta r_o) \\ 0 & T_{43}^{(1)}(\beta r_i) & T_{41}^{(2)}(\alpha r_i) & T_{43}^{(2)}(\beta r_i) \end{vmatrix} \,, \tag{88}$$

$$\triangle_{n2} = \begin{vmatrix} T_{11}^{(1)}(\alpha r_o) & \frac{r_o^2}{2\mu} f_n & T_{11}^{(2)}(\alpha r_o) & T_{13}^{(2)}(\beta r_o) \\ T_{11}^{(1)}(\alpha r_i) & 0 & T_{11}^{(2)}(\alpha r_i) & T_{13}^{(2)}(\beta r_i) \\ T_{41}^{(1)}(\alpha r_o) & 0 & T_{41}^{(2)}(\alpha r_o) & T_{43}^{(2)}(\beta r_o) \\ T_{41}^{(1)}(\alpha r_i) & 0 & T_{41}^{(2)}(\alpha r_i) & T_{43}^{(2)}(\beta r_i) \end{vmatrix} \,, \tag{89}$$

$$\triangle_{n3} = \begin{vmatrix} T_{11}^{(1)}(\alpha r_o) & T_{13}^{(1)}(\beta r_o) & \frac{r_o^2}{2\mu} f_n & T_{13}^{(2)}(\beta r_o) \\ T_{11}^{(1)}(\alpha r_i) & T_{13}^{(1)}(\beta r_i) & 0 & T_{13}^{(2)}(\beta r_i) \\ T_{41}^{(1)}(\alpha r_o) & T_{43}^{(1)}(\beta r_o) & 0 & T_{43}^{(2)}(\beta r_o) \\ T_{41}^{(1)}(\alpha r_i) & T_{43}^{(1)}(\beta r_i) & 0 & T_{43}^{(2)}(\beta r_i) \end{vmatrix} \,, \tag{90}$$

$$\triangle_{n4} = \begin{vmatrix} T_{11}^{(1)}(\alpha r_o) & T_{13}^{(1)}(\beta r_o) & T_{11}^{(2)}(\alpha r_o) & \frac{r_o^2}{2\mu} f_n \\ T_{11}^{(1)}(\alpha r_i) & T_{13}^{(1)}(\beta r_i) & T_{11}^{(2)}(\alpha r_i) & 0 \\ T_{41}^{(1)}(\alpha r_o) & T_{43}^{(1)}(\beta r_o) & T_{41}^{(2)}(\alpha r_o) & 0 \\ T_{41}^{(1)}(\alpha r_i) & T_{43}^{(1)}(\beta r_i) & T_{41}^{(2)}(\alpha r_i) & 0 \end{vmatrix} \,. \tag{91}$$

As in the shell theory, it is useful to define a modal mechanical impedance in vacuum for the 3-D theory as follows:

$$Z_n = \frac{f_n}{-i\omega u_{rn}} \tag{92}$$

with

$$u_{rn} = \frac{1}{r_o}\left[A_n U_1^{(1)}(\alpha r) + C_n U_1^{(2)}(\alpha r) + B_n l U_3^{(1)}(\beta r) + D_n l U_3^{(2)}(\beta r)\right] \tag{93}$$

and the coefficients A_n, B_n, C_n, and D_n given by (84) and (86). Note that Z_n is independent of f_n.

3.4. Radiation problem

The radiation pressure distribution for the outer surface $r = r_o$ vibrating with acceleration $\ddot{u}_r$ can be written as

$$p(r,\theta) = -\frac{\rho_w}{k}\sum_{n=0}^{\infty} \ddot{u}_{rn} P_n(\cos\theta)\frac{h_n(kr)}{h_n'(kr_o)}. \tag{94}$$

We emphasize that the relationship between pressure and normal displacement by the 3-D theory is the same as that for the shell theory. By setting $r = r_o$, we obtain the surface pressure in terms of the modal specific acoustic impedance,

$$p(r_o,\theta) = +\sum_{n=0}^{\infty} z_n \dot{u}_{rn} P_n(\cos\theta), \qquad z_n = i\rho_w c_w \frac{h_n(kr_o)}{h_n'(kr_o)}. \tag{95}$$

3.5. Free vibrations in fluid

We begin by recalling the formula for the normal displacement component

$$u_r = \frac{1}{r}\left[A_n U_1^{(1)}(\alpha r) + C_n U_1^{(2)}(\alpha r) + B_n l U_3^{(1)}(\beta r) + D_n l U_3^{(2)}(\beta r)\right] P_n(cos\theta) \tag{96}$$

where

$$U_1^{(i)}(\alpha r) = nZ_n^{(i)}(\alpha r) - \alpha r Z_{n+1}^{(i)}(\alpha r), \qquad U_3^{(i)}(\beta r) = n(n+1)Z_n^{(i)}(\beta r). \tag{97}$$

Substituting (96) into (95), we get the normal stress

$$\begin{aligned}\sigma_{rr}(r_o,\theta) &= -\sum_{n=0}^{\infty} z_n \dot{u}_{rn} P_n(\cos\theta)\\ &= \sum_{n=0}^{\infty}(i\omega r_n + \omega^2 m_n)\frac{1}{r_o}\left[A_n U_1^{(1)}(\alpha r_o) + C_n U_1^{(2)}(\alpha r_o) + B_n l U_3^{(1)}(\beta r_o) + D_n l U_3^{(2)}(\beta r_o)\right]\\ &\qquad \times P_n(\cos\theta).\end{aligned} \tag{98}$$

Finally, we impose the traction boundary condition

$$\sigma_{rr}\,|_{r=r_0} = \frac{2\mu}{r_o^2}\left[A_n T_{11}^{(1)}(\alpha r_o) + B_n l T_{13}^{(1)}(\beta r_o) + C_n T_{11}^{(2)}(\alpha r_o) + D_n l T_{13}^{(2)}(\beta r_o)\right] P_n(\cos\theta). \tag{99}$$

The resulting systems of modal equations look as follows. For $n = 0$, we get

$$\begin{aligned} A_n T_{11}^{(1)}(\alpha r_o) + C_n T_{11}^{(2)}(\alpha r_o) &= \frac{r_0}{2\mu}(i\omega r_n + \omega^2 m_n)\left[A_n U_1^{(1)}(\alpha r) + C_n U_1^{(2)}(\alpha r)\right],\\ A_n T_{11}^{(1)}(\alpha r_i) + C_n T_{11}^{(2)}(\alpha r_i) &= 0.\end{aligned} \tag{100}$$

For $n > 0$, we obtain

$$\begin{aligned}
&A_n T_{11}^{(1)}(\alpha r_o) + B_n l T_{13}^{(1)}(\beta r_o) + C_n T_{11}^{(2)}(\alpha r_o) + D_n l T_{13}^{(2)}(\beta r_o) \\
&\quad = \frac{r_0}{2\mu}(i\omega r_n + \omega^2 m_n)\left[A_n U_1^{(1)}(\alpha r) + C_n U_1^{(2)}(\alpha r) + B_n l U_3^{(1)}(\beta r) + D_n l U_3^{(2)}(\beta r)\right], \\
&A_n T_{11}^{(1)}(\alpha r_i) + B_n l T_{13}^{(1)}(\beta r_i) + C_n T_{11}^{(2)}(\alpha r_i) + D_n l T_{13}^{(2)}(\beta r_i) = 0, \\
&A_n T_{41}^{(1)}(\alpha r_o) + B_n l T_{43}^{(1)}(\beta r_o) + C_n T_{41}^{(2)}(\alpha r_o) + D_n l T_{43}^{(2)}(\beta r_o) = 0, \\
&A_n T_{41}^{(1)}(\alpha r_i) + B_n l T_{43}^{(1)}(\beta r_i) + C_n T_{41}^{(2)}(\alpha r_i) + D_n l T_{43}^{(2)}(\beta r_i) = 0.
\end{aligned} \tag{101}$$

As in the shell theory, the resulting resonant frequencies are complex and their evaluation involves Bessel's functions of complex argument.

3.6. Forced vibrations in fluid

As in the shell theory model, the problem is equivalent to forced vibrations with the loading term equal to the sum of actual loading and radiation loading,

$$Z_n \dot{u}_{rn} = f_n - z_n \dot{u}_{rn}. \tag{102}$$

Solving for $\dot{u}_{rn}$ we get

$$\dot{u}_{rn} = \frac{f_n}{Z_n + z_n} \tag{103}$$

where the mechanical impedance corresponding to the 3-D theory has been defined by (92).

3.7. Rigid scattering problem

Exactly the same as before.

3.8. Elastic scattering problem

Using the same decomposition as in the shell theory,

$$p = p^{inc}(r,\theta) + p^{s,\infty}(r,\theta) + p^{s,r}(r,\theta), \tag{104}$$

we obtain the final pressure field with

$$p^{inc}(r,\theta) = P_{inc} e^{ikr\cos\theta} = P_{inc} \sum_{n=0}^{\infty} (2n+1) i^n P_n(\cos\theta) j_n(kr), \tag{105}$$

$$p^{s,\infty}(r,\theta) = -P_{inc} \sum_{n=0}^{\infty} (2n+1) i^n P_n(\cos\theta) \frac{j_n'(kr_o)}{h_n'(kr_o)} h_n(kr), \tag{106}$$

$$p^{s,r}(r,\theta) = i\rho_w c_w \sum_{n=0}^{\infty} \dot{u}_{rn} P_n(\cos\theta) \frac{h_n(kr)}{h_n'(kr_o)}, \tag{107}$$

$$\dot{u}_{rn} = -\frac{p_n}{Z_n + z_n} = -\frac{i^{n+1}(2n+1)P_{inc}}{(kr_o)^2 h_n'(kr_o)(Z_n + z_n)}, \tag{108}$$

with modal mechanical impedance Z_n defined by (92).

4. Numerical experiments and conclusions

We conclude our investigations by presenting a series of numerical experiments aimed at comparing modal mechanical impedances and pressure at outer boundary of the vibrating, elastic sphere using both the full 3-D elastic theory and the Kirchhoff-Love shell theory approximation.

4.1. Comparison of modal mechanical impedances

The following data have been chosen: $h/a = 0.01$, $\rho_w = 1000\,\mathrm{kg/m^3}$, $c_w = 1460\,\mathrm{m/s}$, $\rho_s = 7800\,\mathrm{kg/m^3}$, $\nu = 0.3$, $P_{inc} = 1$, $a = 1\,\mathrm{m}$ and $E = 2.0 \times 10^{11}\ \mathrm{N/m^2}$. Mechanical impedances for $n = 0, 1, 2, 5, 10, 15, 20$ are shown in Figs. 1–8, respectively. The results of calculations are compared in terms of the inverse of the imaginary part of the final complex mechanical impedance. The results obtained with the 3-D and shell theories are similar. The only noticeable difference takes place near the resonant frequencies.

4.2. Comparison of pressure on outer boundary

Nine cases were investigated. The first four cases, shown in Figs. 9–12, with $k = 1$, $\rho_w = 1000\,\mathrm{kg/m^3}$, $c_w = 1460\,\mathrm{m/s}$, $\rho_s = 7800\,\mathrm{kg/m^3}$, $\nu = 0.3$, $P_{inc} = 1$, $a = 1\,\mathrm{m}$ and $E = 2.1 \times 10^{11}\,\mathrm{N/m^2}$, correspond to different ratios of thickness h and midsurface radius a: $h/a = 0.01, 0.05, 0.1$ and 0.5, respectively. The second four cases, shown in Figs. 13–16, with $h/a = 0.01$, correspond to an increasing wave number from 5 to 20. The final example, for wave number $k = 1.156353$ (Fig. 17), corresponds to the first minimum of the LBB constant (compare [7]). The horizontal axis corresponds to angle θ measured in degrees from 0 to 180, with the incident wave propagating from $\theta = 0$ direction.

As it is shown in Fig. 18, for $h/a = 0.01$, the results obtained with the shell and 3-D theories are almost indistinguishable with the total error equal to 1.27%. The corresponding relative error is measured in L^2-norm on the outer boundary and defined as

$$error = \left\{ \frac{\displaystyle\int_0^{\pi} |p_{\mathrm{sh}} - p_{\mathrm{ex}}|^2 \sin\theta \,\mathrm{d}\theta}{\displaystyle\int_0^{\pi} |p_{\mathrm{ex}}|^2 \sin\theta \,\mathrm{d}\theta} \right\}^{\frac{1}{2}} \tag{109}$$

where p_{sh} is the obtained pressure by the shell theory, and p_{ex} is the pressure by the 3-D theory.

The difference reaches the minimum value ($error = 0.36\%$) for $h/a = 0.03$, then grows slightly to 2% for $h/a = 0.5$. Here, the shell theory still gives the almost accurate pressure even at $h/a = 0.5$ which corresponds to the fact that the elastic part of the scattering pressure $p^{s,r}$ becomes negligible. In other words, with high ratio h/a, the shell behaves almost as a rigid body (see Fig. 19). For all practical purposes, the shell theory gives satisfactory results for $h/a \leq 0.01$. The results presented in Figs. 13–16 prove that the validity of the shell theory is not limited to small wave numbers.

Finally, results shown in Fig. 17 confirm our earlier study on the mechanical impedances and show that the shell theory may diverge from the 3-D results when the wave number is close to one of resonant (complex) eigenfrequencies.

As a final illustration we present a comparison of the 3-D elasticity exact solution with numerical solutions obtained using the BE/FE method described in [2]. The results presented in Fig. 19 correspond to a thin ($h/a = 0.01$) and a thick ($h/a = 0.5$) shell and wave number $k = 1$. The two curves close to each other on Fig. 19a represent the exact and numerical solutions to the elastic scattering problem and the third one represents the rigid scattering case. For the thick shell case, shown in Fig. 19b, all three curves practically coincide with each other. The numerical solutions were obtained with rather a coarse with of 32 isoparametric, quadratic elements.

Concluding, we can say that for the ratio $h/a \leq 0.01$ and away from the resonant k yielding local minimum of the LBB constant (see [7]), the shell theory gives satisfactory results when compared with 3-D elasticity.

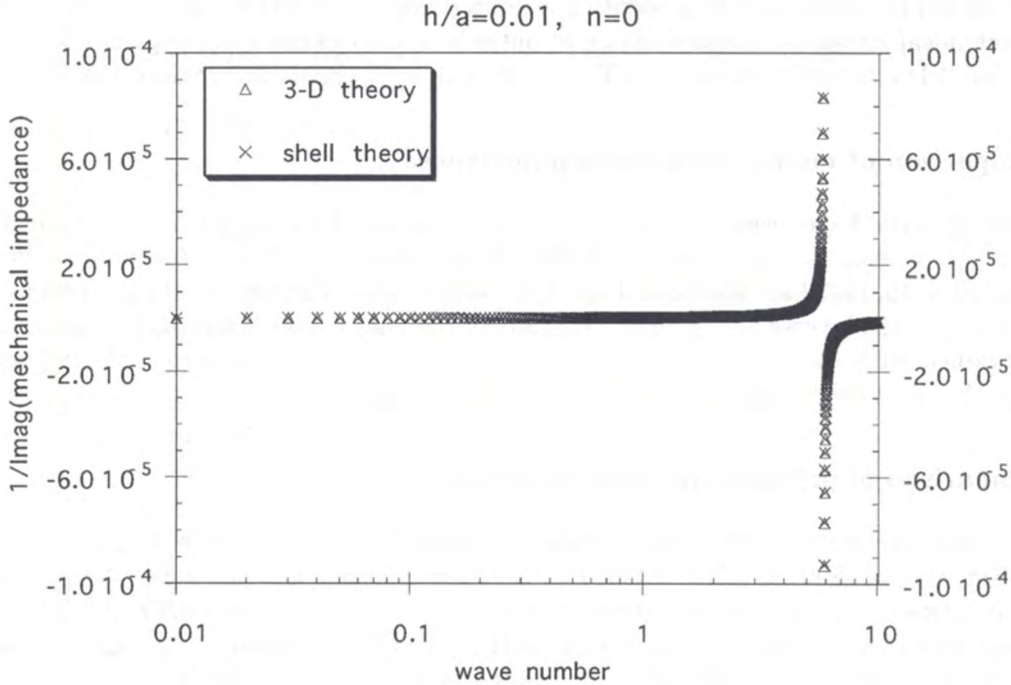
h/a=0.01, n=0
3-D theory
shell theory
1/Imag(mechanical impedance)
1.0 10-4
6.0 10-5
2.0 10-5
-2.0 10-5
-6.0 10-5
-1.0 10-4
0.01
0.1
1
10
wave number

Fig. 1

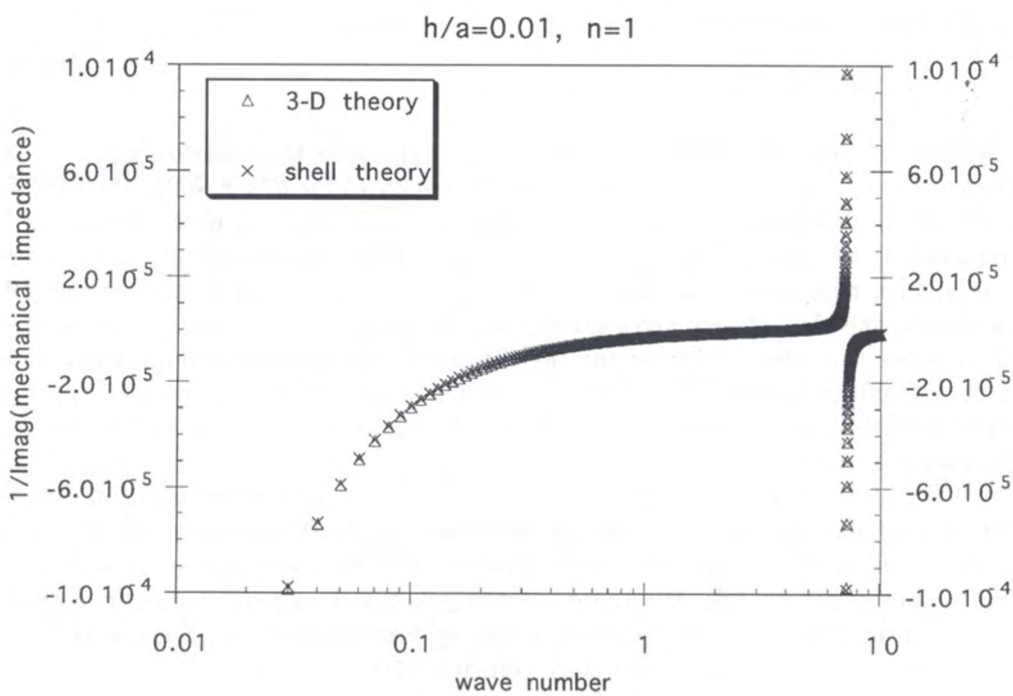
h/a=0.01, n=1
3-D theory
shell theory
1/Imag(mechanical impedance)
1.0 10-4
6.0 10-5
2.0 10-5
-2.0 10-5
-6.0 10-5
-1.0 10-4
0.01
0.1
1
10
wave number

Fig. 2

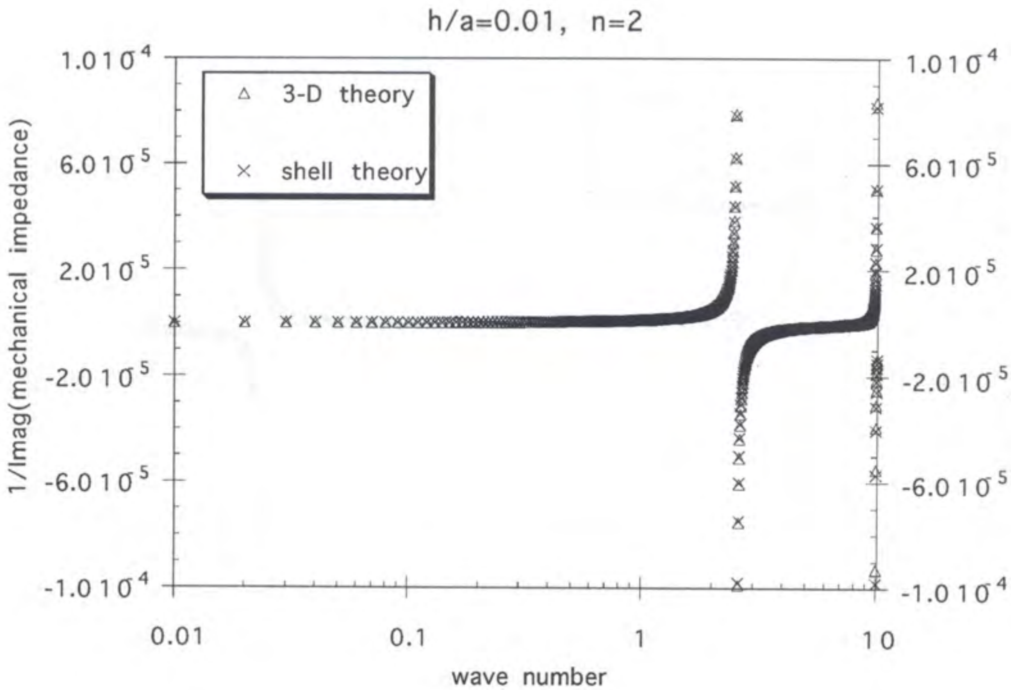
h/a=0.01, n=2
1.0 10^4
6.0 10^5
2.0 10^5
-2.0 10^5
-6.0 10^5
-1.0 10^4
3-D theory
shell theory
1/Imag(mechanical impedance)
0.01
0.1
1
10
wave number

Fig. 3

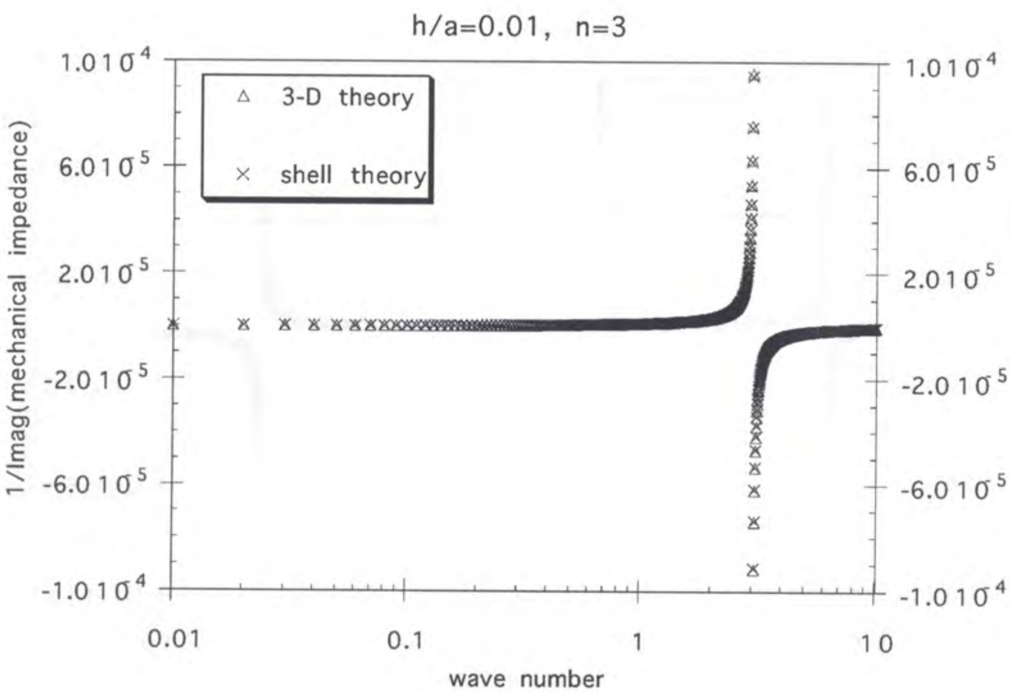
h/a=0.01, n=3
1.0 10^4
6.0 10^5
2.0 10^5
-2.0 10^5
-6.0 10^5
-1.0 10^4
3-D theory
shell theory
1/Imag(mechanical impedance)
0.01
0.1
1
10
wave number

Fig. 4

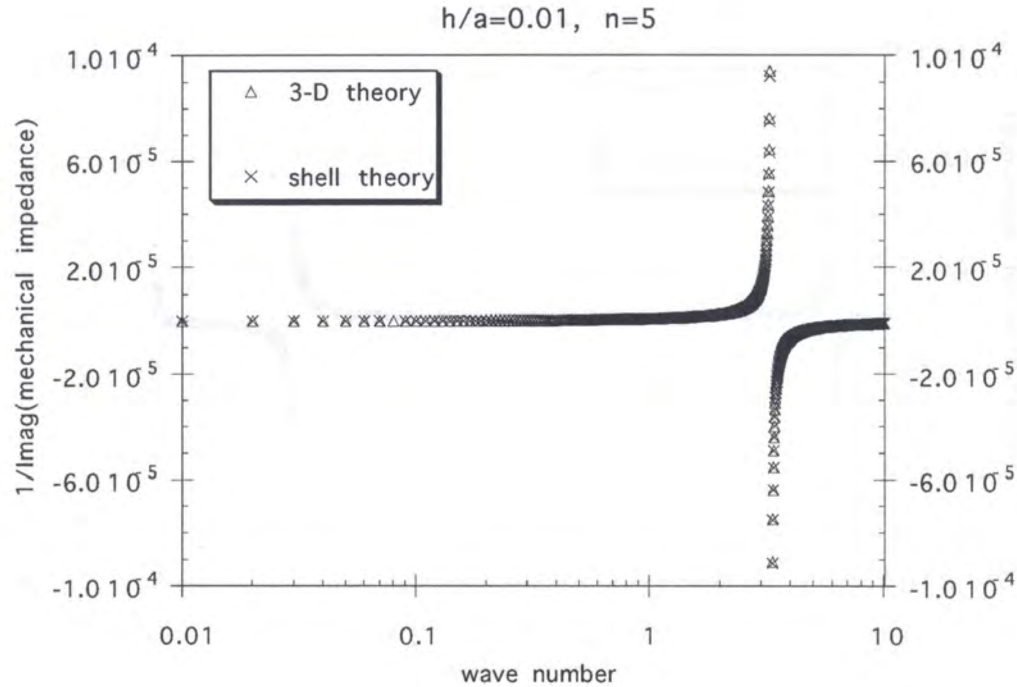
h/a=0.01, n=5
1.0 10⁻⁴
6.0 10⁻⁵
2.0 10⁻⁵
-2.0 10⁻⁵
-6.0 10⁻⁵
-1.0 10⁻⁴
1/Imag(mechanical impedance)
△ 3-D theory
× shell theory
0.01
0.1
1
10
wave number

Fig. 5

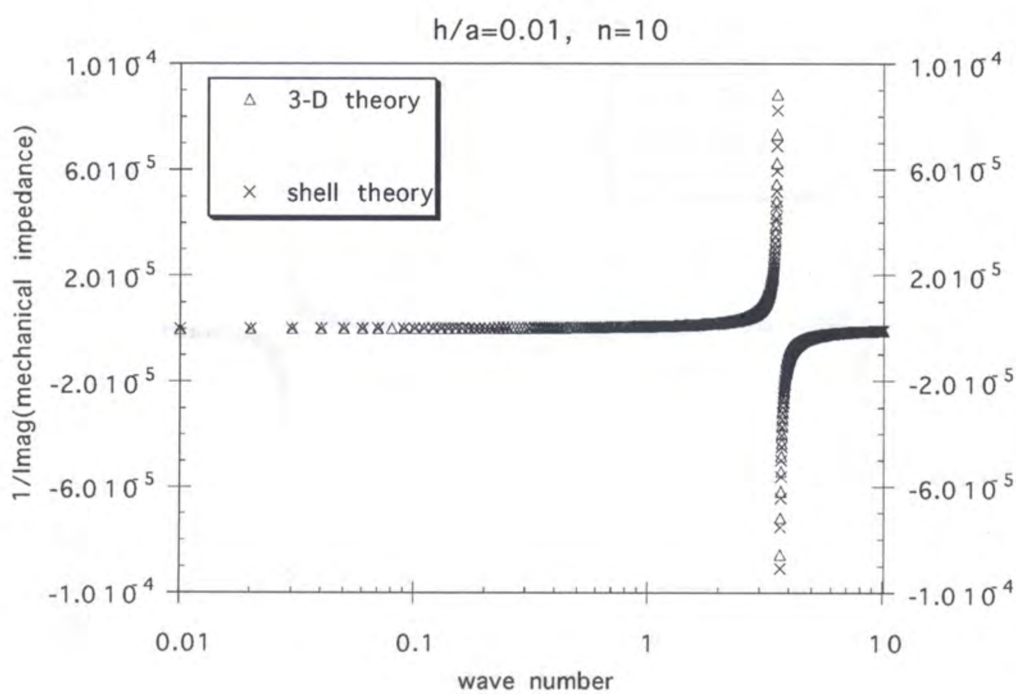
h/a=0.01, n=10
1.0 10⁻⁴
6.0 10⁻⁵
2.0 10⁻⁵
-2.0 10⁻⁵
-6.0 10⁻⁵
-1.0 10⁻⁴
1/Imag(mechanical impedance)
△ 3-D theory
× shell theory
0.01
0.1
1
10
wave number

Fig. 6

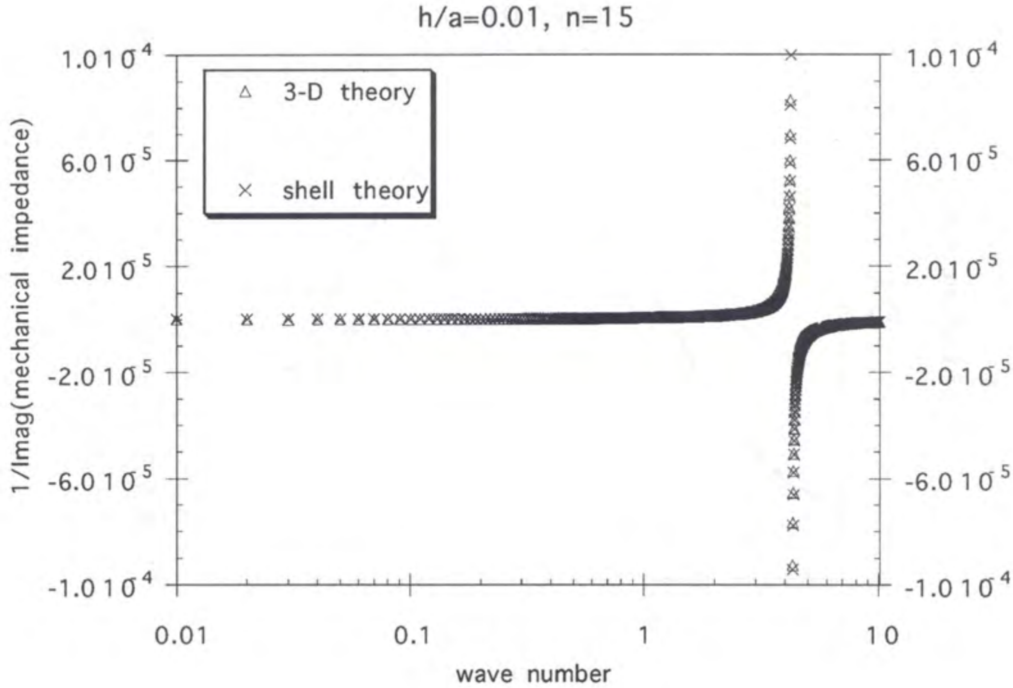
h/a=0.01, n=15
1.0 10^4
6.0 10^5
2.0 10^5
-2.0 10^5
-6.0 10^5
-1.0 10^4
1/Imag(mechanical impedance)
3-D theory
shell theory
0.01
0.1
1
10
wave number

Fig. 7

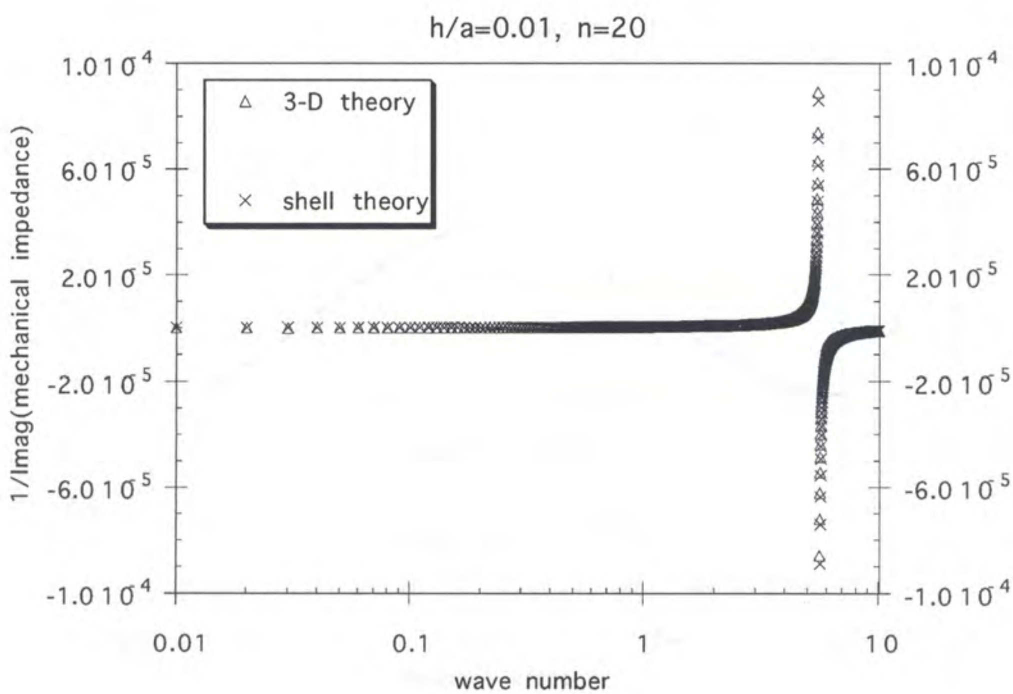
h/a=0.01, n=20
1.0 10^4
6.0 10^5
2.0 10^5
-2.0 10^5
-6.0 10^5
-1.0 10^4
1/Imag(mechanical impedance)
3-D theory
shell theory
0.01
0.1
1
10
wave number

Fig. 8

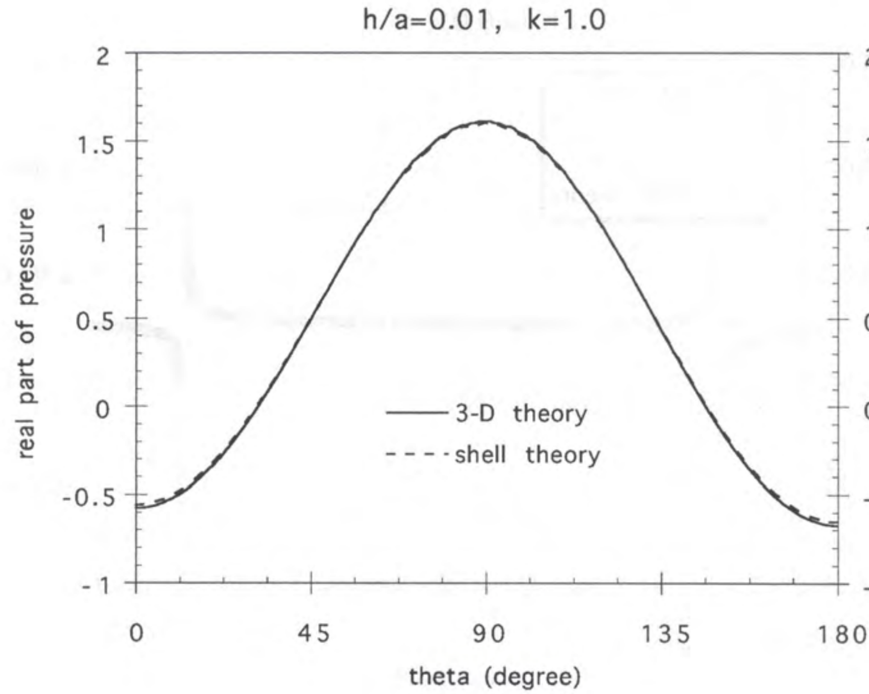
h/a=0.01, k=1.0
real part of pressure
theta (degree)
3-D theory
shell theory
2
1.5
1
0.5
0
-0.5
-1
0
45
90
135
180

Fig. 9

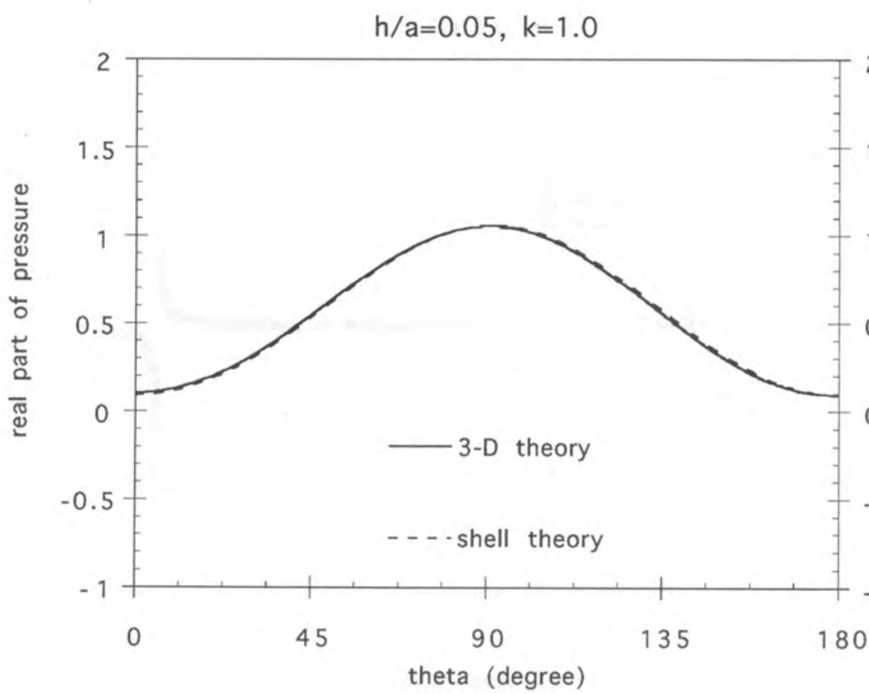
h/a=0.05, k=1.0
real part of pressure
theta (degree)
3-D theory
shell theory
2
1.5
1
0.5
0
-0.5
-1
0
45
90
135
180

Fig. 10

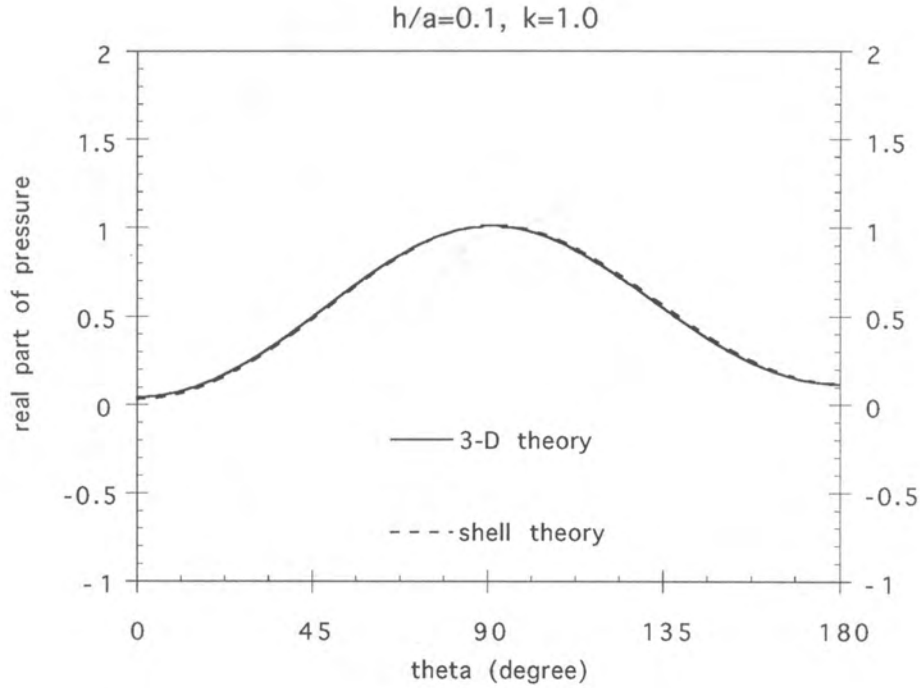
h/a=0.1, k=1.0
real part of pressure
theta (degree)
3-D theory
shell theory
2
1.5
1
0.5
0
-0.5
-1
0
45
90
135
180

Fig. 11

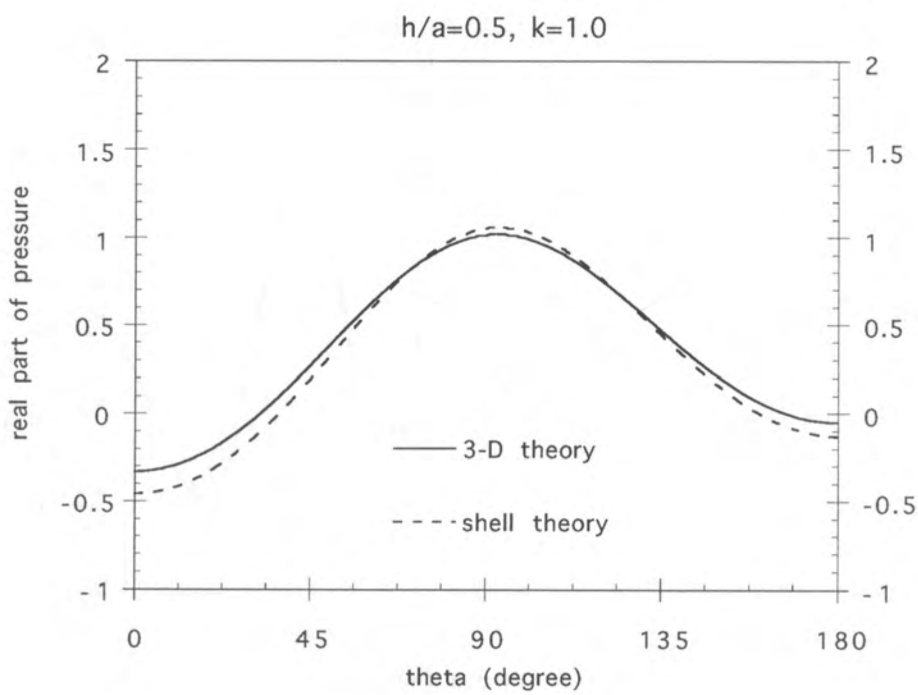
h/a=0.5, k=1.0
real part of pressure
theta (degree)
3-D theory
shell theory
2
1.5
1
0.5
0
-0.5
-1
0
45
90
135
180

Fig. 12

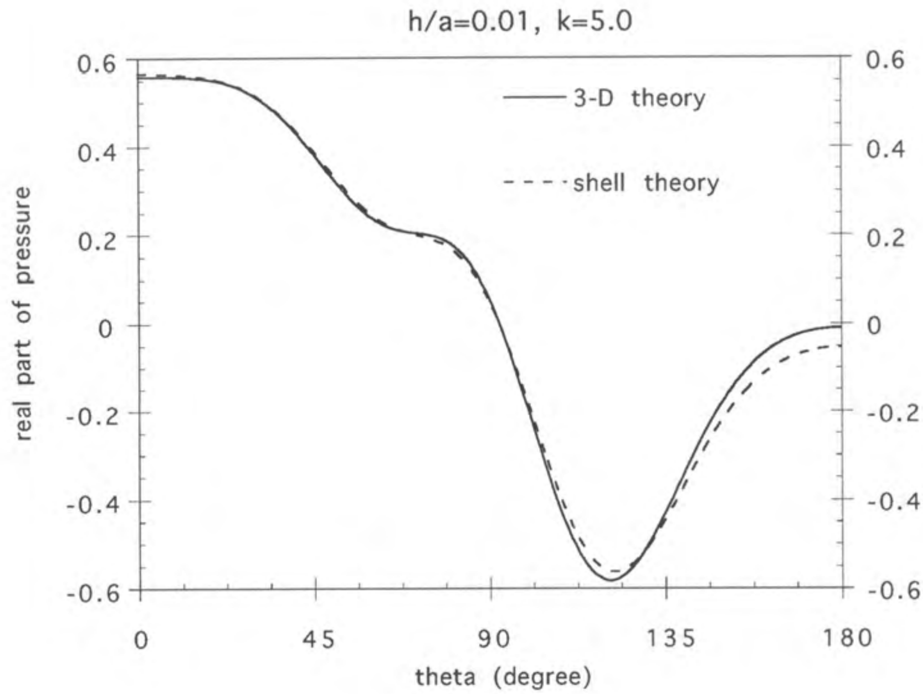
h/a=0.01, k=5.0
3-D theory
shell theory
real part of pressure
theta (degree)
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0
45
90
135
180

Fig. 13

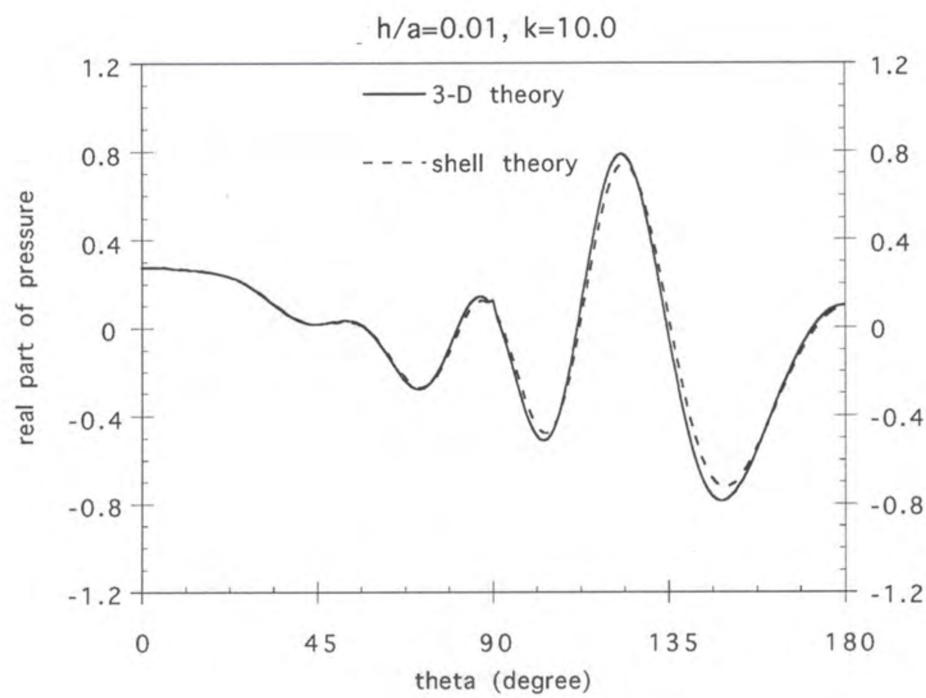
h/a=0.01, k=10.0
3-D theory
shell theory
real part of pressure
theta (degree)
1.2
0.8
0.4
0
-0.4
-0.8
-1.2
0
45
90
135
180

Fig. 14

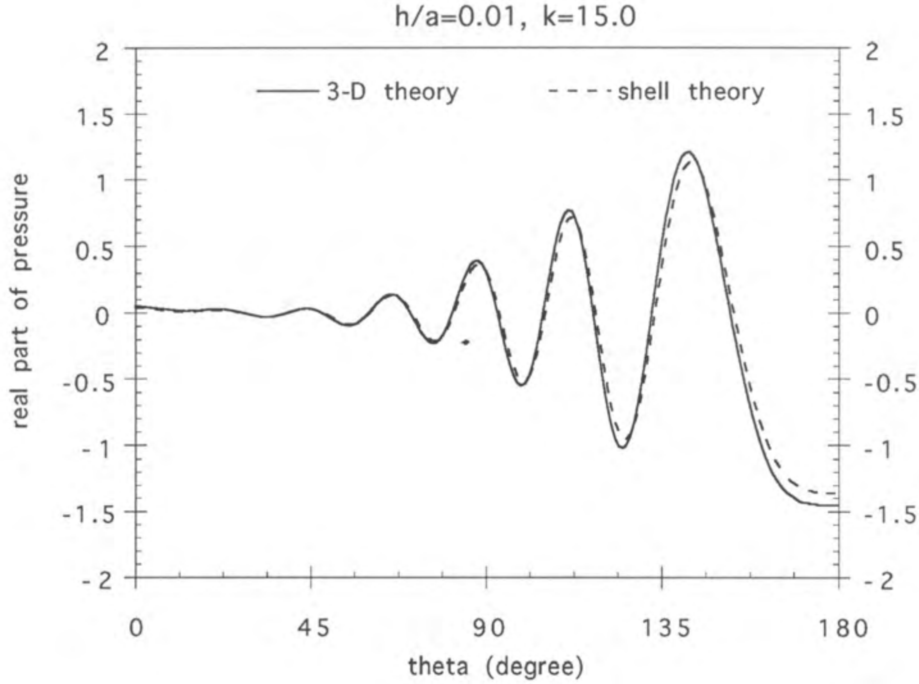
h/a=0.01, k=15.0
3-D theory
shell theory
real part of pressure
theta (degree)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
45
90
135
180

Fig. 15

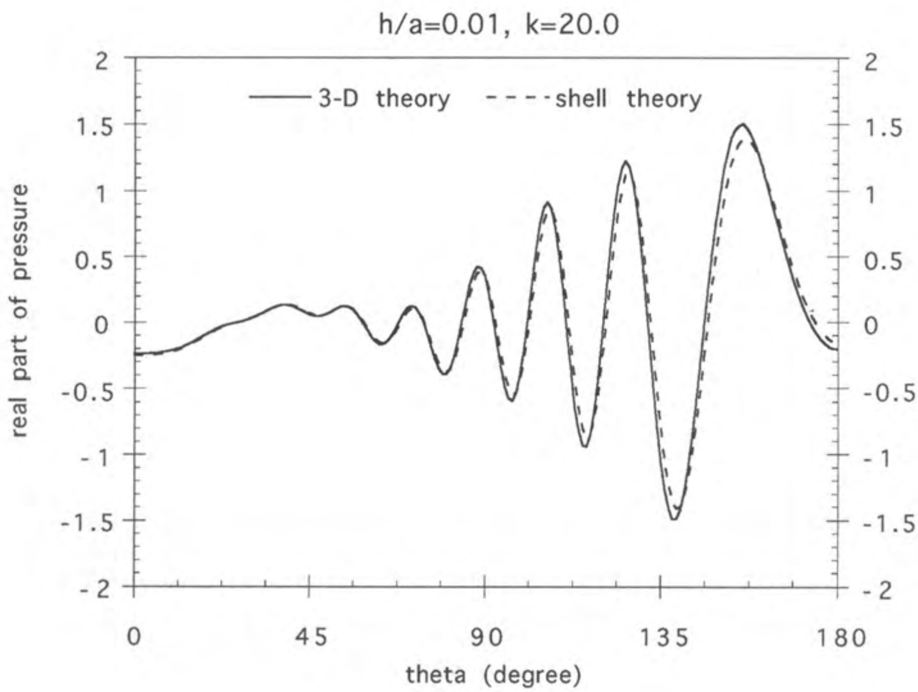
h/a=0.01, k=20.0
3-D theory
shell theory
real part of pressure
theta (degree)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
45
90
135
180

Fig. 16

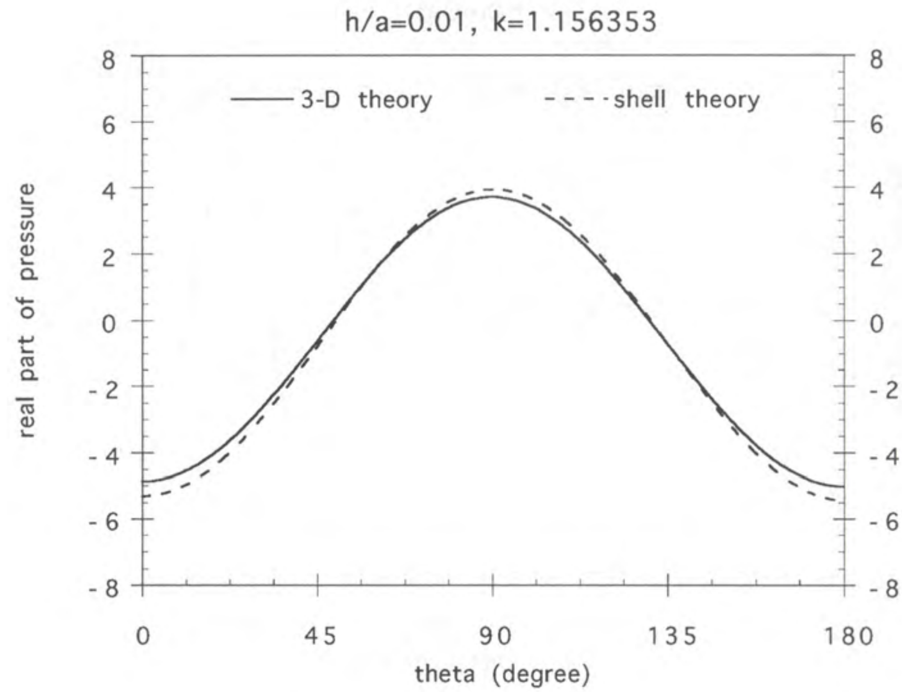

Fig. 17

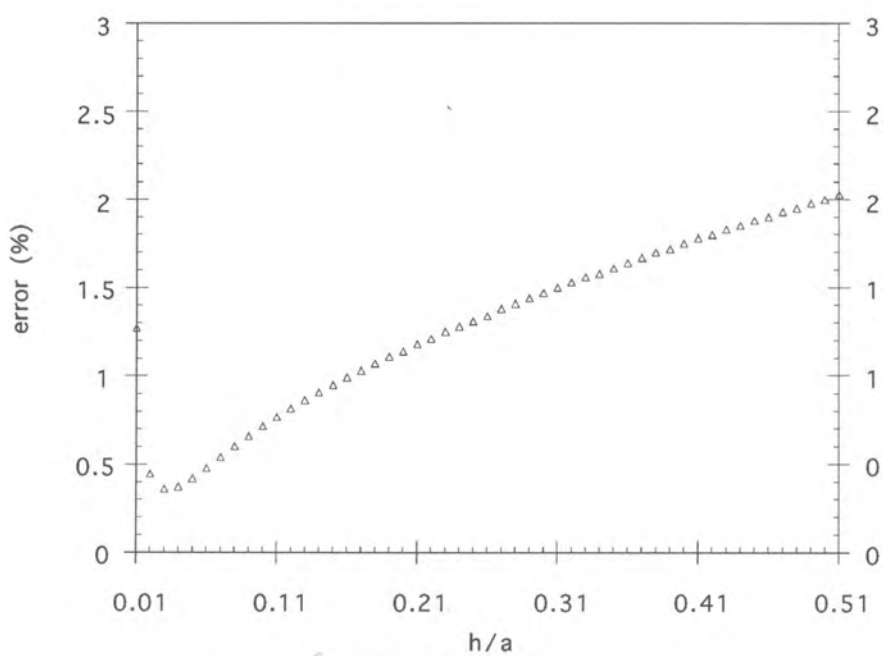

Fig. 18. Error estimate from $h/a = 0.01$ to 0.51

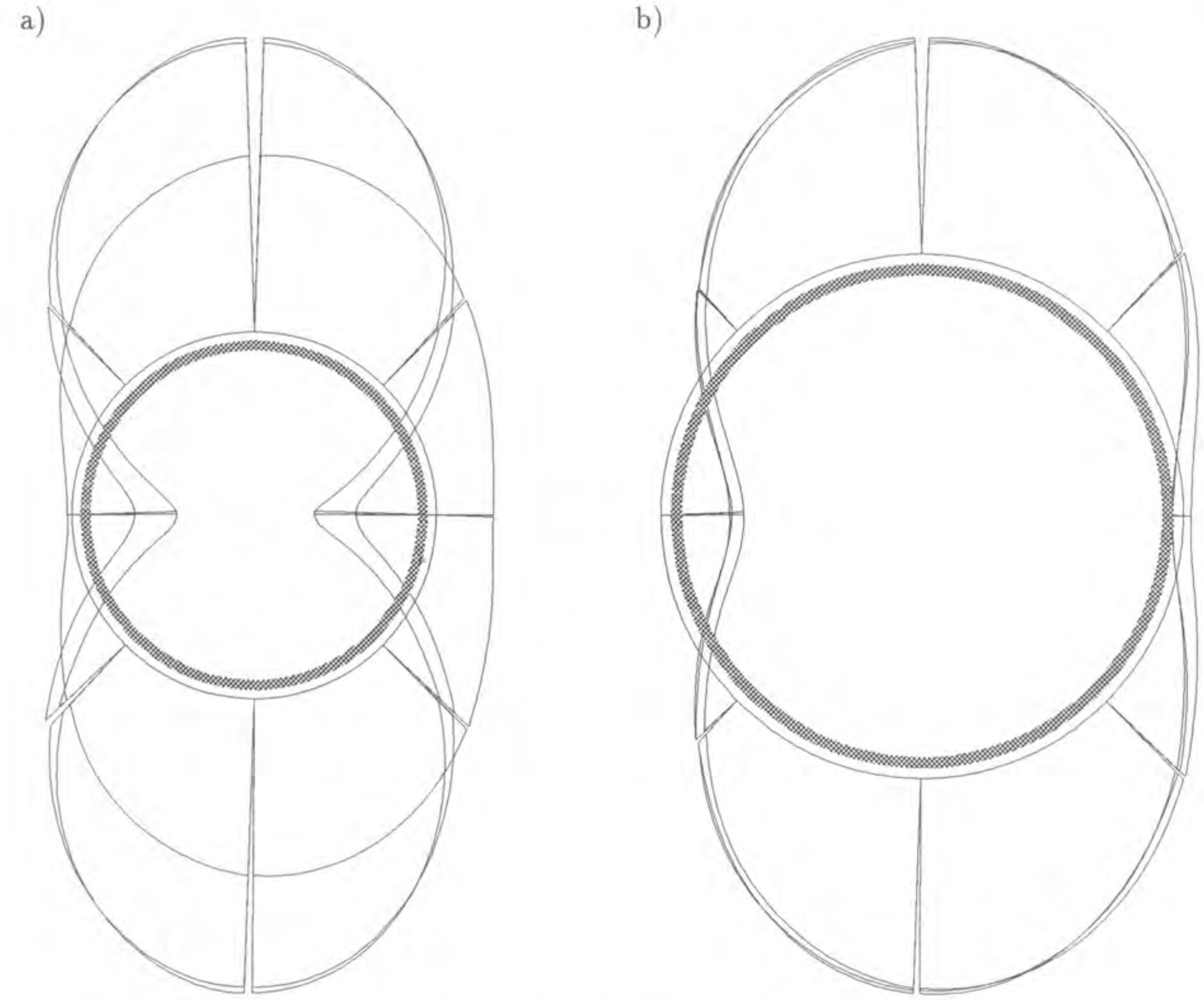

Fig. 19. Numerical results of elastic scattering and exact solution of rigid scattering and elastic scattering with $h/a = 0.01$ (a) and $h/a = 0.5$ (b). h-p BEM, pressure real component, $k = 1.00$

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REFERENCES

[1] Y.C. Chang, L. Demkowicz, Vibrations of spherical shell. Comparison of 3-D elasticity and Kirchhoff shell theory results. *CAMES*, **2**: 187–206, 1995.

[2] L. Demkowicz, J.T. Oden. Elastic scattering problems in linear acoustic using an h-p boundary/finite element method. In: *Adaptive Finite and Boundary Element Methods*, C.A. Brebbia, M.H. Aliabadi (eds.), Computational Mechanics Publications 1993.

[3] M.C. Junger, D. Feit. *Sound, Structure and Their Interaction.* MIT Press, 1972.

[4] A.C. Eringen, E.S. Suhubi. *Elastodynamics*, Vol. 2. Academic Press, New York, 1975.

[5] W.H. Press, S.A. Teukolsky, W.T. Vetterling, B.P. Flannery. *Numerical Recipes in Fortran*, 2nd edition. Press Syndicate of the University of Cambridge, New York, 1992.

[6] J.S. Hubert, E. S. Palencia. *Vibration and Coupling of Continuous Systems. Asymptotic Methods.* Springer--Verlag, New York, 1989.

[7] L. Demkowicz. Asymptotic convergence in finite and boundary element methods. Part 2: The LBB constant for rigid and elastic scattering problems. *Computers and Mathematics with Applications*, in print.