Computer Assisted Mechanics and Engineering Sciences, 2: 149–160, 1995.

The algebraic moments of vorticity. Theory and numerical tests[1]

Robert Bresiński and Andrzej Styczek
Institute of Aeronautics and Applied Mechanics, Warsaw University of Technology
Nowowiejska 22/24, 00-665 Warsaw, Poland

(Received January 2, 1995)

The paper presents the method of algebraic vorticity moments. It may be used to solve problems of viscous liquid motion in 2-D and 3-D cases. Its essence lies in the integration of a set of ordinary differential equations. The unknown functions of those equations defined as $\frac{1}{m!n!}\int_{E_2} \omega x^m y^n \,dx dy$ allow to find the vorticity field and next the velocity. We also show a number of 2-D numerical examples.

1. Introduction

The motion of viscous liquid can be investigated by the application of vortex method. This method is based on the vorticity equation (Helmholtz equation) whose 2-D and 3-D forms are essentially different. Nevertheless, it is important to formulate the general vortex method covering both 2-D and 3-D cases. The algebraic moments method appears to serve that purpose.

The substance of the moments method is to reduce the original problem to Cauchy's formulation for a set of ordinary differential equations. It was initially applied to the investigation of nonviscous liquid motion [1, 2]. A certain number of vortices were assumed to exist. They moved and interacted with each other. But the liquid was accepted as nonviscous. The method was also generalized to the case of viscous fluid and unbounded vorticity support [3]. In the present paper the number of vortices is chosen arbitrarily. Each is mainly concentrated in the area close to the centre, but the support of each is in the whole plane/space. The total vorticity is assumed to be the sum of many contributions, each of them defining one evolving vortex. This approach seems to be more natural when considering viscous fluid motion.

2. Formulation of the problem

The plane motion of viscous fluid in the infinite plane can be described by the following system of equations:

$$\frac{d\omega}{dt} = \nu\Delta\omega\,, \qquad \Delta\psi = -\omega\,, \tag{1}$$

supplemented by the initial value of vorticity field ω. The motion vanishes in infinity and the initial value of vorticity field is assumed to be known and given.

Let us assume ω as a sum of certain contributions ω_i,

$$\omega = \omega_1 + \omega_2 + \omega_3 + \dots \tag{2}$$

[1] The paper has been presented at Japan–Central Europe Joint Workshop on Advanced Computing in Engineering, Pułtusk, Poland, September 26–29, 1994.

The system of equations

$$\frac{\partial\omega_i}{\partial t}+u\frac{\partial\omega_i}{\partial x}+v\frac{\partial\omega_i}{\partial y}=\nu\Delta\omega_i\,, \tag{3}$$

$$\Delta\psi_i=-\omega_i\,,\qquad \psi=\psi_1+\psi_2+\psi_3+\ldots \tag{4}$$

$$u=\frac{\partial\psi}{\partial y}\,,\qquad v=-\frac{\partial\psi}{\partial x}\,, \tag{5}$$

yields the system (1). The 3-D case can be formulated as follows:

$$\frac{\partial\omega_i}{\partial t}+(\mathbf{v}\cdot\nabla)\omega_i-(\omega_i\cdot\nabla)\mathbf{v}=\nu\Delta\omega_i\,, \tag{6}$$

$$\nabla\times\mathbf{v}_i=\omega_i\,,\qquad \nabla\cdot\mathbf{v}_i=0\,, \tag{7}$$

$$\mathbf{v}=\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3+\ldots \tag{8}$$

3. THE MOMENTS AND MOMENT EQUATIONS

Let the vorticity contribution ω_i be a function of time and coordinates: $\omega_i=\omega_i(t,x-x_i,y-y_i)$. The point $x_i(t)$, $y_i(t)$ is the i-th vortex centroid. It means that

$$\int_{E_2}\omega_i(t,\xi,\eta)\xi\,\mathrm{d}\xi\mathrm{d}\eta=\int_{E_2}\omega_i(t,\xi,\eta)\eta\,\mathrm{d}\xi\mathrm{d}\eta=0\,. \tag{9}$$

The moments of vorticity are defined by the following expression

$$I_i^{(m,n)}=\frac{1}{m!n!}\int_{E_2}\omega_i(t,x-x_i,y-y_i)x^my^n\,dxdy\,, \tag{10}$$

$$J_{i/i}^{(m,n)}=\frac{1}{m!n!}\int_{E_2}\omega_i(t,\xi,\eta)\xi^m\eta^n\,\mathrm{d}\xi\mathrm{d}\eta\,. \tag{11}$$

The Steiner's relations allow to write

$$I_i^{(m,n)}=\sum_{k=0}^{m}\sum_{p=0}^{n}\frac{x_i^{m-k}y_i^{n-p}}{(m-k)!(n-p)!}J_{i/i}^{(k,p)}\,. \tag{12}$$

Also, it is possible to find the i-th vortex moments in relation to α-th vortex centroid coordinate system,

$$J_{i/\alpha}^{(m,n)}=\sum_{k=0}^{m}\sum_{p=0}^{n}\frac{(x_i-x_\alpha)^{m-k}(y_i-y_\alpha)^{n-p}}{(m-k)!(n-p)!}J_{i/i}^{(k,p)}\,. \tag{13}$$

Taking into account the total vorticity definition one can find the moments of total vorticity ω at any coordinates,

$$J_{*/\alpha}^{(m,n)}=\sum_{(i)}J_{i/\alpha}^{(m,n)}\,. \tag{14}$$

The global coordinate system is assumed to be an inertial system of coordinates. Taking the time derivative of $I_i^{(m,n)}$ one can write

$$\frac{\mathrm{d}I_i^{(m,n)}}{\mathrm{d}t}=\frac{1}{m!n!}\int_{E_2}\left[\frac{\mathrm{d}\omega_i}{\mathrm{d}t}x^my^n+\omega_i\Big(mx^{m-1}y^nu+nx^my^{n-1}v\Big)\right]dxdy\,. \tag{15}$$

The substitution of $\frac{d\omega_i}{dt} = \nu\Delta\omega_i$ and integration by parts lead to the following formula,

$$\frac{dI_i^{(m,n)}}{dt} = \frac{1}{m!n!}\int_{E_2} \omega_i \left[\nu\left(m(m-1)x^{m-2}y^n + n(n-1)x^m y^{n-2}\right) + mx^{m-1}y^n u + nx^m y^{n-1}v\right] dxdy\,. \tag{16}$$

Also, the substitution $x = x_i - \xi$, $y = y_i - \eta$ and shifting of indices bring the result

$$\frac{dI_i^{(m,n)}}{dt} = \sum_{k=0}^{m}\sum_{p=0}^{n}\frac{x_i^k y_i^p}{k!p!}\left\{ \nu\left(J_{i/i}^{(m-k-2,n-p)} + J_{i/i}^{(m-k,n-p-2)}\right) + \frac{1}{(m-k)!(n-p)!}\int_{E_2}\omega_i\left[(m-k)\xi^{m-k-1}\eta^{n-p}u + (n-p)\xi^{m-k}\eta^{n-p-1}v\right]d\xi d\eta\right\}. \tag{17}$$

The same derivative can be obtained by differentiation of (12),

$$\frac{dI_i^{(m,n)}}{dt} = \sum_{k=0}^{m}\sum_{p=0}^{n}\frac{x_i^k y_i^p}{k!p!}\left[\dot{x}_i J_{i/i}^{(m-k-1,n-p)} + \dot{y}_i J_{i/i}^{(m-k,n-p-1)} + \frac{dJ_{i/i}^{(m-k,n-p)}}{dt}\right]. \tag{18}$$

Comparing both formulas for $\frac{dI_i^{(m,n)}}{dt}$ we receive the set of moment equations:

$$\frac{dJ_{i/i}^{(m,n)}}{dt} = \nu\left[J_{i/i}{}^{(m-2,n)} + J_{i/i}{}^{(m,n-2)}\right] + \int_{E_2}\omega_i\left[\frac{\xi^{m-1}\eta^n(u-\dot{x}_i)}{(m-1)!n!} + \frac{\xi^m\eta^{n-1}(v-\dot{y}_i)}{m!(n-1)!}\right]d\xi d\eta\,. \tag{19}$$

It is easy to see that

$$J_{i/i}^{(0,0)} = I_i^{(0,0)} = J_{\alpha/i}^{(0,0)} = \text{const} \tag{20}$$

and, from the definition of local coordinates,

$$J_{i/i}^{(1,0)} = J_{i/i}^{(0,1)} = 0\,. \tag{21}$$

Moreover, the motion equations of vortex centroids result from their definitions:

$$\dot{x}_i\, J_{i/i}^{(0,0)} = \int_{E_2}\omega_i u\, dxdy\,, \qquad \dot{y}_i\, J_{i/i}^{(0,0)} = \int_{E_2}\omega_i v\, dxdy\,. \tag{22}$$

The system (19–22) defines the moments and locations of vortex centroids. It is possible to obtain the 3-D analogy of this system. Firstly, we write the moments of one of vorticity components, for example the moments of ω_{yi},

$$J_{yi/i}^{(m,n,p)} = \frac{1}{m!n!p!}\int_{E_3}\omega_{yi}(t,\xi,\eta,\zeta)\xi^m\eta^n\zeta^p\, d\xi d\eta d\zeta\,. \tag{23}$$

Then, the Steiner's formula analogous to Eq. (12) can be written. Following the specified method of calculation one can write:

$$J_{y\,i/i}^{(0,0,0)} = \text{const}, \tag{24}$$

$$\dot{x}_i\, J_{y\,i/i}^{(0,0,0)} = \int_{E_3}(\omega_{yi}v_x - \omega_{xi}v_y)\, dxdydz\,, \tag{25}$$

$$\dot{y}_i\, J_{y\,i/i}^{(0,0,0)} = 0\,, \tag{26}$$

$$\dot{z}_i\, J_{y\,i/i}^{(0,0,0)} = \int_{E_3}(\omega_{yi}v_z - \omega_{zi}v_y)\, dxdydz\,, \tag{27}$$

$$
\begin{aligned}
\frac{\mathrm{d}J_{y\,i/i}^{(\alpha,\beta,\gamma)}}{\mathrm{d}t} &= \nu\left[J_{y\,i/i}^{(\alpha-2,\beta,\gamma)} + J_{y\,i/i}^{(\alpha,\beta-2,\gamma)} + J_{y\,i/i}^{(\alpha,\beta,\gamma-2)}\right] \\
&+ \int_{E_3} \frac{\xi^{\alpha-1}\eta^{\beta}\zeta^{\gamma}}{(\alpha-1)!\beta!\gamma!}\left[\omega_{yi}\left(v_x - \dot{x}_i\right) - \omega_{xi}v_y\right] \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta \\
&+ \int_{E_3} \frac{\xi^{\alpha}\eta^{\beta-1}\zeta^{\gamma}}{\alpha!(\beta-1)!\gamma!}\left[\omega_{yi}\left(v_z - \dot{z}_i\right) - \omega_{zi}v_y\right] \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta\,.
\end{aligned} \tag{28}
$$

It is seen, that the set of moment equations valid for 3-D case does not differ essentially from that for 2-D case.

4. Vorticity and moments of vorticity

There is a fundamental problem: to determine the vorticity field having all of its moments. Kantorovitch [6] investigated this problem in the following context: find an element of Banach space if the set of functionals calculated at it is given.

It is possible to solve the problem of vorticity field in a more natural way. At first, we observe that the asymptotic behaviour of initially singular Gaussian vortex with vorticity is given as

$$
\omega \sim \frac{1}{4\nu t}\exp\left(\frac{-r^2}{4\nu t}\right). \tag{29}
$$

This means, that the Gaussian function may be introduced as a factor describing the vorticity distribution. Thus, we assume that

$$
\omega = \frac{1}{4\nu t}\exp\left(\frac{-r^2}{8\nu t}\right) f\left(t, \frac{x}{\sqrt{4\nu t}}, \frac{y}{\sqrt{4\nu t}}\right) \tag{30}
$$

where $f(t,\xi,\eta)$ is a smooth function at any time in the whole plane. It should satisfy some restrictions. The minimum of them is

$$
f \in \mathbf{L}_2\,, \qquad \int_{E_2} f^2(t,\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta < \infty.
$$

Following that, we express f as a series of Hermite functions,

$$
f(t,\xi,\eta) = \sum_{k=0}^{\infty}\sum_{p=0}^{\infty} f_{kp}(t)\mathcal{H}_k(\xi)\mathcal{H}_p(\eta)\,. \tag{31}
$$

The Hermite functions $\mathcal{H}_k$ are defined by Hermite polynomials H_k:

$$
\mathcal{H}_k(x) = \frac{1}{\mu_k}e^{-\frac{x^2}{2}}H_k(x)\,, \qquad H_k(x) = (-1)^k e^{x^2}\frac{\mathrm{d}^k}{\mathrm{d}x^k}e^{-x^2}, \qquad {\mu_k}^2 = \sqrt{\pi}2^k k!\,. \tag{32}
$$

They form an orthonormal set of functions and define a basis of $\mathbf{L}_2(\mathbf{E})$ space. It is easy to see, that

$$
e^{-\frac{x^2}{2}}\mathcal{H}_k(x) = \frac{1}{\mu_k}(-1)^k\frac{\mathrm{d}^k}{\mathrm{d}x^k}e^{-x^2}. \tag{33}
$$

Using these formulas, we rewrite ω in the form

$$
\omega = \frac{1}{4\nu t}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}(-1)^{k+p}\frac{f_{kp}}{\mu_k\mu_p}\frac{\partial^k}{\partial\left(\frac{x}{\sqrt{4\nu t}}\right)^k}\frac{\partial^p}{\partial\left(\frac{y}{\sqrt{4\nu t}}\right)^p}e^{-\frac{r^2}{4\nu t}}. \tag{34}
$$

The vorticity should be defined as a weakly differentiable function. This requirement is fulfilled when the set of coefficients $\{f_{kp}\}$ satisfies the following condition,

$$\sum_{k=0}^{\infty}\sum_{p=0}^{\infty} f_{kp}^2(1+k)(1+p) = f_1^2 < \infty\,. \tag{35}$$

Moreover, it makes sense to define the vorticity as a bounded function. Doing that we write

$$\sum_{k=0}^{\infty}\sum_{p=0}^{\infty} |f_{kp}| = f_0 < \infty\,. \tag{36}$$

Now, it is possible to obtain

$$|f| \le \sum_{k=0}^{\infty}\sum_{p=0}^{\infty} |f_{kp}|\,|\mathcal{H}_k|\,|\mathcal{H}_p| < f_0 \tag{37}$$

as a result of the property $|\mathcal{H}_k| < 1$, [4]. Coming back to the vorticity definition (34) we introduce a linear differential operator $\mathcal{L}$

$$\mathcal{L} = \sum_{k=0}^{\infty}\sum_{p=0}^{\infty}(-1)^{k+p}\frac{f_{kp}}{\mu_k\mu_p}(4\nu t)^{\frac{k+p}{2}}\frac{\partial^k}{\partial x^k}\frac{\partial^p}{\partial y^p}\,. \tag{38}$$

The vorticity ω will be shortly denoted as follows:

$$\omega = \frac{1}{4\nu t}\mathcal{L}\,e^{-\frac{r^2}{4\nu t}}\,. \tag{39}$$

Now, the moments can be calculated. We express them as combinations of f_{kp}. Following the definition we write

$$J^{(m,n)} = (4\nu t)^{\frac{m+n}{2}}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\frac{f_{kp}}{\mu_k\mu_p}\frac{1}{m!}\int_{-\infty}^{+\infty} x^m(-1)^k\frac{\partial^k}{\partial x^k}e^{-x^2}\,\mathrm{d}x\;\frac{1}{n!}\int_{-\infty}^{+\infty} y^m(-1)^p\frac{\partial^p}{\partial y^p}e^{-y^2}\,\mathrm{d}y\,. \tag{40}$$

Integration by parts yields

$$J^{(m,n)} = (4\nu t)^{\frac{m+n}{2}}j^{(m,n)}\,, \tag{41}$$

$$j^{(m,n)} = \sum_{k=0}^{m}\sum_{p=0}^{n} b_{m-k}b_{n-p}\frac{f_{kp}}{\mu_k\mu_p}\,, \tag{42}$$

$$b_{m-k} = \frac{1}{m!}\int_{-\infty}^{+\infty} x^m(-1)^k\frac{\mathrm{d}^k}{\mathrm{d}x^k}e^{-x^2}\,\mathrm{d}x\,, \tag{43}$$

$$b_{2k} = \frac{\sqrt{\pi}}{4^k k!}\,,\qquad b_{2k+1} = 0\,,\qquad b_{-(k+1)} = 0\,,\qquad k = 0,1,2\ldots \tag{44}$$

It is necessary to find the reverse relations. We do that in the following way. The set of coefficients $\{b_k\}$ defines an analytic function B,

$$B(X) = \sum_{m=0}^{\infty} b_m\,e^{imX} = \sqrt{\pi}\exp\left(\frac{e^{2iX}}{4}\right). \tag{45}$$

One can see that $\sqrt{\pi}e^{-\frac{1}{4}} \le |B| \le \sqrt{\pi}e^{\frac{1}{4}}$. It means that B^{-1} exists, is bounded and can be written in the form of a Fourier series,

$$B^{-1}(X) = \sum_{m=0}^{\infty} d_m e^{imX}\,, \tag{46}$$

$$d_{2m} = \frac{(-1)^m}{\sqrt{\pi}4^m m!}\,,\qquad d_{2m+1} = 0\,,\qquad d_{-(m+1)} = 0\,,\qquad m = 0,1,2\ldots \tag{47}$$

Now, we define another function $\mathcal{F}$,

$$\mathcal{F}(X,Y) = \sum_{k=0}^{\infty}\sum_{p=0}^{\infty} \frac{f_{kp}}{\mu_k \mu_p} e^{(kX+pY)} . \tag{48}$$

The expression (44) shows that there exists a product

$$j(X,Y) = \sum_{k=0}^{\infty}\sum_{p=0}^{\infty} j^{(k,p)} e^{i(kX+pY)} = B(X)B(Y)\mathcal{F}(X,Y) . \tag{49}$$

The reverse formula $\mathcal{F} = B^{-1}(X)B^{-1}(Y)j(X,Y)$ defines the reverse transformation $j^{(m,n)} \to f_{kp}$,

$$\frac{f_{kp}}{\mu_k \mu_p} = \sum_{m=0}^{k}\sum_{n=0}^{p} d_{k-m} d_{p-n} j^{(m,n)} , \tag{50}$$

which is also an algebraic convolution. It allows to find the vorticity when the system of moments is given. We note, that relation (44), Steiner's formula and transformation (50) allow to express the i-th vorticity, which was connected with i-th local coordinates, in any α-th coordinate system. It means, that the total vorticity $\omega = \omega_1 + \omega_2 + \ldots$ can also be expressed in any local coordinate system.

The 3-D case appears to be more complicated. The reason for that ensues from the following fact,

$$\nabla \cdot \omega = \frac{\partial \omega_x}{\partial x} + \frac{\partial \omega_y}{\partial y} + \frac{\partial \omega_z}{\partial z} = 0 , \tag{51}$$

The vorticity vector $\boldsymbol{\omega}$ satisfies this condition if it is proposed in the form

$$\begin{aligned}
\omega_x &= \frac{1}{(4\nu t)^{\frac{3}{2}}} e^{-\frac{r^2}{8\nu t}} \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\sum_{\gamma=0}^{\infty} f^x_{\alpha\beta\gamma} \mathcal{H}_\alpha\left(\frac{x}{\sqrt{4\nu t}}\right) \mathcal{H}_{\beta+1}\left(\frac{y}{\sqrt{4\nu t}}\right) \mathcal{H}_{\gamma+1}\left(\frac{z}{\sqrt{4\nu t}}\right), \\
\omega_y &= \frac{1}{(4\nu t)^{\frac{3}{2}}} e^{-\frac{r^2}{8\nu t}} \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\sum_{\gamma=0}^{\infty} f^y_{\alpha\beta\gamma} \mathcal{H}_{\alpha+1}\left(\frac{x}{\sqrt{4\nu t}}\right) \mathcal{H}_{\beta}\left(\frac{y}{\sqrt{4\nu t}}\right) \mathcal{H}_{\gamma+1}\left(\frac{z}{\sqrt{4\nu t}}\right), \\
\omega_z &= \frac{1}{(4\nu t)^{\frac{3}{2}}} e^{-\frac{r^2}{8\nu t}} \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\sum_{\gamma=0}^{\infty} f^z_{\alpha\beta\gamma} \mathcal{H}_{\alpha+1}\left(\frac{x}{\sqrt{4\nu t}}\right) \mathcal{H}_{\beta+1}\left(\frac{y}{\sqrt{4\nu t}}\right) \mathcal{H}_{\gamma}\left(\frac{z}{\sqrt{4\nu t}}\right),
\end{aligned} \tag{52}$$

where the set of coefficients $f^i_{\alpha\beta\gamma}$ is described by optional three-indices arrays $\mathbf{A}, \mathbf{B}, \mathbf{C}$,

$$\begin{aligned}
f^x_{\alpha\beta\gamma} &= \frac{\mu_\alpha}{\mu_{\alpha+1}} \left[2A_{\alpha\beta\gamma} - B_{\alpha\beta\gamma} - C_{\alpha\beta\gamma}\right] , \\
f^y_{\alpha\beta\gamma} &= \frac{\mu_\beta}{\mu_{\beta+1}} \left[-A_{\alpha\beta\gamma} + 2B_{\alpha\beta\gamma} - C_{\alpha\beta\gamma}\right] , \\
f^z_{\alpha\beta\gamma} &= \frac{\mu_\gamma}{\mu_{\gamma+1}} \left[-A_{\alpha\beta\gamma} - B_{\alpha\beta\gamma} + 2C_{\alpha\beta\gamma}\right] .
\end{aligned} \tag{53}$$

It may be seen that a "vector" operator $\vec{\mathcal{L}}$ is defined. It contains the set of coefficients $\left\{f^x_{\alpha\beta\gamma}, f^y_{\alpha\beta\gamma}, f^z_{\alpha\beta\gamma}\right\}$ and appropriate derivatives. Using it, we write

$$\omega = \frac{1}{(4\nu t)^{\frac{3}{2}}} \vec{\mathcal{L}} e^{-\frac{r^2}{4\nu t}} . \tag{54}$$

Note, that the set of moments $\left\{J_x^{(m,n,p)},\ J_y^{(m,n,p)},\ J_z^{(m,n,p)}\right\}$ which fulfills the restriction

$$J_x^{(m-1,n,p)} + J_y^{(m,n-1,p)} + J_z^{(m,n,p-1)} = 0 \tag{55}$$

allows to find the arrays $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$.

5. THE VELOCITY FIELD

The velocity field in 2-D motion can be found in the following way:

$$\Delta\psi = -\omega = -\frac{1}{4\nu t}\mathcal{L}e^{-\frac{r^2}{4\nu t}}, \tag{56}$$

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}. \tag{57}$$

Now, we substitute $\psi = \mathcal{L}\psi_0$ and observe, that the operators $\mathcal{L}$ and Laplacian Δ commute. Making use of this property we obtain

$$\begin{aligned}\Delta\psi_0 &= -\frac{1}{4\nu t}e^{\frac{r^2}{4\nu t}},\\ \psi_0 &= \frac{1}{2}\int_0^1 \frac{e^{-\frac{(rs)^2}{4\nu t}}-1}{s}\,\mathrm{d}s,\\ \psi &= \mathcal{L}\psi_0.\end{aligned} \tag{58}$$

Determination of ψ_0 leads to the following formulas:

$$\begin{aligned}u &= -\frac{1}{2\sqrt{4\nu t}}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\frac{f_{kp}}{\mu_k\mu_p}\int_0^1 s^{k+p}e^{-\frac{(rs)^2}{4\nu t}}\,H_k\left(\frac{xs}{\sqrt{4\nu t}}\right)H_{p+1}\left(\frac{ys}{\sqrt{4\nu t}}\right)\mathrm{d}s,\\ v &= \frac{1}{2\sqrt{4\nu t}}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\frac{f_{kp}}{\mu_k\mu_p}\int_0^1 s^{k+p}e^{-\frac{(rs)^2}{4\nu t}}\,H_{k+1}\left(\frac{xs}{\sqrt{4\nu t}}\right)H_p\left(\frac{ys}{\sqrt{4\nu t}}\right)\mathrm{d}s.\end{aligned} \tag{59}$$

It is possible to obtain the following estimation,

$$\max\{|u|,|v|\} < \frac{f_0}{\sqrt{t}}. \tag{60}$$

Here, the simple estimation was applied, [4],

$$|\mathcal{H}_k| = \left|\frac{1}{\mu_k}e^{-\frac{(xs)^2}{8\nu t}}\,H_k\left(\frac{xs}{\sqrt{4\nu t}}\right)\right| < 1. \tag{61}$$

Moreover, both components of velocity are L_2 class functions. This corollary, however, results from an additional condition

$$\int_{E_2}\omega|_{t=0}\,\mathrm{d}x\mathrm{d}y = \int_{E_2}\omega\,\mathrm{d}x\mathrm{d}y = 0. \tag{62}$$

We have

$$\int_{E_2}u^2\,\mathrm{d}x\mathrm{d}y = \frac{1}{16\nu t}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}f_{kp}f_{mn}\sqrt{1+p}\sqrt{1+n}\int_0^1\int_0^1 s^{k+p}z^{m+n}g_{kpmn}(s,z)\,\mathrm{d}s\mathrm{d}z, \tag{63}$$

$$g_{kpmn}(s,z) := \int_{E_2}e^{-\frac{s^2+z^2}{4\nu t}r^2}\,\mathcal{H}_k(x)\,\mathcal{H}_{p+1}(y)\,\mathcal{H}_m(x)\,\mathcal{H}_{n+1}(y)\,\mathrm{d}x\mathrm{d}y, \tag{64}$$

and next, after using the inequality $|\mathcal{H}_k| < 1$, we get

$$\left|\int_0^1\int_0^1 s^{k+p}z^{m+n}\int_{E_2}e^{-(s^2+z^2)r^2}\,\mathrm{d}x\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}z\right| \le \text{const}\int_0^1\int_0^1\frac{s^{k+p}z^{m+n}}{s^2+z^2}\,\mathrm{d}s\mathrm{d}z. \tag{65}$$

From the restriction involving the total charge of vorticity we see that $f_{00} = 0$, which means that both $k+p > 0$ and $m+n > 0$. It also means that the part of series components can be estimated as

$$\sqrt{1+p}\sqrt{1+k}\int_0^1\int_0^1\frac{s^{k+p}z^{m+n}}{s^2+z^2}\,\mathrm{d}s\mathrm{d}z < \frac{\sqrt{(1+p)(1+n)}}{k+p+m+n}\int_0^{\frac{\pi}{2}}\cos^{k+p}(\theta)\sin^{m+n}(\theta)\,\mathrm{d}\theta < 1 \tag{66}$$

which assures the final estimation

$$\|u\|^2\|v\|^2 < \text{const}\cdot f_1^2\,. \tag{67}$$

The 3-D velocity field can be found in a similar way. The following equations are fulfilled,

$$\nabla\times\mathbf{v}=\omega\,,\qquad \nabla\cdot\mathbf{v}=0\,. \tag{68}$$

Substituting $\mathbf{v}=\nabla\times\mathbf{B}$, $\nabla\cdot\mathbf{B}$, we obtain

$$\Delta\mathbf{B}=\Delta\vec{\mathcal{L}}B_0=-\omega=-\frac{1}{\sqrt{4\nu t^3}}\vec{\mathcal{L}}e^{-\frac{(rs)^2}{4\nu t}}\,. \tag{69}$$

This allows to calculate

$$B_0=\frac{1}{2\sqrt{4\nu t}}\int_0^1 e^{-\frac{(rs)^2}{4\nu t}}\,\mathrm{d}s\,, \tag{70}$$

$$\mathbf{B}=\vec{\mathcal{L}}B_0\,. \tag{71}$$

The operator $\vec{\mathcal{L}}$ was introduced earlier. It is determined entirely by the set of coefficients $\left\{f^x_{\alpha\beta\gamma}, f^y_{\alpha\beta\gamma}, f^y_{\alpha\beta\gamma}\right\}$.

6. THE INTEGRAL TERMS

The moment equations (19) contain terms similar to $\frac{1}{m!n!}\int_{E_2}\omega u\xi^m\eta^n\,\mathrm{d}\xi\mathrm{d}\eta$. The velocity is bounded and estimated by $\frac{f_0}{\sqrt{t}}$. Besides, the vorticity was earlier written as the product of function f and an exponential function. We write

$$\begin{aligned}\frac{1}{m!n!}\left|\int_{E_2}\omega u\xi^m\eta^n\,d\xi d\eta\right| &\le \frac{\max|u|}{m!n!}(4\nu t)^{\frac{m+n}{2}}\int_{E_2}e^{-\frac{x^2+y^2}{2}}|f(t,\xi,\eta)\xi^m\eta^n|\,\mathrm{d}\xi\mathrm{d}\eta\\ &\le \frac{\max|u|}{m!n!}(4\nu t)^{\frac{m+n}{2}}\|f\|_{L_2}\left\|e^{-\frac{\xi^2}{2}}\xi^m\right\|_{L_2}\left\|e^{-\frac{\eta^2}{2}}\eta^n\right\|_{L_2}.\end{aligned} \tag{72}$$

In view of

$$\|e^{-\frac{\eta^2}{2}}\eta^m\|_{L_2}=\pi^{\frac14}\sqrt{\frac{(2m-1)!!}{2^m}}$$

and one of "binomial" inequalities [5]

$$\frac{(2m-1)!!}{2^m m!}=\frac{1\cdot3\cdot5\cdot\ldots\cdot(2m-1)}{2^m m!}\le\left(\frac{3/4}{2m+1}\right)^{\frac12} \tag{73}$$

it is possible to ascertain that

$$\frac{1}{m!n!}\left|\int_{E_2}\omega u\xi^m\eta^n\,\mathrm{d}\xi\mathrm{d}\eta\right|\le\text{const}\frac{\max|u|(4\nu t)^{\frac{m+n}{2}}\|f\|_{L_2}}{\sqrt{m!n!}[(2m+1)(2n+1)]^{\frac14}}\,. \tag{74}$$

Moreover, it is possible to prove, that $\frac{\omega\xi^m\eta^n}{m!n!}$ is an L_2-class function. Doing that, we use the inequality

$$e^{-\eta^2}\eta^{2m}\le\frac{m!}{\sqrt{2\pi m}}$$

which results from the maximum of the estimated function and Stirling's formula.

7. Numerical calculations

A number of calculations have been done. The results are shown in Figs. 1–4. The figures present the time evolution of several vortices. An initial condition for each vortex is

$$\omega(0,x,y) = Ce^{-(\frac{x^2+y^2}{\sigma})}.$$

The first three examples were calculated for two vortices, for Reynolds numbers equal to 800, 400 and 450, respectively, while the fourth example — for three vortices and for Reynolds number equal to 1225. In each case Reynolds number was defined as $Re = \frac{\omega d^2}{\nu}$, where ν denotes viscosity coefficient, ω is defined as a maximum of initial vorticity, and d is given as a distance between vortices in the first three examples (Figs. 1–3). In the fourth case, d is given as a distance between the "large" vortex and the centre of vorticity of both "small" vortices. It is important to note that limit Reynolds number Re_{kr} exists for each initial condition. For Reynolds numbers greater than Re_{kr} the system of equations (19) is not stable.

Each figure presents distribution of vorticity. One color means that vorticity is greater than known value a_i and less than a_{i+1}. Minimum value of vorticity is equal to 0 and maximum is equal to 1.

Numerical calculations were made in order to solve a set of equations for moments $j^{(m,n)}$. The moments are defined by the formula

$$j^{(m,n)} = \frac{1}{m!n!}\int_{E_2} \omega(t,s,z)s^m z^n \,\mathrm{d}s\mathrm{d}z$$

where $s = \frac{x}{\sqrt{4\nu t}}$, $z = \frac{y}{\sqrt{4\nu t}}$. Link between $j^{(m,n)}$ and $J^{(m,n)}$ is given by (41). The system of equations for $j^{(m,n)}$ follows from the system of equations for $J^{(m,n)}$ and from the relation (41). Integrals (72) which appear in the system of equations (19) may be calculated implicitly or numerically with the use of quadratures. In the first case we obtain

$$\frac{1}{m!n!}\int_{E_2} \omega_i u x^m y^n \,\mathrm{d}x\mathrm{d}y = \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty} R^{mn}_{\alpha\beta kp} J^{(\alpha,\beta)}_{*/i} J^{(k,p)}_{i/i} \tag{75}$$

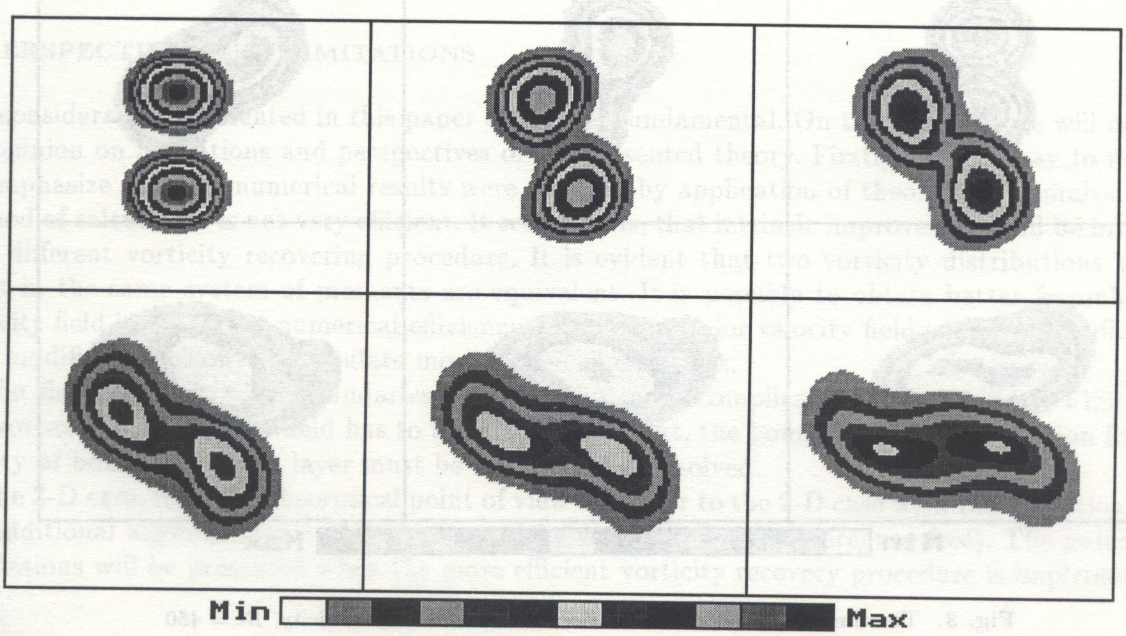

Fig. 1. The time evolution of two vortices via moments of vorticity, $Re = 800$

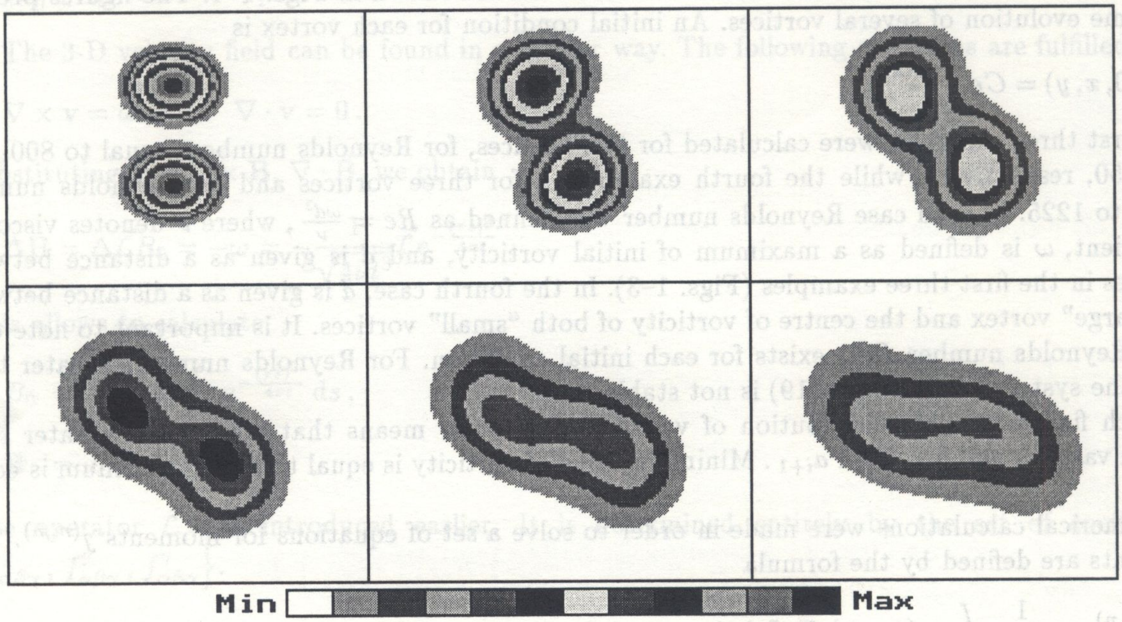

Fig. 2. The time evolution of two vortices via moments of vorticity, $Re = 400$

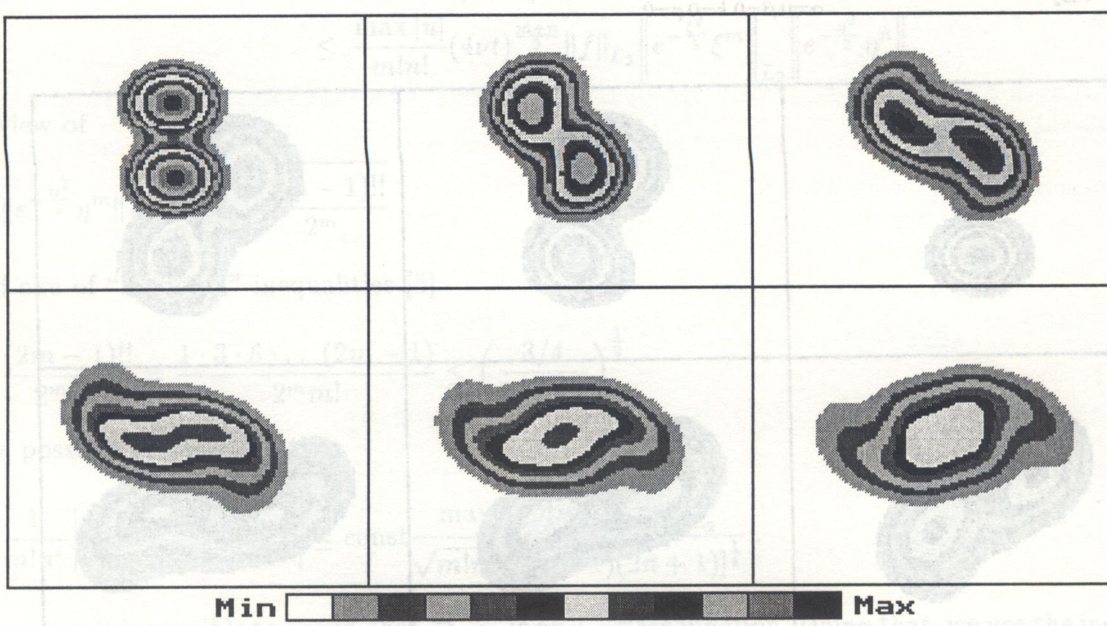

Fig. 3. The time evolution of two vortices via moments of vorticity, $Re = 450$

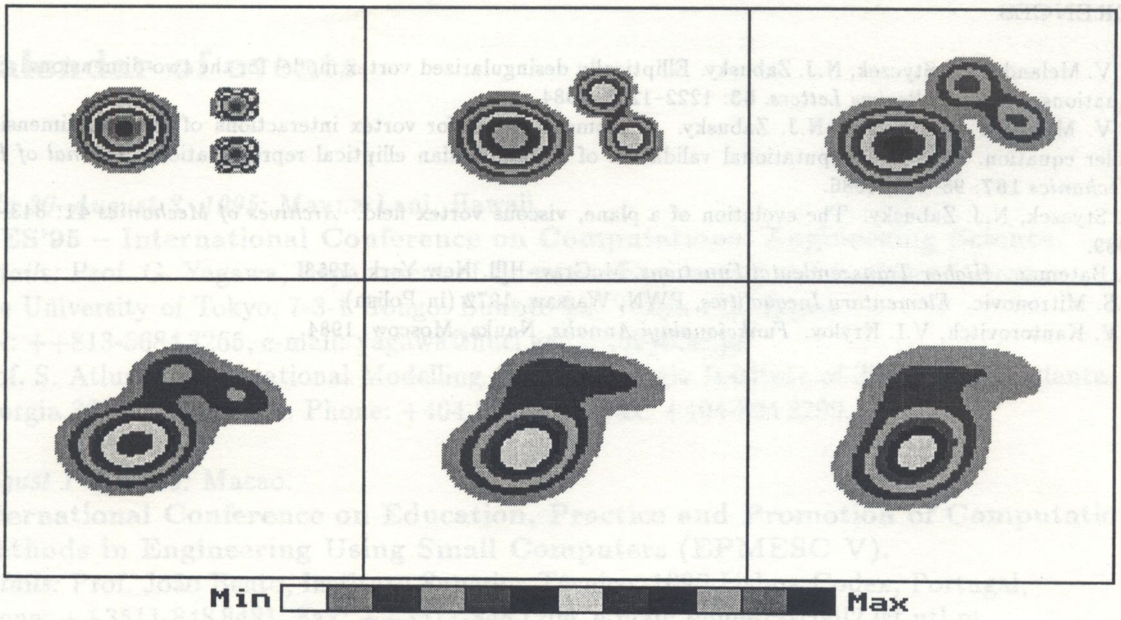

Fig. 4. The time evolution of three vortices via moments of vorticity, $Re = 1225$

and the system (19) becomes an ordinary differential bilinear system of equations with a universal six-indices array R being constant. We note that there exist analytic formulas for elements of R. In the second case we obtain

$$\frac{1}{m!n!}\int_{E_2}\omega_i u x^m y^n \,dx dy = \sum_{k=0}^{N}\sum_{p=0}^{N} A_k A_p \frac{x_k^m y_p^n}{m!n!}\,\omega_i u|_{x=x_k,y=y_p} . \qquad (76)$$

The first method is not effective for numerical calculations with computers equipped with small memory. In such a case the second method is more efficient regarding available computer memory and time of calculations.

8. Perspectives and limitations

The considerations presented in this paper are rather fundamental. On that account we will deliver our opinion on limitations and perspectives of the presented theory. Firstly, as it is easy to notice, we emphasize that the numerical results were received by application of theoretical formulas. This method of calculation is not very efficient. It seems to us, that intrinsic improvement will be brought by a different vorticity recovering procedure. It is evident that two vorticity distributions which result in the same system of moments are equivalent. It is possible to obtain better formulas for vorticity field in a point of numerical efficiency. The formulas for velocity field will also be different. This modification allows to calculate more advanced examples.

The flow problem with boundaries seems to be more complicated for treatment. First, the non-vortex part of velocity field has to be introduced. Next, the boundary integral equation for the density of boundary vortex layer must be formulated and solved.

The 3-D case, from the theoretical point of view is similar to the 2-D case with the exception that one additional algebraic constraint is present (the vorticity field is divergence free). The numerical calculations will be presented when the more efficient vorticity recovery procedure is implemented.

Acknowledgments

This work was supported by KBN Grant 3 3314 9203.

REFERENCES

[1] M.V. Melander, A. Styczek, N.J. Zabusky. Elliptically desingularized vortex model for the two-dimensional Euler equations. *Physical Review Letters.* **53**: 1222–1225, 1984.

[2] M.V. Melander, A. Styczek, N.J. Zabusky. A moment model for vortex interactions of the two-dimensional Euler equation. Part 1. Computational validation of a Hamiltonian elliptical representation. *Journal of Fluid Mechanics* **167**: 95–115, 1986.

[3] A. Styczek, N.J. Zabusky. The evolution of a plane, viscous vortex field. *Archives of Mechanics* **41**: 343–350, 1989.

[4] H. Bateman. *Higher Transcendental Functions.* McGraw-Hill, New York, 1953.

[5] D.S. Mitronovic. *Elementary Inequalities*, PWN, Warsaw, 1972 (in Polish).

[6] L.V. Kantorovitch, V.I. Krylov. *Funkcionalnyi Annaliz.* Nauka, Moscow, 1984.