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Quasi-periodic solutions: analytical and numerical investigations

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First, an analytical asymptotic method to construct quasi-periodic solutions in autonomous dynamical systems governed by a nonlinear second order set of ordinary differential equations with delay is presented. The approach is based on the double asymptotic expansion of two independent perturbation parameters and is supported by symbolic computation using *Mathematica* package. Both resonance and non-resonance cases are successfully analyzed and the catastrophes of the torus solutions are classified and discussed. Second, a new method for numerical calculations of the quasi-periodic orbits, which is based on a concept of the general Poincaré map, is addressed. In both cases considered examples support the introduced theory.

1. Introduction

In this paper quasi-periodic attractors, which play the fundamental role in nonlinear dynamics, are reconsidered. Landau [32] suggested that an infinite sequence of tori doubling bifurcations can lead to chaotic orbits. This idea has been used also by Ruelle, Takens [38] and Newhouse et al. [36], who have shown that a sequence of the bifurcations of a torus may cause an occurrence of chaos. Recently many other specific problems dealing with transition from quasi-periodicity to chaos are widely described in the literature [1–4, 21, 24, 25, 29, 37, 41]. On the other hand, the quasi-periodic solutions have a long history in nonlinear dynamics, although serious formal approach to investigate invariant orbits have been developed in a rather episodic way. In the isolated nonlinear dynamical systems such attractors appear in rare cases and usually by the slight change of the right sides of the governing equations they are supplanted by periodic or even chaotic solutions. Thus, the problem of finding the set of parameters for which quasi-periodic solution exists occur.

One of the earliest attempts to calculate the quasi-periodic solutions belong to Krilov, Bogolubov, Mitropolski and Samoilenko [15–17, 31], where the asymptotic methods of nonlinear dynamics have been used. Some fundamental theoretical results, also with relation to chaos, have been given in [19, 22, 23, 25, 33, 34, 39].

In the field of computational dynamics one of the most popular method to find such solutions is based on solving an initial value problem. Projection of the attractor, Poincaré maps, time series analysis (Fast Fourier Transform) and Lyapunov exponents are used to identify such solutions. This rough approach, however, does not allow for the systematical study of the torus behaviour in the change of the control parameter and cannot be used to follow the unstable torus. For these reasons, Kaas-Petersen [26–28] has presented another approach based on numerical evidence and in an algorithmic spirit, developing a method often used to calculate periodic orbits by solving a boundary value problem. He has shown that a torus solution may be treated as a fixed point of a certain generalized Poincaré map. He demonstrated numerically that the stability of the torus is equivalent to the stability of the fixed point. This approach gives the possibility to establish a bifurcation of a torus as well as allowing the construction of the bifurcation diagrams.

There are also some works, similar in spirit to the Kaas-Peterson's papers, in which an attempt to locate a single point on the invariant curve and the corresponding quasi-periodic solution is demonstrated [44,45]. Another approach is to search for many points on the invariant curve simultaneously with a collocation method [30,46].

In the paper this problem is also addressed. We will show, how an analysis of a quasi-periodic solution of the original autonomous system reduces to the analysis of a periodic solution of the nonautonomous one with a periodic excitation. We propose a method of transformation from a one dimensional curve to a two-dimensional torus. Simple example confirms the introduced theory.

Many aspects of the quasi-periodic orbits, including theorems and their proofs, are discussed in a seminal book by Samoilenko [39]. He showed that a problem of finding quasi-periodic attractors in a system of autonomous nonlinear differential equations can be reduced to the consideration of the solution

$$x = u(\phi)\,, \qquad \phi = (\phi_1, \ldots, \phi_m) \in T_m\,, \qquad x \in R^n\,, \tag{1}$$

of the system of equations

$$\frac{d\phi}{dt} = a(\phi)\,, \qquad \frac{dx}{dt} = P(\phi)x + F(\phi)\,. \tag{2}$$

The determination of the torus requires the application of approximate methods. One such method is a version of Galerkin's iteration method. In accordance with this method a function $u(\phi)$, which defines the invariant torus (1) is sought as a limit of the sequence of Galerkin approximations to the periodic solution of

$$Lu(\phi) = F(\phi)\,, \tag{3}$$

where a differential operator is defined as

$$Lu = \sum_{\nu=1}^{m} \frac{\partial u}{\partial \phi_\nu} a_\nu(\phi) - P(\phi)u\,. \tag{4}$$

The Galerkin approximations are sought in the form of the following trigonometric polynomials

$$W_N(\phi) = \sum_{\|k\| \ll N} W_k^{(N)} e^{i(k,\phi)}\,, \tag{5}$$

where the unknown coefficients are found from the system of algebraic equations

$$\left(LW_N(\phi), e^{i(k,\phi)}\right)_0 = \left(F(\phi), e^{i(k,\phi)}\right)_0\,, \qquad \| k \| \leq N\,. \tag{6}$$

Samoilenko formulates precise conditions under which such approximations are possible for the arbitrary $N \geq 0$ and when they approach the solutions of (3) for $N \to +\infty$.

Such an idea has been used by Chua and Ushida [18]. They assume a generalized Fourier series

$$x = a_0 + \sum_{i=1}^{\infty} (a_{2i-1} \cos \nu_i t + a_{2i} \sin \nu_i t)\,, \tag{7}$$

$$\nu_i = m_{1i}\omega_i + \ldots + m_\pi \omega_\pi\,, \tag{8}$$

where $m_{1i}, \ldots, m_\pi$ are integers chosen in such a way that $\nu_i > 0$, and then they calculate a_i.

Here, another analytical method to find quasi-periodic solutions is presented. This approach is organized in the spirit of the asymptotic techniques described in [20,35]. It possesses some benefits in comparison with the above mentioned methods. First of all, it considers system of differential equations of second order, which is obtained directly from Newton's laws and do not need further transformations to obtain first order equations. The presented method also does not require transformations leading to the system (2). Double asymptotic expansion is used, and this approach can be easily generalized to multiple asymptotic expansions. (As has been shown earlier by the author [5–13], such an approach can lead to new qualitative results which cannot be found using single asymptotic expansion.) Finally, it will be shown that the domain of existence of quasi-periodic solutions in two parameter space can be found. These considerations are supported by symbolic computation with the use of *Mathematica* package.

2. ANALYTICAL METHOD

2.1. Non-resonance case

We consider the following dynamical system

$$\ddot{x}_s + \omega_s^2 x_s = \epsilon F_s\left(x_1, \ldots, x_n,\ \dot{x}_1, \ldots \dot{x}_n,\ x_1(t-\tau), \ldots \dot{x}_n(t-\tau),\ \epsilon\right), \qquad s = 1, \ldots, n\,, \tag{9}$$

where F_s is continuous and fulfils a Lipshitz condition in a certain domain D of the n-dimensional Euclidean space. ϵ and τ (time delay) are small positive parameters and $\epsilon > \tau > 0$. Under such an assumption we take

$$x_i(t-\tau) = x_i(t) - \tau\dot{x}_i + \frac{1}{2}\tau^2\ddot{x}_i + \ldots, \qquad i = 1, \ldots, n\,, \tag{10}$$

and from (10) we obtain

$$\ddot{x}_s + \omega_s^2 x_s = \epsilon F_s\left(x_1, \ldots, x_n,\ \dot{x}_1, \ldots \dot{x}_n,\ \epsilon, \tau\right), \qquad s = 1, \ldots, n\,. \tag{11}$$

We consider first a non-resonance case in which

$$\dot{\phi}_s = \omega_s + \eta_s(\epsilon, \tau), \qquad s = 1, \ldots, n\,, \tag{12}$$

and the frequencies $\omega = (\omega_1, \ldots, \omega_n)$ are positive and are incommensurable, i. e.

$$((k, \omega)) = k_1\omega_1 + k_2\omega_2 + \ldots + k_n\omega_n \neq 0 \tag{13}$$

or equivalently

$$|((k, \omega))| = \alpha > 0\,, \tag{14}$$

where k_i of $k = (k_1, \ldots, k_n)$ are integer and $|k| = \sum_{i=1}^{n} |k_i| \neq 0$.

Suppose that we have found such a function $\eta(\epsilon, \tau)$ that the dynamical system (10) has the following quasi-periodic solutions

$$x_s = Q^s(\phi_1, \ldots, \phi_n, \epsilon, \tau), \tag{15}$$

$$\lambda_s \equiv \dot{\phi}_s = \omega_s + \eta_s\,, \qquad s = 1, \ldots, n\,, \tag{16}$$

where:

$$Q^s(\phi_1, \ldots, \phi_n, \epsilon, \tau) = Q^s_{00}(\phi_1, \ldots, \phi_n) + \sum_{k=1}^{\infty}\sum_{l=0}^{\infty} \epsilon^k \tau^l Q^s_{kl}(\phi_1, \ldots, \phi_n)\,. \tag{17}$$

The function $Q^s(\phi_1, \ldots, \phi_n, \epsilon, \tau)$ is 2π-periodic in regard to all the independent variables $\phi_1, \ldots, \phi_n$. Because for $\epsilon = 0$ the autonomous equations (11) are uncoupled, we take

$$Q^s_{00} = A_s \cos\phi_s \tag{18}$$

and, additionally, we choose such a family of quasi-periodic solutions that the following arbitrary condition [14] is fulfilled

$$\frac{\partial Q^s(0, \ldots, 0, \epsilon, \tau)}{\partial \phi_s} = 0\,, \qquad s = 1, \ldots, n\,. \tag{19}$$

We expand $\epsilon\eta$ into the double series

$$\eta_s(\epsilon, \tau) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \epsilon^k \tau^l \eta^s_{kl}\,. \tag{20}$$

Similarly, we develop λ_s^2 and ϵF_s.

After comparing the terms at parameters $\epsilon^k\tau^l$ we have obtained the following recurrent set of equations

$$\sum_{m=1}^{n}\sum_{r=1}^{n}\frac{\partial^2 Q_{10}^s}{\partial\phi_m\partial\phi_r}\omega_m\omega_r+\omega_s^2 Q_{10}^s=$$
$$=F_{10}^s(A_1\cos\phi_1,\ldots,A_n\cos\phi_n,-A_1\omega_1\sin\phi_1,\ldots,-A_n\omega_n\sin\phi_n)-2\omega_s\eta_{10}^s A_s\cos\phi_s\,,$$

$$\sum_{m=1}^{n}\sum_{r=1}^{n}\frac{\partial^2 Q_{20}^s}{\partial\phi_m\partial\phi_r}\omega_m\omega_r+\omega_s^2 Q_{20}^s=$$
$$=-2\omega_s\eta_{20}^s A_s\cos\phi_s-2\omega_s\eta_{10}^s(B_{10}^s\cos\phi_s+C_{10}^s\sin\phi_s)$$
$$+\sum_{l=1}^{n}\frac{\partial F_s}{\partial x_l}(B_{10}^l\cos\phi_l+C_{10}^l\sin\phi_l)+\sum_{l=1}^{n}\frac{\partial F_s}{\partial\dot{x}_l}(-B_{10}^l\sin\phi_l+C_{10}^l\cos\phi_l)\omega_l+\bar{F}_{20}^s\,,$$

$$\sum_{m=1}^{n}\sum_{r=1}^{n}\frac{\partial^2 Q_{30}^s}{\partial\phi_m\partial\phi_r}\omega_m\omega_r+\omega_s^2 Q_{30}^s= \tag{21}$$
$$=-2\omega_s\eta_{30}^s A_s\cos\phi_s-2\omega_s\eta_{10}^s(B_{20}^s\cos\phi_s+C_{20}^s\sin\phi_s)$$
$$+\sum_{l=1}^{n}\frac{\partial F_s}{\partial x_l}(B_{20}^l\cos\phi_l+C_{20}^l\sin\phi_l)+\sum_{l=1}^{n}\frac{\partial F_s}{\partial\dot{x}_l}(-B_{20}^l\sin\phi_l+C_{20}^l\cos\phi_l)\omega_l+\bar{F}_{30}^s\,,$$

$$\sum_{m=1}^{n}\sum_{r=1}^{n}\frac{\partial^2 Q_{11}^s}{\partial\phi_m\partial\phi_r}\omega_m\omega_r+\omega_s^2 Q_{11}^s=0,$$

$$\sum_{m=1}^{n}\sum_{r=1}^{n}\frac{\partial^2 Q_{21}^s}{\partial\phi_m\partial\phi_r}\omega_m\omega_r+\omega_s^2 Q_{21}^s=-2\omega_s\eta_{21}^s A_s\cos\phi_s+\bar{F}_{21}^s\,,$$

where:

$$F_{kl}^s=\int_0^{2\pi}\ldots\int_0^{2\pi}F_{kl}^s(\sin\phi_s+\cos\phi_s)\,\mathrm{d}\phi_1\ldots\mathrm{d}\phi_n+\bar{F}_{kl}^s\,,\qquad kl=10,\,20,\,30,\,21\,. \tag{22}$$

Based on the obtained recurrent set of equations (21) we now demonstrate an algorithm to obtain the solution. From the first equation of (21) we find

$$h_{10}^p(A_{10}^1\ldots,A_{10}^n)=\int_0^{2\pi}\ldots\int_0^{2\pi}F_{10}^p\sin\phi_p\,\mathrm{d}\phi_1\ldots\mathrm{d}\phi_n=0\,,$$
$$\eta_{10}^p=\frac{1}{\omega_p A_p^{10(0)}}\left(\frac{1}{2\pi}\right)^n\int_0^{2\pi}\ldots\int_0^{2\pi}F_{10}^p\cos\phi_p\,\mathrm{d}\phi_1\ldots\mathrm{d}\phi_n=0\,,\qquad p=1,\ldots,n\,. \tag{23}$$

We expand $\bar{F}_{10}^s$ into the following multiple series

$$\bar{F}_{10}^s=\sum_k^s R_{k,10}^s e^{i((k,\phi))}, \tag{24}$$

and from the first equation of (21) we find

$$Q_{10}^s=\bar{Q}_{10}^s+B_{10}^s\cos\phi_s+C_{10}^s\sin\phi_s\,,\qquad s=1,\ldots,n\,, \tag{25}$$

where:

$$\bar{Q}_{10}^s=\sum_k^s\frac{R_{k,10}^s}{\omega_s^2-((k,\omega))^2}e^{i((k,\phi))}. \tag{26}$$

From (18) we calculate C_{10}^s, $s=1,\ldots,n$.

In order to avoid resonance terms, from the second equation of (21) we obtain

$$\begin{aligned}
&\omega_s\eta_{10}^s C_{10}^s + \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi}\cdots\int_0^{2\pi}\left[\sum_{l=1}^n \frac{\partial F_s}{\partial x_l}(B_{10}^l\cos\phi_l + C_{10}^l\sin\phi_l)\right.\\
&\qquad\left.+\sum_{l=1}^n \frac{\partial F_s}{\partial \dot{x}_l}(-B_{10}^l\sin\phi_l + C_{10}^l\cos\phi_l)\omega_l + \bar{F}_{20}^s\right]\sin\phi_s\,\mathrm{d}\phi_1\ldots\mathrm{d}\phi_n = 0,\\
&-A_s\omega_s\eta_{20}^s - \omega_s\eta_{10}^s B_{10}^s + \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi}\cdots\int_0^{2\pi}\left[\sum_{l=1}^n \frac{\partial F_s}{\partial x_l}(B_{10}^l\cos\phi_l + C_{10}^l\sin\phi_l)\right.\\
&\qquad\left.+\sum_{l=1}^n \frac{\partial F_s}{\partial \dot{x}_l}(-B_{10}^l\sin\phi_l + C_{10}^l\cos\phi_l)\omega_l + \bar{F}_{20}^s\right]\cos\phi_s\,\mathrm{d}\phi_1\ldots\mathrm{d}\phi_n = 0,\\
&\qquad\qquad s = 1,\ldots,n,
\end{aligned}\tag{27}$$

and

$$\bar{F}_{20}^s = -(\eta_{10}^s)^2 A_s\cos\phi_s + \sum_{l=1}^s\left(\frac{\partial F_s}{\partial x_l}\bar{Q}_{10}^l + \frac{\partial F_s}{\partial \dot{x}_l}\sum_{j=1}^s\frac{\partial \bar{Q}_{10}^l}{\partial \phi_j}\omega_j\right) + \frac{\partial F_s}{\partial \epsilon} - 2\omega_s\eta_{10}^s\bar{Q}_{10}^s\,. \tag{28}$$

From the first equation of (27) we obtain $B_{10}^1,\ldots,B_{10}^n$ whereas from the second one $\eta_{20}^1,\ldots,\eta_{20}^n$.

As in the previous case, we expand

$$\bar{F}_{20}^s = \sum_k^s R_{k,20}^s\, e^{i((k,\phi))}, \tag{29}$$

and we find

$$Q_{20}^s = \bar{Q}_{20}^s + B_{20}^s\cos\phi_s + C_{20}^s\sin\phi_s\,, \qquad s = 1,\ldots,n, \tag{30}$$

where:

$$\bar{Q}_{20}^s = \sum_k^s \frac{R_{k,20}^s}{\omega_s^2 - ((k,\omega))^2}\, e^{i((k,\phi))}. \tag{31}$$

C_{20}^s are obtained from (18), whereas η_{30}^s and B_{20}^s are found from the following equation (analogous to (27))

$$\begin{aligned}
&\omega_s\eta_{10}^s C_{20}^s + \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi}\cdots\int_0^{2\pi}\left[\sum_{l=1}^n \frac{\partial F_s}{\partial x_l}(B_{20}^l\cos\phi_l + C_{20}^l\sin\phi_l)\right.\\
&\qquad\left.+\sum_{l=1}^n \frac{\partial F_s}{\partial \dot{x}_l}(-B_{20}^l\sin\phi_l + C_{20}^l\cos\phi_l)\omega_l + \bar{F}_{30}^s\right]\sin\phi_s\,\mathrm{d}\phi_1\ldots\mathrm{d}\phi_n = 0,\\
&-A_s\omega_s\eta_{30}^s - \omega_s\eta_{10}^s B_{20}^s + \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi}\cdots\int_0^{2\pi}\left[\sum_{l=1}^n \frac{\partial F_s}{\partial x_l}(B_{20}^l\cos\phi_l + C_{20}^l\sin\phi_l)\right.\\
&\qquad\left.+\sum_{l=1}^n \frac{\partial F_s}{\partial \dot{x}_l}(-B_{20}^l\sin\phi_l + C_{20}^l\cos\phi_l)\omega_l + \bar{F}_{30}^s\right]\cos\phi_s\,\mathrm{d}\phi_1\ldots\mathrm{d}\phi_n = 0,\\
&\qquad\qquad s = 1,\ldots,n,
\end{aligned}\tag{32}$$

where:

$$\begin{aligned}\bar{F}^s_{30} &= -2\eta^s_{10}\eta^s_{20}A_s\cos\phi_s - (\eta^s_{10})^2 - 2\omega_s\eta^s_{20}Q_{10} - 2\omega_s\eta^s_{10}\bar{Q}_{20} \\ &+ \sum_{l=1}^{n}\Big[\sum_{m=1}^{n}\frac{\partial^2 F_s}{\partial x_l\partial x_m}Q^l_{10}Q^m_{10} + \frac{\partial F_s}{\partial x_l}\bar{Q}^l_{20} + \frac{\partial^2 F_s}{\partial x_l\partial\epsilon}Q^l_{10} \\ &+ \sum_{m=1}^{n}\left(\frac{\partial^2 F_s}{\partial\dot{x}_l\partial\dot{x}_m}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial Q^l_{10}}{\partial\phi_j}\frac{\partial Q^m_{10}}{\partial\phi_i}\omega_i\omega_j + \frac{\partial F_s}{\partial\dot{x}_l}\sum_{j=1}^{n}\frac{\partial\bar{Q}^l_{20}}{\partial\phi_j}\omega_j + \frac{\partial^2 F_s}{\partial\dot{x}_l\partial\epsilon}\sum_{j=1}^{n}\frac{\partial Q^l_{10}}{\partial\phi_j}\omega_j\right) \\ &+ 2\sum_{m=1}^{n}\left(\frac{\partial^2 F_s}{\partial x_l\partial\dot{x}_m}Q^l_{10}\sum_{z=1}^{n}\frac{\partial Q^m_{10}}{\partial\phi_z}\omega_z\right)\Big] + \frac{\partial^2 F_s}{\partial\epsilon^2}\,. \end{aligned} \tag{33}$$

From the fourth equation of (21) we get

$$Q^s_{11} = B^s_{11}\cos\phi_s + C^s_{11}\sin\phi_s\,. \tag{34}$$

The C^s_{11} obtained from (18) are equal to zero, whereas η^s_{21} and B^s_{11} are found from the following equations

$$\begin{aligned} &\left(\frac{1}{2\pi}\right)^n\int_0^{2\pi}\!\!\cdots\int_0^{2\pi}\Big[\sum_{l=1}^{n}\frac{\partial F_s}{\partial x_l}B^l_{11}\cos\phi_l \\ &\qquad + \sum_{l=1}^{n}\frac{\partial F_s}{\partial\dot{x}_l}(-B^l_{11}\sin\phi_l)\omega_l + \bar{F}^s_{21}\Big]\sin\phi_s\,\mathrm{d}\phi_1\ldots\mathrm{d}\phi_n = 0\,, \\ &-A_s\omega_s\eta^s_{10} - \omega_s\eta^s_{21}B^s_{11} + \left(\frac{1}{2\pi}\right)^n\int_0^{2\pi}\!\!\cdots\int_0^{2\pi}\Big[\sum_{l=1}^{n}\frac{\partial F_s}{\partial x_l}B^l_{11}\cos\phi_l \\ &\qquad + \sum_{l=1}^{n}\frac{\partial F_s}{\partial\dot{x}_l}(-B^l_{11}\sin\phi_l)\omega_l + \bar{F}^s_{21}\Big]\cos\phi_s\,\mathrm{d}\phi_1\ldots\mathrm{d}\phi_n = 0\,, \\ &\qquad s = 1,\ldots,n\,, \end{aligned} \tag{35}$$

where:

$$\bar{F}^s_{21} = \sum_{l=1}^{n}\left(\frac{\partial^2 Q^l_{10}}{\partial x_l\partial\tau} + \frac{\partial^2 F_s}{\partial\dot{x}_l\partial\tau}\sum_{j=1}^{n}\frac{\partial Q^l_{10}}{\partial\phi_j}\omega_j\right). \tag{36}$$

2.2. Resonance case

Consider now a case, when a subset of frequencies $\omega^* = (\omega_1,\ldots,\omega_p)$ fulfils the following condition

$$|((\omega^*, k^*))| \geq 0\,, \tag{37}$$

where $k^* = (k_1,\ldots,k_p)$ is a vector of integer components and, additionally, the following resonances occur

$$\omega_s = ((\omega^*, k^{(s-p)}))\,, \qquad s = p+1,\ldots,n\,. \tag{38}$$

Similarly to the non-resonance case we have

$$\lambda_s = \omega_s + \eta_s(\epsilon,\tau)\,, \qquad s = 1,\ldots,n\,, \tag{39}$$

where

$$\eta_s(\epsilon,\tau) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\epsilon^k\tau^l\eta^s_{kl}\,, \qquad s = 1,\ldots,p\,. \tag{40}$$

Suppose that we have found $\eta(\epsilon,\tau)$ such that

$$x_s = Q^s(\phi_1,\ldots,\phi_p,\epsilon,\tau)\,, \qquad s=1,\ldots,n\,, \qquad \lambda_s \equiv \dot{\phi}_l = \omega_l + \eta_l\,, \qquad l=1,\ldots,p\,, \tag{41}$$

where:

$$\begin{aligned} Q^s(\phi_1,\ldots,\phi_n,\epsilon,\tau) &= Q_{00}(\phi_s) + \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\epsilon^k\tau^l Q^s_{kl}(\phi_1,\ldots,\phi_p)\,,\\ Q_{00}(\phi_s) &= A_s\cos\phi_s\,, \qquad s=1,\ldots,n\,. \end{aligned} \tag{42}$$

$Q^s_{kl}(\phi_1,\ldots,\phi_p)$ are periodic functions in regard to all independent variables $\phi_1,\ldots,\phi_p$. Similarly to the non-resonance case we take arbitrarily

$$\frac{\partial Q^s(0,\ldots,0,\epsilon,\tau)}{\partial\phi_s} = 0\,, \qquad s=1,\ldots,p\,. \tag{43}$$

For $s=p+1,\ldots,n$ we have

$$\phi_s = ((\phi^*, k^{(s-p)})) + \psi_{s-p}\,, \qquad \phi^* = (\phi_1^*,\ldots,\phi_p^*)\,. \tag{44}$$

We assume that small corrections of frequencies do not change the resonance order. From (39) we obtain

$$\begin{aligned} \lambda_1 &= \omega_1 + \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\epsilon^k\tau^l\eta^{(1)}_{kl}\,,\\ \lambda_p &= \omega_p + \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\epsilon^k\tau^l\eta^{(p)}_{kl}\,,\\ \lambda_{p+1} &= \omega_{p+1} + \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\epsilon^k\tau^l\eta^{(p+1)}_{kl}\,,\\ \lambda_n &= \omega_n + \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\epsilon^k\tau^l\eta^{(n)}_{kl}\,. \end{aligned} \tag{45}$$

Taking into account (38) and the given above assumption we get

$$\lambda_s = ((\lambda^*, k^{(s-p)}))\,, \tag{46}$$

$$\lambda^* = (\lambda_1^*,\ldots,\lambda_p^*)\,, \qquad s=p+1,\ldots,n\,. \tag{47}$$

From (45)–(47) we have

$$\begin{aligned} &\left(\left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\epsilon^k\tau^l\eta^{(1)}_{kl},\ldots,\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\epsilon^k\tau^l\eta^{(p)}_{kl}, k^{(1)}\right)\right) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\epsilon^k\tau^l\eta^{(p+1)}_{kl}\,,\\ &\vdots\\ &\left(\left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\epsilon^k\tau^l\eta^{(1)}_{kl},\ldots,\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\epsilon^k\tau^l\eta^{(p)}_{kl}, k^{(n-p)}\right)\right) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\epsilon^k\tau^l\eta^{(n)}_{kl}\,. \end{aligned} \tag{48}$$

Equations (48) show that $\eta^{(p+1)}_{kl},\ldots,\eta^{(n)}_{kl}$ are defined.

Comparing the terms standing at the parameters $\epsilon^k\tau^l$ we obtain the following system of recurrent equations

$$\begin{aligned} &\sum_{m=1}^{n}\sum_{r=1}^{n}\frac{\partial^2 Q^s_{10}}{\partial\phi_m\partial\phi_r}\omega_m\omega_r + \omega_s^2 Q^s_{10} =\\ &= \bar{F}^s_{10}(A_1\cos\phi_1,\ldots,A_n\cos\phi_n,-A_1\omega_1\sin\phi_1,\ldots,-A_n(\omega^*,k^{n-p})\sin\phi_n) - 2\omega_s\eta^s_{10}A_s\cos\phi_s\,,\\ &\qquad\qquad \text{where } \phi_s = (\phi^*, k^{(s-p)}),\ \text{for } p+1\le s\le n, \end{aligned}$$

$$\sum_{m=1}^{n}\sum_{r=1}^{n}\frac{\partial^2 Q_{20}^s}{\partial\phi_m\partial\phi_r}\omega_m\omega_r+\omega_s^2 Q_{20}^s =$$
$$= -2\omega_s\eta_{20}^s A_s\cos\phi_s - 2\omega_s\eta_{10}^s(B_{10}^s\cos\phi_s + C_{10}^s\sin\phi_s)$$
$$+\sum_{l=1}^{n}\frac{\partial F_s}{\partial x_l}(B_{10}^l\cos\phi_l + C_{10}^l\sin\phi_l)+\sum_{l=1}^{n}\frac{\partial F_s}{\partial\dot x_l}(-B_{10}^l\sin\phi_l + C_{10}^l\cos\phi_l)\omega_l+\bar F_{20}^s\,,$$
$$\text{for } 1\le l\le p,\quad 1\le s\le p,$$

$$\sum_{m=1}^{n}\sum_{r=1}^{n}\frac{\partial^2 Q_{20}^s}{\partial\phi_m\partial\phi_r}\omega_m\omega_r+\omega_s^2 Q_{20}^s =$$
$$= -2\omega_s\eta_{20}^s A_s\cos((\phi^*,k^{(s-p)})) - 2\omega_s\eta_{10}^s\left(B_{10}^s\cos((\phi^*,k^{(s-p)})) + C_{10}^s\sin((\phi^*,k^{(s-p)}))\right)$$
$$+\sum_{l=1}^{n}\frac{\partial F_s}{\partial x_l}\left(B_{10}^l\cos((\phi^*,k^{(l-p)})) + C_{10}^l\sin((\phi^*,k^{(l-p)}))\right)$$
$$+\sum_{l=1}^{n}\frac{\partial F_s}{\partial\dot x_l}\left(-B_{10}^l\sin((\phi^*,k^{(l-p)})) + C_{10}^l\cos((\phi^*,k^{(l-p)}))\right)\omega_l+F_{20}^s\,,$$
$$\text{for } p+1\le l\le n,\quad p+1\le s\le n,$$

$$\sum_{m=1}^{n}\sum_{r=1}^{n}\frac{\partial^2 Q_{30}^s}{\partial\phi_m\partial\phi_r}\omega_m\omega_r+\omega_s^2 Q_{30}^s =$$
$$= -2\omega_s\eta_{30}^s A_s\cos\phi_s - 2\omega_s\eta_{10}^s(B_{20}^s\cos\phi_s + C_{20}^s\sin\phi_s) \tag{49}$$
$$+\sum_{l=1}^{n}\frac{\partial F_s}{\partial x_l}(B_{20}^l\cos\phi_l + C_{20}^l\sin\phi_l)+\sum_{l=1}^{n}\frac{\partial F_s}{\partial\dot x_l}(-B_{20}^l\sin\phi_l + C_{20}^l\cos\phi_l)\omega_l+\bar F_{30}^s\,,$$
$$\text{for } 1\le l\le p,\quad 1\le s\le p,$$

$$\sum_{m=1}^{n}\sum_{r=1}^{n}\frac{\partial^2 Q_{30}^s}{\partial\phi_m\partial\phi_r}\omega_m\omega_r+\omega_s^2 Q_{30}^s =$$
$$= -2\omega_s\eta_{30}^s A_s\cos((\phi^*,k^{(s-p)})) - 2\omega_s\eta_{10}^s\left(B_{20}^s\cos((\phi^*,k^{(s-p)})) + C_{20}^s\sin((\phi^*,k^{(s-p)}))\right)$$
$$+\sum_{l=1}^{n}\frac{\partial F_s}{\partial x_l}\left(B_{20}^l\cos((\phi^*,k^{(l-p)})) + C_{20}^l\sin((\phi^*,k^{(l-p)}))\right)$$
$$+\sum_{l=1}^{n}\frac{\partial F_s}{\partial\dot x_l}\left(-B_{20}^l\sin((\phi^*,k^{(l-p)})) + C_{20}^l\cos((\phi^*,k^{(l-p)}))\right)\omega_l+\bar F_{30}^s\,,$$
$$\text{for } p+1\le l\le n,\quad p+1\le s\le n,$$

$$\sum_{m=1}^{n}\sum_{r=1}^{n}\frac{\partial^2 Q_{11}^s}{\partial\phi_m\partial\phi_r}\omega_m\omega_r+\omega_s^2 Q_{11}^s = 0\,,$$

$$\sum_{m=1}^{n}\sum_{r=1}^{n}\frac{\partial^2 Q_{21}^s}{\partial\phi_m\partial\phi_r}\omega_m\omega_r+\omega_s^2 Q_{21}^s = -2\omega_s\eta_{21}^s A_s\cos\phi_s+\bar F_{21}^s\,,$$
$$\text{for } 1\le l\le p,\quad 1\le s\le p,$$

$$\sum_{m=1}^{n}\sum_{r=1}^{n}\frac{\partial^2 Q_{21}^s}{\partial\phi_m\partial\phi_r}\omega_m\omega_r+\omega_s^2 Q_{21}^s = -2\omega_s\eta_{21}^s A_s\cos((\phi^*,k^{(s-p)}))+\bar F_{21}^s\,,$$
$$\text{for } p+1\le l\le n,\quad p+1\le s\le n.$$

In a manner similar to that discussed for the non-resonance case we obtain the following equations

$$
\begin{aligned}
&G^s_{10}(A_1,\ldots,A_n,\psi^1_{10},\ldots,\psi^{n-p}_{10}) = \int_0^{2\pi}\cdots\int_0^{2\pi} F^s_{10}\sin\phi_s \,\mathrm{d}\phi_1\ldots\mathrm{d}\phi_{p1} = 0\,,\\
&H^s_{10}(A_1,\ldots,A_n,\psi^1_{10},\ldots,\psi^{n-p}_{10},\eta^s_{10},\ldots,\eta^p_{10}) =\\
&\quad = \omega_s\eta^s_{10}A_s - \left(\frac{1}{2\pi}\right)^p\int_0^{2\pi}\cdots\int_0^{2\pi} F^s_{10}\cos\phi_s\,\mathrm{d}\phi_1\ldots\mathrm{d}\phi_{p1} = 0\,,\\
&G^s_{20}(B^1_{10},\ldots,B^n_{10},\psi^1_{20},\ldots,\psi^{n-p}_{20}) =\\
&\quad = -\omega_s\eta^s_{10}C^s_{10} + \left(\frac{1}{2\pi}\right)^n\int_0^{2\pi}\cdots\int_0^{2\pi}\Bigg[\sum_{l=1}^n\frac{\partial F_s}{\partial x_l}(B^l_{10}\cos\phi_l + C^l_{10}\sin\phi_l)\\
&\qquad + \sum_{l=1}^n\frac{\partial F_s}{\partial\dot{x}_l}(-B^l_{10}\sin\phi_l + C^l_{10}\cos\phi_l)\omega_l + \bar{F}^s_{20}\Bigg]\sin\phi_s\,\mathrm{d}\phi_1\ldots\mathrm{d}\phi_{p1} = 0\,,\\
&H^s_{20}(B^1_{10},\ldots,B^n_{10},\psi^1_{20},\ldots,\psi^{n-p}_{20},\eta^1_{20},\ldots,\eta^p_{20}) =\\
&\quad = -\omega_s\eta^s_{20}A_s - \omega_s\eta^s_{10}B^s_{10} + \left(\frac{1}{2\pi}\right)^n\int_0^{2\pi}\cdots\int_0^{2\pi}\Bigg[\sum_{l=1}^n\frac{\partial F_s}{\partial x_l}(B^l_{10}\cos\phi_l + C^l_{10}\sin\phi_l)\\
&\qquad + \sum_{l=1}^n\frac{\partial F_s}{\partial\dot{x}_l}(-B^l_{10}\sin\phi_l + C^l_{10}\cos\phi_l)\omega_l + \bar{F}^s_{20}\Bigg]\cos\phi_s\,\mathrm{d}\phi_1\ldots\mathrm{d}\phi_{p1} = 0\,,\\
&G^s_{30}(B^1_{20},\ldots,B^n_{20},\psi^1_{30},\ldots,\psi^{n-p}_{30}) =\\
&\quad = -\omega_s\eta^s_{10}C^s_{20} + \left(\frac{1}{2\pi}\right)^n\int_0^{2\pi}\cdots\int_0^{2\pi}\Bigg[\sum_{l=1}^n\frac{\partial F_s}{\partial x_l}(B^l_{20}\cos\phi_l + C^l_{20}\sin\phi_l) \qquad (50)\\
&\qquad + \sum_{l=1}^n\frac{\partial F_s}{\partial\dot{x}_l}(-B^l_{20}\sin\phi_l + C^l_{20}\cos\phi_l)\omega_l + \bar{F}^s_{30}\Bigg]\sin\phi_s\,\mathrm{d}\phi_1\ldots\mathrm{d}\phi_{p1} = 0\,,\\
&H^s_{30}(B^1_{20},\ldots,B^n_{20},\psi^1_{30},\ldots,\psi^{n-p}_{30},\eta^1_{30},\ldots,\eta^p_{30}) =\\
&\quad = -\omega_s\eta^s_{30}A_s - \omega_s\eta^s_{10}B^s_{20} + \left(\frac{1}{2\pi}\right)^n\int_0^{2\pi}\cdots\int_0^{2\pi}\Bigg[\sum_{l=1}^n\frac{\partial F_s}{\partial x_l}(B^l_{20}\cos\phi_l + C^l_{20}\sin\phi_l)\\
&\qquad + \sum_{l=1}^n\frac{\partial F_s}{\partial\dot{x}_l}(-B^l_{20}\sin\phi_l + C^l_{20}\cos\phi_l)\omega_l + \bar{F}^s_{30}\Bigg]\cos\phi_s\,\mathrm{d}\phi_1\ldots\mathrm{d}\phi_{p1} = 0\,,\\
&G^s_{21}(B^1_{11},\ldots,B^n_{11},\psi^1_{21},\ldots,\psi^{n-p}_{21}) =\\
&\quad = \left(\frac{1}{2\pi}\right)^n\int_0^{2\pi}\cdots\int_0^{2\pi}\Bigg[\sum_{l=1}^n\frac{\partial F_s}{\partial x_l}B^l_{11}\cos\phi_l\\
&\qquad + \sum_{l=1}^n\frac{\partial F_s}{\partial\dot{x}_l}(-B^l_{11}\sin\phi_l)\omega_l + \bar{F}^s_{21}\Bigg]\sin\phi_s\,\mathrm{d}\phi_1\ldots\mathrm{d}\phi_{p1} = 0\,,\\
&H^s_{21}(B^1_{11},\ldots,B^n_{11},\psi^1_{21},\ldots,\psi^{n-p}_{21},\eta^1_{21},\ldots,\eta^{n-p}_{21}) =\\
&\quad = -\omega_s\eta^s_{21}A_s - \omega_s\eta^s_{21}B^s_{11} + \left(\frac{1}{2\pi}\right)^n\int_0^{2\pi}\cdots\int_0^{2\pi}\Bigg[\sum_{l=1}^n\frac{\partial F_s}{\partial x_l}B^l_{11}\cos\phi_l\\
&\qquad + \sum_{l=1}^n\frac{\partial F_s}{\partial\dot{x}_l}(-B^l_{11}\sin\phi_l)\omega_l + \bar{F}^s_{21}\Bigg]\cos\phi_s\,\mathrm{d}\phi_1\ldots\mathrm{d}\phi_{p1} = 0\,,\\
&\qquad\qquad s = 1,\ldots,n\,.
\end{aligned}
$$

During the calculations we have expanded the rest of the right hand side of (49) into the following Fourier series

$$\bar{F}^s_{kl} = R^s_{k^*,kl} \sum_{k^*}^{s} e^{i((k^*,\phi^*))} , \tag{51}$$

and we have obtained the following solution

$$\begin{aligned} Q^s_{kl} &= \bar{Q}^s_{kl}(\phi^*) + B^s_{kl}\cos\phi_s + C^s_{kl}\sin\phi_s , && \text{for } 1 \le s \le p , \\ Q^s_{kl} &= \bar{Q}^s_{kl}(\phi^*) + B^s_{kl}\cos((\phi^*, k^{s-p})) + C^s_{kl}\sin((\phi^*, k^{s-p})) , && \text{for } p+1 \le s \le n , \end{aligned} \tag{52}$$

where:

$$\bar{Q}^s_{kl}(\phi^*) = \sum_{k^*}^{s} \frac{R^s_{k^*,kl}}{\omega_s^2 - ((k^*,\omega^*))^2} e^{i((k^*,\phi^*))} . \tag{53}$$

From the first equations of (49) we have obtained $2n$ equations to find $2n$ unknown quantities $A_1, \ldots, A_n$, $\psi^1_{10}, \ldots, \psi^{n-p}_{10}$, $\eta^1_{10}, \ldots, \eta^p_{10}$. Then, from the next recurrent equations we have found $B^1_{kl}, \ldots, B^n_{kl}$, $\psi^1_{kl}, \ldots, \psi^{n-p}_{kl}$, $\eta^1_{kl}, \ldots, \eta^p_{kl}$ and $C^1_{kl}, \ldots, C^n_{kl}$.

2.3. Catastrophes

The presented formal analytical approach to construct a quasi-periodic solution allows for practical application, as well as for discovery of singularities and branching phenomena.

In many branches of practical application the problem of control of the new frequencies λ_s, $s = 1, \ldots, n$, arises. Suppose that we require the following conditions to be fulfilled

$$\lambda_s = r_s \omega_s , \qquad s = 1, \ldots, n . \tag{54}$$

From (11) we obtain

$$g_s(\epsilon, \tau, A_1, \ldots, A_n, \Lambda_1, \ldots, \Lambda_m) = \epsilon \eta^s_{kl}(\epsilon, \tau, A_1, \ldots, A_n, \Lambda_1, \ldots, \Lambda_m) = 0 . \tag{55}$$

The amplitudes $A_1, \ldots, A_n$ are generated by the functions F_s and a set of parameters $(\Lambda_1, \ldots, \Lambda_m)$ attached to them. For some values of $(\epsilon, \tau, A_1, \ldots, A_n, \Lambda_1, \ldots, \Lambda_m)$ singularities can appear. We define them as frequency-catastrophes by the following equations

$$\begin{aligned} h^s_{10}(A_1, \ldots, A_n, \Lambda_1, \ldots, \Lambda_m) &= 0 , \\ g_s(\epsilon, \tau, A_1, \ldots, A_n, \Lambda_1, \ldots, \Lambda_m) &= 0 , \\ \frac{\partial g_s}{\partial A_s}(\epsilon, \tau, A_1, \ldots, A_n, \Lambda_1, \ldots, \Lambda_m) &= 0 , \\ s = 1, \ldots, n . \end{aligned} \tag{56}$$

From (56), one or few catastrophe sets $(\epsilon^c, \tau^c, \Lambda^c_1, \ldots, \Lambda^c_m)$ can be found, where τ is treated as a branching parameter. The successfully found sets (if any) can further be divided. For example, the n-hysteresis variety [40] is defined by the set

$$\begin{aligned} h^s_{10}(A_1, \ldots, A_n, \Lambda_1, \ldots, \Lambda_m) &= 0 , \\ g_s(\epsilon, \tau, A_1, \ldots, A_n, \Lambda_1, \ldots, \Lambda_m) &= 0 , \\ \frac{\partial g_s}{\partial A_s}(\epsilon, \tau, A_1, \ldots, A_n, \Lambda_1, \ldots, \Lambda_m) &= 0 , \\ \frac{\partial^2 g_s}{\partial A_s^2}(\epsilon, \tau, A_1, \ldots, A_n, \Lambda_1, \ldots, \Lambda_m) &= 0 , \\ s = 1, \ldots, n . \end{aligned} \tag{57}$$

The n-isola centers [40] are defined by the following equations

$$
\begin{aligned}
& h^s_{10}(A_1,\ldots,A_n,\Lambda_1,\ldots,\Lambda_m) && = 0\,,\\
& g_s(\epsilon,\tau,A_1,\ldots,A_n,\Lambda_1,\ldots,\Lambda_m) && = 0\,,\\
& \frac{\partial g_s}{\partial A_s}(\epsilon,\tau,A_1,\ldots,A_n,\Lambda_1,\ldots,\Lambda_m) && = 0\,,\\
& \frac{\partial g_s}{\partial \tau}(\epsilon,\tau,A_1,\ldots,A_n,\Lambda_1,\ldots,\Lambda_m) && = 0\,,\\
& \qquad\qquad s=1,\ldots,n\,.
\end{aligned}
\tag{58}
$$

A similar classification to that given above can be introduced on the base of the vector of independent variables $\phi_1,\ldots,\phi_n$, referred further as the variables of torus. We define the variable-catastrophes [40] by the following equations

$$
\begin{aligned}
& h^s_{10}(A_1,\ldots,A_n,\Lambda_1,\ldots,\Lambda_m) && = 0\,,\\
& Q_s(\epsilon,\tau,\phi_1,\ldots,\phi_n,A_1,\ldots,A_n,\Lambda_1,\ldots,\Lambda_m) && = 0\,,\\
& \frac{\partial Q_s}{\partial \phi_s}(\epsilon,\phi_1,\ldots,\phi_n,\tau,A_1,\ldots,A_n,\Lambda_1,\ldots,\Lambda_m) && = 0\,,\\
& \qquad\qquad s=1,\ldots,n\,.
\end{aligned}
\tag{59}
$$

The variables n-hysteresis variety [40] is defined by the equations

$$
\begin{aligned}
& h^s_{10}(A_1,\ldots,A_n,\Lambda_1,\ldots,\Lambda_m) && = 0\,,\\
& Q_s(\epsilon,\tau,\phi_1,\ldots,\phi_n,A_1,\ldots,A_n,\Lambda_1,\ldots,\Lambda_m) && = 0\,,\\
& \frac{\partial Q_s}{\partial \phi_s}(\epsilon,\phi_1,\ldots,\phi_n,\tau,A_1,\ldots,A_n,\Lambda_1,\ldots,\Lambda_m) && = 0\,,\\
& \frac{\partial^2 Q_s}{\partial \phi_s^2}(\epsilon,\phi_1,\ldots,\phi_n,\tau,A_1,\ldots,A_n,\Lambda_1,\ldots,\Lambda_m) && = 0\,,\\
& \qquad\qquad s=1,\ldots,n\,.
\end{aligned}
\tag{60}
$$

The variables n-isola centers [40] are governed by the equations

$$
\begin{aligned}
& h^s_{10}(A_1,\ldots,A_n,\Lambda_1,\ldots,\Lambda_m) && = 0\,,\\
& Q_s(\epsilon,\tau,\phi_1,\ldots,\phi_n,A_1,\ldots,A_n,\Lambda_1,\ldots,\Lambda_m) && = 0\,,\\
& \frac{\partial Q_s}{\partial \phi_s}(\epsilon,\phi_1,\ldots,\phi_n,\tau,A_1,\ldots,A_n,\Lambda_1,\ldots,\Lambda_m) && = 0\,,\\
& \frac{\partial Q_s}{\partial \tau}(\epsilon,\phi_1,\ldots,\phi_n,\tau,A_1,\ldots,A_n,\Lambda_1,\ldots,\Lambda_m) && = 0\,,\\
& \qquad\qquad s=1,\ldots,n\,.
\end{aligned}
\tag{61}
$$

The catastrophes defined by the Eqs. (56)–(60) can be found by the use of numerical methods. The parameters set $(\epsilon,\tau,\Lambda_1,\ldots,\Lambda_m)$, constituting the solution of the problem is known as a catastrophes set. There are generally three possibilities:

1. The catastrophes equations have no solutions in the considered space of parameters. It means that in the torus analyzed the catastrophes do not appear.

2. The catastrophes equations are satisfied for the arbitrary values of $A_1,\ldots,A_n$ and/or $\phi_1,\ldots,\phi_n$ for a certain set of $(\epsilon,\tau,\Lambda_1,\ldots,\Lambda_m)$.

3. There are one or few isolated solutions for a certain sets of $(\epsilon,\tau,\Lambda_1,\ldots,\Lambda_m)$.

With the use of the above given definitions and considerations one can construct (for a particular case) many special multi-diagrams in the higher-dimensional hyper-space $(\epsilon, \tau, \Lambda_1, \ldots, \Lambda_m)$. Then, by the use of parameter τ, such catastrophes can be controlled.

Finally, we briefly consider another break down of the quasi-periodic solution. Taking into account (12) we calculate

$$((k,\lambda)) = \alpha_1 + ((k, \epsilon\eta(\epsilon,\tau))) \tag{62}$$

where:

$$\alpha_1 = ((k,\omega))\,. \tag{63}$$

We define the break-down of the quasi-periodic solution by the following equation

$$((k,\lambda)) = 0\,. \tag{64}$$

For the given ϵ we can find a critical value of the control parameter τ for which (64) holds. Thus, due to (3.11), the domain of existence of the quasi-periodic solutions in two parameter space is established.

2.4. Example

We would like to support the above given theoretical results presenting an example. From (12) and for $s = 1, 2$ we get

$$\begin{aligned} \lambda_1 - \omega_1 &= \epsilon\eta_1(A_1, A_2, \epsilon, \mu, P^{(1)})\,, \\ \lambda_2 - \omega_2 &= \epsilon\eta_2(A_1, A_2, \epsilon, \mu, P^{(2)})\,, \end{aligned} \tag{65}$$

where A_1, A_2 are the amplitudes, ϵ, μ are the perturbation parameters, while $P^{(1)}$ and $P^{(2)}$ are the vectors of parameters. Suppose that we want to keep the λ_i $(i = 1, 2)$ constant treating ϵ, μ as the variables. We have to solve first the nonlinear equations leading to determination of the unknown A_1 and A_2, which are the functions of parameters (usually we solve the equations numerically). Therefore, the problem can be reduced to the consideration of the following equations

$$\begin{aligned} \omega_1^* &= \epsilon\eta_1(\epsilon, \mu, P^{(1)})\,, \\ \omega_2^* &= \epsilon\eta_2(\epsilon, \mu, P^{(2)})\,, \end{aligned} \tag{66}$$

where: $\omega_1^* = \lambda_1 - \omega_1$ and $\omega_2^* = \lambda_2 - \omega_2$.

Let us discuss briefly the typical issues of the catastrophe theory and how they relate to our present example. It should be emphasized that this theory refers to the Taylor expansion near the equilibrium surface. The vanishing of certain terms from the Taylor expansion can be understood as a singularity leading to the catastrophe. In our approach, instead of the real coordinates, we have perturbation parameters which possess physical meaning (they are combinations of the physical quantities). Additionally, our example results from an utilization of the averaging procedure. Here the catastrophes can be interpreted in relation to the frequency surfaces. We show that each of equations (66) can exhibit at least elementary catastrophes introduced by Thom [42]. If there exists a set of parameters for which Eqs. (66) have (one or more than one) solutions we will call it a branching set. In general, each of Eqs. (66) can be represented by the following polynomials

$$\begin{aligned} &p^{(i)}_{(6,0)}\epsilon^6 + p^{(i)}_{(5,0)}\epsilon^5 + p^{(i)}_{(4,0)}\epsilon^4 + p^{(i)}_{(3,0)}\epsilon^3 + p^{(i)}_{(2,0)}\epsilon^2 + p^{(i)}_{(1,0)}\epsilon \\ &\quad + p^{(i)}_{(5,1)}\epsilon^5\mu + p^{(i)}_{(4,1)}\epsilon^4\mu + p^{(i)}_{(3,1)}\epsilon^3\mu + p^{(i)}_{(2,1)}\epsilon^2\mu + p^{(i)}_{(1,1)}\epsilon\mu \\ &\quad + p^{(i)}_{(4,2)}\epsilon^4\mu^2 + p^{(i)}_{(3,2)}\epsilon^3\mu^2 + p^{(i)}_{(2,2)}\epsilon^2\mu^2 + p^{(i)}_{(1,2)}\epsilon\mu^2 + p^{(i)}_{(3,3)}\epsilon^3\mu^3 + p^{(i)} = 0\,, \end{aligned} \tag{67}$$

where: $p^{(i)} = -\omega_i^*$, $i = 1, 2$.

According to Thompson [43], if $\mu = 0$, whereas only $p^{(i)}_{(2,0)} \neq 0$ and $p^{(i)} \neq 0$, we get the fold catastrophe and a point $p^{(i)} = \epsilon = 0$ is the turning point.

For $p^{(i)}_{(3,0)} \neq 0$, $p^{(i)}_{(2,0)} \neq 0$, $p^{(i)} \neq 0$, we get a cusp manifold, which can be symmetric stable or unstable.

For $\mu = 0$, $p^{(i)}_{(4,0)} \neq 0$, $p^{(i)}_{(2,0)} \neq 0$ and $p^{(i)} \neq 0$, a swallow-tail catastrophe can be realized.

Additionally, thanks to the occurrence of the non-zero parameter μ, a set of two other catastrophes can be detected.

For $p^{(i)}_{(4,0)} > 0$, $p^{(i)}_{(2,2)} < 0$, $p^{(i)}_{(3,0)} > 0$, $p^{(i)}_{(1,2)} > 0$, $p^{(i)}_{(2,0)} < 0$, $p^{(i)}_{(1,1)} < 0$, the elliptic umbilic is realized, whereas for $p^{(i)}_{(5,0)} > 0$, $p^{(i)}_{(3,1)} > 0$, $p^{(i)}_{(3,0)} > 0$, $p^{(i)}_{(1,2)} > 0$, $p^{(i)}_{(2,0)} < 0$, $p^{(i)}_{(1,1)} < 0$, the parabolic umbilic occurs. (Note that in all cases considered the non-mentioned parameters are equal to zero. Pictures which characterize an appropriate catastrophe can be found in references [42, 43].)

One can imagine that if one of equations can exhibit one of the freely chosen catastrophe, then it can be tied to each of the potential catastrophes. This leads to the complicated branches of manifolds in a control space of parameters $P^{(i)}$ and the parameters ϵ, μ play the role of incremental coordinates.

3. Numerical method

We consider now a general numerical technique for tracking the quasi-periodic solutions using a generalized Poincaré map approach. The analysis introduced below is independent of the analytical methods considered before. However, if for some values of control parameters catastrophes appear, the they can be analyzed with the use of analytical methods described earlier. Thus, both of the methods presented support each other in finding a final solution to the problem.

3.1. Theory

Let us consider the following dynamical system

$$\frac{dx}{dt} = F(x), \qquad F: R^n \to R^n. \tag{68}$$

Samoilenko [39] showed that a quasi-periodic orbit lies on an invariant manifold M, which can be covered by the points $x_i \in M$. This manifold can be approximated by the n-dimensional balls situated in the points x_i with the radii r. An m-dimensional invariant torus $m = \{(\phi, y),\ \ y \in R^n,\ \ \phi \in R^m\}$ is approximated by the equation

$$y = y(\phi). \tag{69}$$

In what follows, we get

$$\frac{\partial y}{\partial t} + \sum_{k=1}^{m} \Omega_k(\phi, y) \frac{\partial y}{\partial \phi_k} = F(\phi, y) \tag{70}$$

with the following boundary conditions

$$y(\phi_1, \ldots, \phi_{j-1}, 0, \phi_{j+1}, \ldots, \phi_m) = y(\phi_1, \ldots, \phi_{j-1}, 2\Pi, \phi_{j+1}, \ldots, \phi_m). \tag{71}$$

Let us now introduce the variable $\phi_k = t$, and the period $T = 2\Pi$. If we can eliminate all variables ϕ_j, $j = 1, 2, \ldots, m-1$ except ϕ_k, then

$$y(0, \ldots, 0, t_0, 0, \ldots, 0) = y(0, \ldots, 0, t_0 + T, 0, \ldots, 0). \tag{72}$$

Therefore, based on (70), we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}t} = F(t, y) \tag{73}$$

with the boundary condition

$$y(t_0) = y(t_0 + T). \tag{74}$$

In order to give more light to the above formal consideration we investigate the case of $m = 2$. Having two incommensurable frequencies ω_1 and ω_2 with the corresponding periods $T_i = \frac{2\Pi}{\omega_i}$ $(i = 1, 2)$ we get

$$x(\omega_1 t_k, \omega_2 t_k) = x\left(2\Pi k,\ \omega_2\left(t_k - \mathrm{INT}\left(\frac{t_k}{T_2}\right)T_2\right)\right) = x\left(0,\ \frac{2\Pi}{T_2}\mathrm{MOD}(t_k, T_2)\right) =$$
$$= x\left(0,\ 2\Pi\frac{t_k}{T_2} - 2\Pi\,\mathrm{INT}\left(\frac{t_k}{T_2}\right)\right) = x\left(0,\ 2\Pi\frac{t_k}{T_2}\right) = x\left(0,\ k\frac{\omega_2}{\omega_1}2\Pi\right) = x(0, \tau_k), \quad x \in R^n. \tag{75}$$

We can use a continued fraction expansion in approximation of irrational numbers (winding number $\omega = \frac{\omega_2}{\omega_1}$) by rational numbers

$$\frac{r_l}{s_l} = [\omega_1, \omega_2, \ldots, \omega_l], \tag{76}$$

where ω_i are integers. With enough accuracy of $\frac{r_l}{s_l}$ we get $x(0,0) = x(2\Pi s_l, 2\Pi r_l) = x_0$ for $k = s_l$. In order to use numerical spline interpolation we rescale the map to get period $T = 1$, and the spline approximation is applied to the points τ_k very close to 1. Knowing ω_1 and ω_2 we can very easy find r_l and s_l and the fixed point x_0 of the above described Poincaré map. This is very adequate when starting with nonautonomous system. In order to get a fixed point of the map (75) a spline interpolation is recommended and a shooting method can be used to fulfil the boundary conditions.

Finally, we propose here a route from a one-dimensional curve $x(\tau_i) = x(i)$ to the two-dimensional torus S according to the formula

$$S \equiv (\mathrm{MOD}(i, s_l) + 1, \mathrm{MOD}(i, r_l) + 1) = x_i, \qquad i = 1, 2, \ldots \tag{77}$$

Above, s_l corresponds to ϕ_1 and r_l corresponds to ϕ_2. On the boundaries of S we have periodic solutions.

3.2. Example

As an example we consider a two-dimensional torus with $\omega_1 = \sqrt{2}$ and $\omega_2 = 1$ and a solution is governed by the equation

$$x(\phi_1, \phi_2) = \cos t + \cos\sqrt{2}t. \tag{78}$$

This solution can be found in one degree-of-freedom quasi-periodically excited mechanical systems with the periods $T_i = \frac{2\Pi}{\omega_i}$ $(i = 1, 2)$ and with positive damping. One of such dynamical systems has been considered by Kaas-Petersen [26]. In order to check a validity of the introduced theory we omit here an unsteady process leading to the steady state (78) and we introduce the following denotations:

$$\begin{aligned} \tau_k &= 1 + \bar{t}_k, \qquad \text{if} \quad 0 \le t_k < \frac{1}{2}, \\ \tau_k &= \bar{t}_k, \qquad \text{if} \quad \frac{1}{2} \le t_k < 1, \end{aligned} \tag{79}$$

where:

$$\tau_k = \left(\frac{t_k}{T_2}\right)\mathrm{MOD}\,1 = \left(k\frac{T_1}{T_2}\right)\mathrm{MOD}\,1 = \left(k\frac{\omega_2}{\omega_1}\right)\mathrm{MOD}\,1. \tag{80}$$

In order to apply the spline interpolation we use the following points: $\tau_4 = 0.8284268$, $\tau_7 = 0.94974$, $\tau_{10} = 1.0710675$, $\tau_{13} = 1.1923877$. The fixed point of the generalized Poincaré map is found to be $(x_0 = 1.99,\ \dot{x}_0 = -0.0004)$, which is a very good result in comparison to the exact solution of $(x_0 = 2.0,\ \dot{x}_0 = 0.0)$. The stroboscopic function can be obtained in this simple case analytically and it is governed by

$$x(\bar{t}_k) = 1 + \cos(2\Pi \bar{t}_k)\,, \qquad \dot{x}(\bar{t}_k) = -\sin(2\Pi \bar{t}_k)\,. \tag{81}$$

The computational results are shown in Fig. 1 and they are in a very good agreement with the results presented in the paper [26]. All numerical computations were conducted using the FORTRAN routines from the IMSL-library.

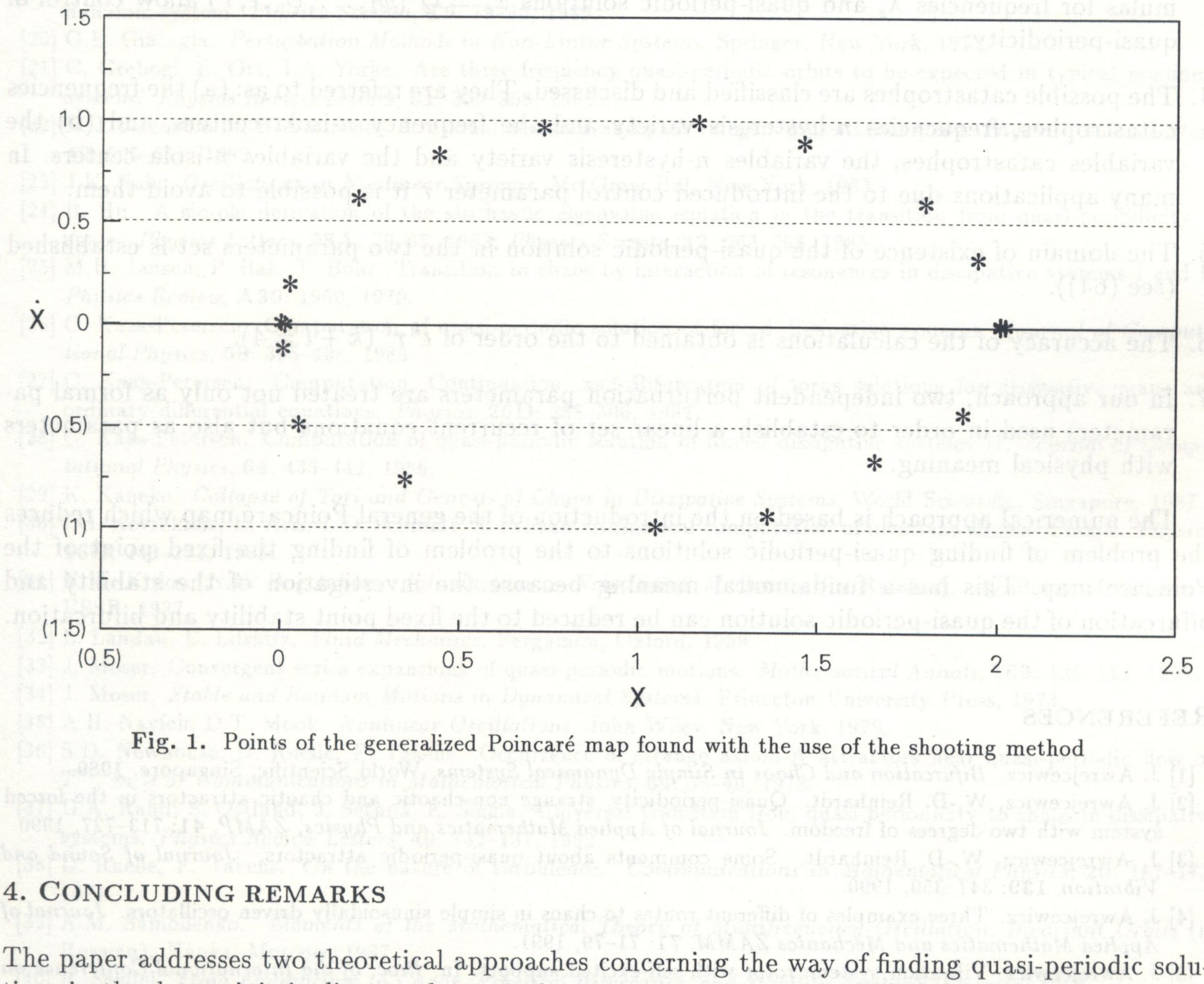

Fig. 1. Points of the generalized Poincaré map found with the use of the shooting method

4. Concluding remarks

The paper addresses two theoretical approaches concerning the way of finding quasi-periodic solutions in the deterministic discrete dynamical systems, i.e. the analytical and numerical techniques. Both of the methods are supported with the computer assisted studies. In the first case symbolic computational analysis (*Mathematica* package) is conducted, whereas in the second case the FORTRAN program has been developed on the basis of the IMSL-library routines.

The analytical approach is based on the double perturbation method, when one of the perturbation parameters (ϵ) is strictly connected with the system, and the other (τ) is treated as a control, or, in the sense of singularity theory, as the bifurcation parameter. It should be pointed out that instead of delay some other parameters can be used as control parameters — particularly those whose small changes cause large changes in the phase flow. The unknown quasi-periodic solutions x_s and the corresponding set of frequencies λ_s is sought in the form of double asymptotically convergent series of the powers of the parameters ϵ and τ. Such an approach possesses some benefits in comparison with the others described in Introduction. They will be now briefly summarized:

1. The method is applied to the set of second-order differential equations. Such equations are derived from the Newton laws or Lagrange equations and they govern the dynamics of almost all physical systems with inertia. There is, however, one limitation. Usually, linear parts of (9) are coupled because of damping. Only for the cases with small enough damping coefficients it is possible to find the transformations leading to (9).

2. This approach can be treated as an extension of the classical single perturbation parameter method applied to periodic orbits and widely discussed in the literature [20, 35]. For the control parameter $\tau = 0$ it is reduced to a method applied to the construction of a quasi-periodic solution [17, 22].

3. Both resonance and non-resonance cases are successfully analyzed. The explicitly given formulas for frequencies λ_s and quasi-periodic solutions $x_s = Q^s(\phi_1, \ldots, \phi_n, \epsilon, \tau)$ allow control of quasi-periodicity.

4. The possible catastrophes are classified and discussed. They are referred to as: (a) the frequencies catastrophes, frequencies n-hysteresis variety and the frequency n-isola centers, and (b) the variables catastrophes, the variables n-hysteresis variety and the variables n-isola centers. In many applications due to the introduced control parameter τ it is possible to avoid them.

5. The domain of existence of the quasi-periodic solution in the two parameters set is established (see (64)).

6. The accuracy of the calculations is obtained to the order of $\epsilon^k \tau^l$ $(k + l = 4)$.

7. In our approach, two independent perturbation parameters are treated not only as formal parameters used in order to establish a linear set of recurrent equations but also as parameters with physical meaning.

The numerical approach is based on the introduction of the general Poincaré map which reduces the problem of finding quasi-periodic solutions to the problem of finding the fixed point of the Poincaré map. This has a fundamental meaning because the investigation of the stability and bifurcation of the quasi-periodic solution can be reduced to the fixed point stability and bifurcation.

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