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Computer analysis of slackened-elastic-plastic beams under non-proportional loads

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A computer method for elastic-plastic continuous beams with rotation constraints is proposed. Such structures belong to a particular class of slackened systems, i.e. systems with gaps at structural joints. The mathematical model of slackened structures represents a discrete form of the Kuhn-Tucker's conditions, and is equivalent to dual Quadratic Programming Problems (QPPs). The uniqueness of a solution to the problem of the beams under considerations is assured, excluding cases where the structure converts into a mechanism, and the solution corresponds to dual Linear Programming Problems (LPPs). In order to calculate both the structure and mechanism a concept of the finite element with elastic rotation supports is used in the computer program. The uniqueness of solutions makes it possible to use a pure elastic analysis with some additional constraints superimposed on state variables. It allows us to avoid the time-consuming mathematical programming methods. Several examples illustrate the behaviour of beams under multiparameter loads. Results relate the cyclic loading (shakedown) as well as elastic, sublimit and limit surfaces. The work presents the characteristic features of slackened structural systems.

1. Introduction

The problem of slackened-elastic-plastic (SEP) structures, i.e. structures with gaps (clearances) at connections, belongs to the mechanics of systems with unilateral constraints. The formulation of this problem, using the discrete matrix description of G. Maier [12], has been presented in [2] (see also [5, 6, 8]). The theory is geometrically linear, load processes are assumed to be quasi-static and all friction effects are neglected. The main theoretical and numerical results are obtained for skeletal structures with convex clearance regions. The original structure, sublimit and limit load problems are considered in detail in [3].

The holonomic and incremental analyses for SEP structures are formulated as linear (LP) and quadratic (QP) mathematical programming problems that correspond to dual extremum principles, [1, 12]. However, the LP and QP codes, even in simple cases, lead to time-consuming procedures and require a considerably large computer memory. It should be mentioned that in the presence of unilateral constraints the incremental methods must be modified in comparison with the classical formulation, and standard FEM-codes cannot be used.

In the cases of one-parameter contact and yield conditions one can use iterative (trial-and-error) procedures of very short computer run-times. These cases correspond to structures made of truss elements with longitudinal gaps or beam elements of ideal I-cross sections with rotation gaps. Then the clearance regions are strictly convex and the uniqueness of solutions is assured, except of particular clearance and/or plastic mechanisms. In each step of calculations the type of the structure changes accordingly to a current state of deformation. Calculations relate to pure elastic structures with various boundary conditions depending on constraints imposed on the state variables. After such modifications solutions of the problem can be easily obtained by means of standard FEM codes without using the inconvenient LP and QP procedures.

In the present work the iterative procedure is developed for elastic-perfectly plastic continuous beams with rotation constraints where "clearance hinges" and classical plastic hinges (due to bending) can arise. Solutions of the Quadratic Programming Problems (QPPs) coincide with the elastic solutions satisfying the prescribed constraints for angles of rotations whereas the Linear Programming Problems (LPPs) are solved with the use of the solutions for rigid "articulated" structures where a "small" elasticity at the end parts of elements is introduced. Note that the continuous beams, despite of their simple layout, exhibit all important characteristic features of slackened systems.

2. Mechanical model

The structures under considerations consist of deformable beam elements and undeformable connecting elements (plates) of very small dimensions. Each beam element is joined with the connecting plate by means of the hinge with rotation constraints ($-l_k^- \le \phi_k \le l_k^+$; $l_k^- \ge 0$, $l_k^+ \ge 0$). The computational model of a continuous beam with rotation clearances is presented in Fig. 1. In the interior of each connecting plate a certain point called "node" is distinguished, and the external load $\mathbf{p}$ is applied only at the nodes. The vector of displacements $\mathbf{u}$ collects all the generalized displacements (vertical translations and rotations) of the connecting plates.

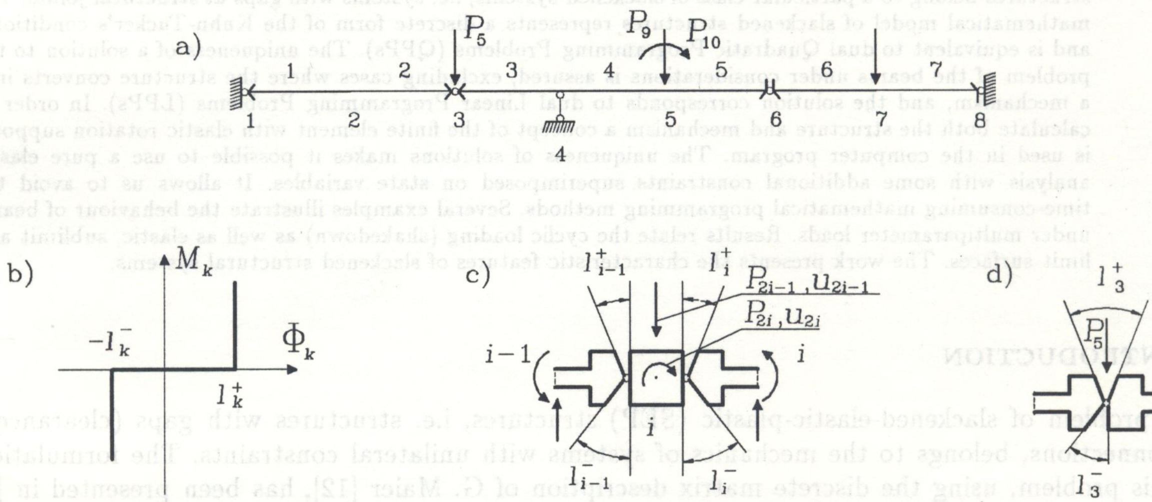

Fig. 1. Discrete model of slackened continuous beam; (a) beam, (b) mechanical characteristic of connection, (c) general model of connection with rotation gaps, (d) simplified model of connection

The relative rotations of the beam element faces and the connecting plates are defined as concentrated clearance strains $\boldsymbol{\varepsilon}_L$. The meaning of these strains is similar to plastic strains $\boldsymbol{\varepsilon}_P$ concentrated at plastic hinges. Elastic strains $\boldsymbol{\varepsilon}_E$ are defined as the angles between the tangents and the secant of deformed beam element. The vector of total strains $\boldsymbol{\varepsilon}$ is assumed to be a sum of clearance, elastic and plastic parts. The present work concerns a particular case, when vertical loads are only applied, and one can use the simplified model of the connection shown in Fig. 1d. The generalized stress vector $\boldsymbol{\sigma}$ consists of particular bending moments at beam element faces, (cf. [1]), and the load, displacement, stress and strain vectors can then be expressed as follows:

$$\mathbf{p} = [P_1, P_2, \ldots, P_n]^T, \qquad \mathbf{u} = [u_1, u_2, \ldots, u_n]^T,$$
$$\boldsymbol{\sigma} = [M_1, M_2, \ldots, M_m]^T, \qquad \boldsymbol{\varepsilon} = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m]^T, \qquad m \ge n. \tag{1}$$

In (1), $P_1, P_2, \ldots, P_n$ and $u_1, u_2, \ldots, u_n$ denote the vertical forces and vertical displacements of all structural nodes, respectively. $M_1, M_2, \ldots, M_m$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ are the bending moments and generalized strains of all beam element ends, respectively. Superscript T denotes the transpose.

Kinematical and statical quantities, shown in Fig. 2, are consistent in the sense of the virtual work equation:

$$\mathbf{p}^T \mathbf{u} = \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}. \tag{2}$$

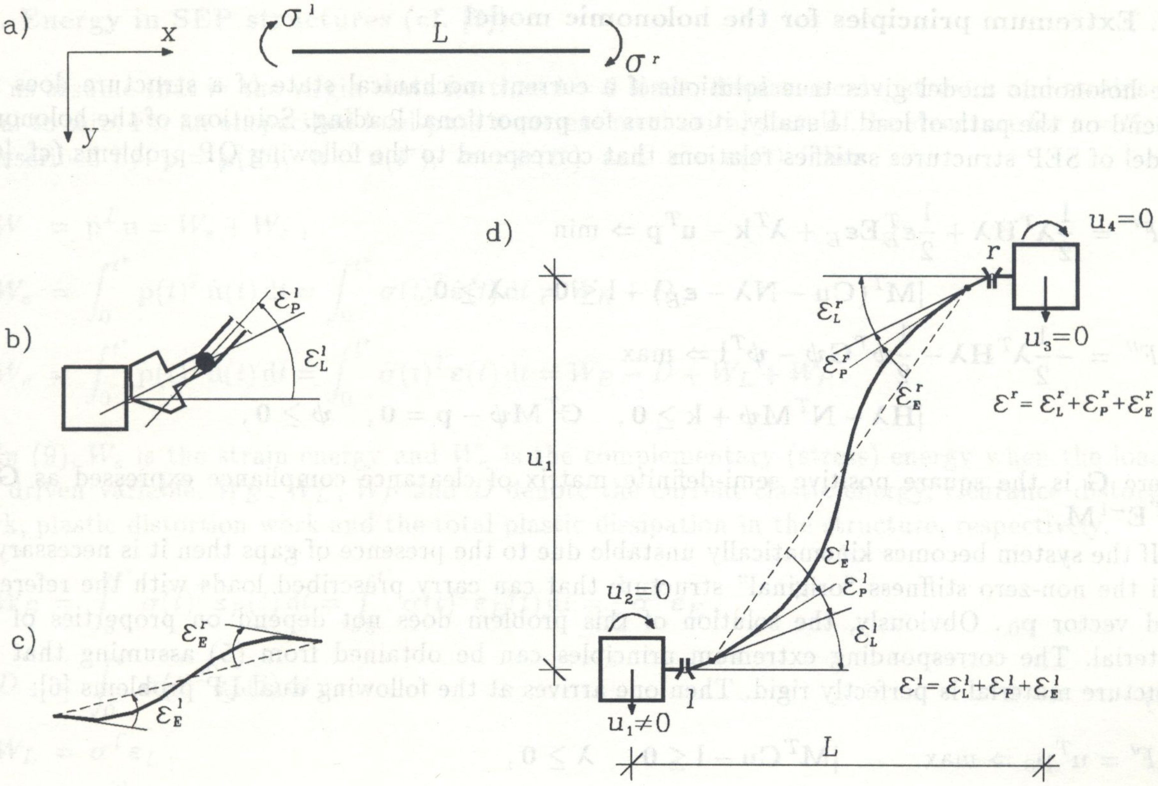

Fig. 2. Generalized stresses $\boldsymbol{\sigma}$ and strains $\boldsymbol{\varepsilon}$

3. MATHEMATICAL FOUNDATIONS

3.1. General matrix relations

The general mathematical model of SEP structures is described by the following matrix relations (cf. [6] for details):

$$
\begin{array}{lll}
1.\ \mathbf{C}\mathbf{u} - \boldsymbol{\varepsilon} = \mathbf{0}, & 5.\ \mathbf{f} = \mathbf{N}^T\boldsymbol{\sigma} - \mathbf{H}\boldsymbol{\lambda} - \mathbf{k} \le \mathbf{0}, & 9.\ \mathbf{g} = \mathbf{M}^T\boldsymbol{\varepsilon}_L - \mathbf{l} \le \mathbf{0}, \\
2.\ \mathbf{C}^T\boldsymbol{\sigma} - \mathbf{p} = \mathbf{0}, & 6.\ \boldsymbol{\varepsilon}_P = \mathbf{N}\boldsymbol{\lambda}, & 10.\ \boldsymbol{\sigma} = \mathbf{M}\boldsymbol{\psi}, \\
3.\ \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_L + \boldsymbol{\varepsilon}_E + \boldsymbol{\varepsilon}_P, & 7.\ \dot{\boldsymbol{\lambda}} \ge \mathbf{0}, & 11.\ \boldsymbol{\psi} \ge \mathbf{0}, \\
4.\ \boldsymbol{\varepsilon}_E = \mathbf{E}^{-1}\boldsymbol{\sigma}, & 8.\ \dot{\boldsymbol{\lambda}}^T\mathbf{f} = 0, & 12.\ \boldsymbol{\psi}^T\mathbf{g} = 0.
\end{array}
\quad (3)
$$

In (3), $\mathbf{C}$ is the geometric compatibility matrix for an "ideal" structure (i.e. structure without clearances). Matrix equation $(3)_4$ represents the generalized Hooke's law while $\mathbf{E}$ denotes the strictly positively defined elasticity matrix. The yield and contact conditions are assumed in the form of linear matrix inequalities $(3)_5$ and $(3)_9$, respectively; $\mathbf{N}$ and $\mathbf{M}$ are rectangular matrices that collect all the external normals to the particular sides of yield and clearance polyhedrons. The linear plastic hardening is described by matrix $\mathbf{H}$; $\boldsymbol{\lambda}$ and $\boldsymbol{\psi}$ are the plastic strain rate and stress multiplier vectors, respectively. Clearance moduli (i.e. limit values of gaps) are described by vector $\mathbf{l}$. Generalized yield limits are gathered in vector $\mathbf{k}$.

At active sides of the yields and contact polyhedrons the following orthogonality conditions hold:

$$\dot{\boldsymbol{\lambda}}^T\dot{\mathbf{f}} \ge 0\,, \qquad \boldsymbol{\psi}^T\dot{\mathbf{g}} = \dot{\boldsymbol{\psi}}^T\mathbf{g} = 0\,.$$

The ideal structure is assumed to be kinematically stable. It corresponds to the requirement

$$\det(\mathbf{C}^T\mathbf{C}) \neq 0\,. \qquad (4)$$

3.2. Extremum principles for the holonomic model

The holonomic model gives true solutions if a current mechanical state of a structure does not depend on the path of load. Usually, it occurs for proportional loading. Solutions of the holonomic model of SEP structures satisfies relations that correspond to the following QP problems (cf. [6]):

$$\begin{aligned}
&F' = \frac{1}{2}\boldsymbol{\lambda}^T\mathbf{H}\boldsymbol{\lambda} + \frac{1}{2}\boldsymbol{\varepsilon}_E^T\mathbf{E}\boldsymbol{\varepsilon}_E + \boldsymbol{\lambda}^T\mathbf{k} - \mathbf{u}^T\mathbf{p} \Rightarrow \min \\
&\qquad |\mathbf{M}^T(\mathbf{C}\mathbf{u} - \mathbf{N}\boldsymbol{\lambda} - \boldsymbol{\varepsilon}_E) - \mathbf{l} \le \mathbf{0}, \quad \boldsymbol{\lambda} \ge \mathbf{0}, \\
&F'' = -\frac{1}{2}\boldsymbol{\lambda}^T\mathbf{H}\boldsymbol{\lambda} - \frac{1}{2}\boldsymbol{\psi}^T\mathbf{G}\boldsymbol{\psi} - \boldsymbol{\psi}^T\mathbf{l} \Rightarrow \max \\
&\qquad |\mathbf{H}\boldsymbol{\lambda} - \mathbf{N}^T\mathbf{M}\boldsymbol{\psi} + \mathbf{k} \ge \mathbf{0}, \quad \mathbf{C}^T\mathbf{M}\boldsymbol{\psi} - \mathbf{p} = \mathbf{0}, \quad \boldsymbol{\psi} \ge \mathbf{0},
\end{aligned} \tag{5}$$

where $\mathbf{G}$ is the square positive semi-definite matrix of clearance compliance expressed as $\mathbf{G} = \mathbf{M}^T\mathbf{E}^{-1}\mathbf{M}$.

If the system becomes kinematically unstable due to the presence of gaps then it is necessary to find the non-zero stiffness "original" structure that can carry prescribed loads with the reference load vector $\mathbf{p}_0$. Obviously, the solution of this problem does not depend on properties of the material. The corresponding extremum principles can be obtained from (5) assuming that the structure material is perfectly rigid. Then one arrives at the following dual LP problems [6]:

$$\begin{aligned}
&F' = \mathbf{u}^T\mathbf{p}_0 \Rightarrow \max \qquad |\mathbf{M}^T\mathbf{C}\mathbf{u} - \mathbf{l} \le \mathbf{0}, \quad \boldsymbol{\lambda} \ge \mathbf{0}, \\
&F'' = \boldsymbol{\psi}^T\mathbf{l} \Rightarrow \min \qquad |\mathbf{C}^T\mathbf{M}\boldsymbol{\psi} - \mathbf{p}_0 \ge \mathbf{0}, \quad \boldsymbol{\psi} \ge \mathbf{0}.
\end{aligned} \tag{6}$$

Note that a good approximation of (6) corresponds to the solution of a slackened-rigid system with a "small" elasticity inside the clearance region.

3.3. Sublimit and limit load problems

In the case of slackened-perfectly plastic (SpP) structures the formulation of the limit load problem is much more complex than that for perfectly plastic structures. The mathematical model for SpP structures can be written in the form:

$$\begin{array}{lll}
1.\ \ \mathbf{C}\dot{\mathbf{u}} - \dot{\boldsymbol{\varepsilon}} = \mathbf{0}, & 5.\ \ \boldsymbol{\sigma} = \mathbf{M}_a\boldsymbol{\psi}_a, & 9.\ \ \dot{\boldsymbol{\varepsilon}}_P = \mathbf{N}\dot{\boldsymbol{\lambda}}, \\
2.\ \ \mathbf{C}^T\boldsymbol{\sigma} = \mu\mathbf{p}, & 6.\ \ \boldsymbol{\psi}_a \ge \mathbf{0}, & 10.\ \ \dot{\boldsymbol{\lambda}} \ge \mathbf{0}, \\
3.\ \ \dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}_L + \dot{\boldsymbol{\varepsilon}}_P, & 7.\ \ \dot{\mathbf{g}}_a = \mathbf{M}_a^T\dot{\boldsymbol{\varepsilon}}_L \le \mathbf{0}, & 11.\ \ \mathbf{f} = \mathbf{N}^T\boldsymbol{\sigma} - \mathbf{k} \le \mathbf{0}, \\
4.\ \ \dot{\mathbf{u}}^T\mathbf{p} = 1, & 8.\ \ \boldsymbol{\psi}_a^T\dot{\mathbf{g}}_a = 0, & 12.\ \ \dot{\boldsymbol{\lambda}}^T\mathbf{f} = 0,
\end{array} \tag{7}$$

where the subscript a indicates the active submatrices determined from the solution of the original structure problem. Relations (7) represent the Kuhn-Tucker's conditions, and can be modified to the following dual LPPs:

$$\begin{aligned}
&F' = \dot{\boldsymbol{\lambda}}^T\mathbf{k} \Rightarrow \min \qquad |\mathbf{M}_a^T(\mathbf{C}\dot{\mathbf{u}} - \mathbf{N}\dot{\boldsymbol{\lambda}}) \le \mathbf{0}, \quad \dot{\mathbf{u}}^T\mathbf{p} = 1, \quad \dot{\boldsymbol{\lambda}} \ge \mathbf{0}, \\
&F'' = \mu \Rightarrow \max \qquad | -\mathbf{N}^T\mathbf{M}_a^T\boldsymbol{\psi}_a + \mathbf{k} \ge \mathbf{0}, \quad \mathbf{C}^T\mathbf{M}_a\boldsymbol{\psi}_a - \mu\mathbf{p} = \mathbf{0}, \quad \boldsymbol{\psi}_a \ge \mathbf{0}.
\end{aligned} \tag{8}$$

The presence of gaps can induce clearance-plastic mechanisms at so-called "sublimit" states corresponding to lower values of the load multiplier μ. So, the ultimate limit load is attained in a step-wise way and is identical with that of the ideal structure. This problem has been considered in [3].

3.4. Energy in SEP structures (cf. [7])

Let us assume that in the virgin state for time $t = 0$ loads, displacements, stresses and strains are equal to zero. For an unspecified load-path a current mechanical state of the structure for $t = t^* > 0$ is described by: $\mathbf{p} = \mathbf{p}(t^*)$, $\mathbf{u} = \mathbf{u}(t^*)$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(t^*)$ and $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(t^*)$. Then

$$\begin{aligned} W &= \mathbf{p}^T\mathbf{u} = W_\varepsilon + W_\sigma\,,\\ W_\varepsilon &= \int_0^{t^*} \mathbf{p}(t)^T\dot{\mathbf{u}}(t)\,\mathrm{d}t = \int_0^{t^*} \boldsymbol{\sigma}(t)^T\dot{\boldsymbol{\varepsilon}}(t)\,\mathrm{d}t = W_E + D\,,\\ W_\sigma &= \int_0^{t^*} \dot{\mathbf{p}}(t)^T\mathbf{u}(t)\,\mathrm{d}t = \int_0^{t^*} \dot{\boldsymbol{\sigma}}(t)^T\boldsymbol{\varepsilon}(t)\,\mathrm{d}t = W_E - D + W_L + W_P\,. \end{aligned} \tag{9}$$

In (9), W_ε is the strain energy and W_σ is the complementary (stress) energy when the load is the driven variable. W_E, W_L, W_P and D denote the current elastic energy, clearance distortion work, plastic distortion work and the total plastic dissipation in the structure, respectively,

$$\begin{aligned} W_E &= \int_0^{t^*} \boldsymbol{\sigma}(t)^T\dot{\boldsymbol{\varepsilon}}_E(t)\,\mathrm{d}t = \int_0^{t^*} \dot{\boldsymbol{\sigma}}(t)^T\boldsymbol{\varepsilon}_E(t)\,\mathrm{d}t = \frac{1}{2}\boldsymbol{\sigma}^T\boldsymbol{\varepsilon}_E\,,\\ D &= \int_0^{t^*} \boldsymbol{\sigma}(t)^T\dot{\boldsymbol{\varepsilon}}_P(t)\,\mathrm{d}t\,,\\ W_L &= \boldsymbol{\sigma}^T\boldsymbol{\varepsilon}_L\,,\\ W_P &= \boldsymbol{\sigma}^T\boldsymbol{\varepsilon}_P\,. \end{aligned} \tag{10}$$

For the holonomic model, relations (9) take the more simple form,

$$W_\varepsilon = W_E + D\,, \qquad W_\sigma = W_E + W_L\,, \qquad D = W_P = \boldsymbol{\sigma}^T\boldsymbol{\varepsilon}_P\,. \tag{11}$$

3.5. Incremental analysis

The incremental analysis problem is formulated as follows: "For a given load vector $\mathbf{p}$ the stresses $\boldsymbol{\sigma}$, strains $\boldsymbol{\varepsilon}_L$, $\boldsymbol{\varepsilon}_E$, $\boldsymbol{\varepsilon}_P$ and displacements $\mathbf{u}$ are known. Find the increments of these vectors $\Delta\boldsymbol{\sigma}$, $\Delta\boldsymbol{\varepsilon}_L$, $\Delta\boldsymbol{\varepsilon}_E$, $\Delta\boldsymbol{\varepsilon}_P$, $\Delta\mathbf{u}$ if the load increment is equal to $\Delta\mathbf{p}$".

At the beginning of each increment the vectors $\mathbf{f}$ and $\mathbf{g}$ are known. Particular components of these vectors are equal to zero. It means that they are active, and their increments have to be non-positive, i.e. $\Delta\mathbf{f}_a \le \mathbf{0}$ and $\Delta\mathbf{g}_a \le \mathbf{0}$. The mathematical model of such an incremental problem can be written as

$$\begin{array}{llll} 1. & \mathbf{C}\Delta\mathbf{u} - \Delta\boldsymbol{\varepsilon} = \mathbf{0}\,, & 7. & \Delta\mathbf{g}_a = \mathbf{M}_a^T\Delta\boldsymbol{\varepsilon}_L \le \mathbf{0}\,,\\ 2. & \mathbf{C}^T\Delta\boldsymbol{\sigma} = \Delta\mathbf{p}\,, & 8. & (\boldsymbol{\psi}+\Delta\boldsymbol{\psi})_a^T\Delta\mathbf{g}_a = 0\,,\\ 3. & \Delta\boldsymbol{\varepsilon} = \Delta\boldsymbol{\varepsilon}_L + \Delta\boldsymbol{\varepsilon}_E + \Delta\boldsymbol{\varepsilon}_P\,, & 9. & \Delta\boldsymbol{\varepsilon}_P = \mathbf{N}_a\Delta\boldsymbol{\lambda}_a\,,\\ 4. & \Delta\boldsymbol{\sigma} = \mathbf{E}\Delta\boldsymbol{\varepsilon}_E\,, & 10. & \Delta\boldsymbol{\lambda}_a \ge \mathbf{0}\,,\\ 5. & \mathbf{M}_a\Delta\boldsymbol{\psi}_a = \mathbf{E}\Delta\boldsymbol{\varepsilon}_E\,, & 11. & \Delta\mathbf{f}_a = \mathbf{N}^T\Delta\boldsymbol{\sigma} - \mathbf{H}_a\Delta\boldsymbol{\lambda}_a \le \mathbf{0}\,,\\ 6. & (\boldsymbol{\psi}+\Delta\boldsymbol{\psi})_a \ge \mathbf{0}\,, & 12. & \Delta\boldsymbol{\lambda}_a^T\Delta\mathbf{f}_a = 0\,. \end{array} \tag{12}$$

Mathematical model (12) is equivalent to a saddle point problem or to dual problems of QP. Note that $\Delta\boldsymbol{\psi}_a$ can be arbitrary in sign whereas the "global" stress multiplier must be non-negative, $(\boldsymbol{\psi}+\Delta\boldsymbol{\psi})_a \ge \mathbf{0}$. Relations (12) have to be check within each step of loading.

4. Incremental procedure

4.1. General concept of incremental procedure

The further incremental analysis is carried out assuming that the material of the beam is elastic-perfectly plastic ($\mathbf{H} = \mathbf{0}$). In the case of rotation gaps the solution uniqueness is assured. So, the solution is correct if all relations of the mathematical model are satisfied.

The partial plastification of the ideal I-cross sections does not happen, and yielding is caused only by bending moments. Therefore one can introduce a concept of plastic and clearances hinges where mutual rotations of adjacent beam elements take place. The location of these hinges determines a current type of the beam. So, the general concept of the procedure proposed consists in an identification of a proper type of the structure. Within a current load-increment a linear elastic structure or a mechanism can then be calculated. To solve the problem, procedures for elastic structures and mechanisms are needed.

4.2. Loads

The incremental analysis allows us to determine the state variables for arbitrary load-path. The load program can be described by means of a certain curve in the load space with a fixed initial point and sense (cf. Fig. 3). The piece-wise linear approximation of this curve is assumed. Particular segments represent known, nominal load increments. The nominal value of the load increment at step "j" can be written as

$$\Delta \mathbf{p}_j = \mu_j \mathbf{p}_{0j} \,, \tag{13}$$

where $\mathbf{p}_{0j}$ and μ_j are the reference load vector and the nominal load multiplier, respectively. It is convenient to assume that the nominal load multiplier $\mu_j = 1$.

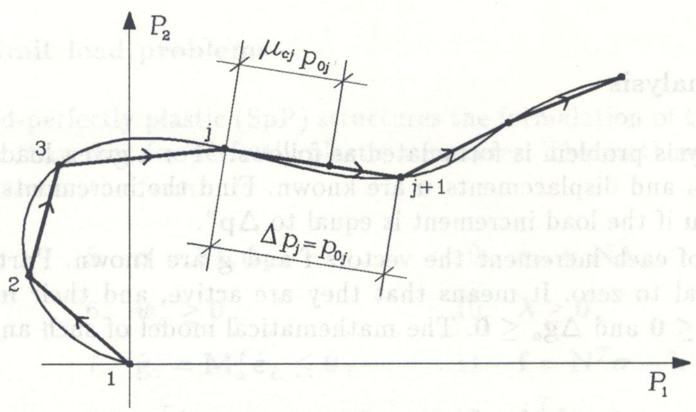

Fig. 3. Load program

The solution of the problem is assembled of solutions of consecutive subproblems relating to a current type of structure and load increment. Loads, displacements, stresses and strains are accumulated in each increment. The load step is evaluated with the use of the scaling procedure that allows us to calculate a current load multiplier which is usually less than the nominal one, $\mu_{cj} < \mu_j$. The current load multiplier is accepted if one of the following cases occurs:

- the appearance of a new contact,
- the loss of an existing contact,
- the occurrence of a plastic hinge,

• the unloading of an existing plastic hinge;

otherwise the nominal load multiplier is applied.

4.3. Description of the incremental procedure

The incremental analysis of a slackened beam starts by reading in the data for the problem. The prescribed load program consists of n nominal load increments. A flow diagram for the computer program is shown in Fig. 4.

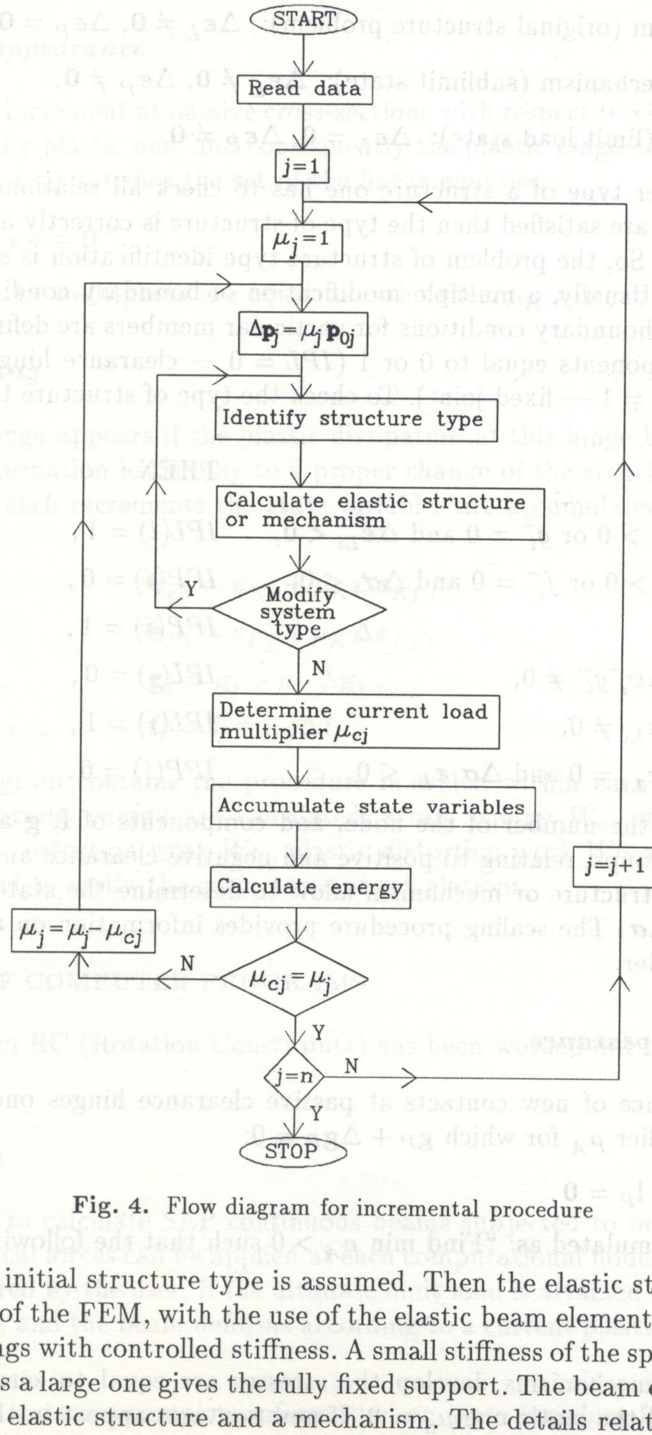

Fig. 4. Flow diagram for incremental procedure

In the beginning an initial structure type is assumed. Then the elastic structure (or mechanism) is calculated by means of the FEM, with the use of the elastic beam element supported at both ends on elastic rotation springs with controlled stiffness. A small stiffness of the springs corresponds to the pin-end (hinge) whereas a large one gives the fully fixed support. The beam element proposed allows us to calculate both an elastic structure and a mechanism. The details related to the beam element are contained in Appendix. The fundamental question is: whether the system under consideration is a mechanism or a structure. To answer this question one compares the external energy, i.e. a half of the scalar product of external loads and displacements, with the internal energy calculated as the energy of stresses and strains, excluding the contribution of support springs. A large difference between these energies corresponds to a mechanism. However, one has to be aware that the solution in this case is equivalent to the solution of LPP and therefore can be non-unique. Usually, the solution non-uniqueness arises for symmetrical modes of kinematics. On the other hand in the computer algorithm a unique solution is needed. So, in order to assure the solution uniqueness the loads and beam element stiffness are randomly perturbed. In each mechanism the increments of clearance $\Delta\boldsymbol{\varepsilon}_L$ and/or plastic $\Delta\boldsymbol{\varepsilon}_P$ strains occur, and the kind of mechanism depends on their values. The following three kinds of mechanism can be distinguished:

- a clearance mechanism (original structure problem): $\Delta\boldsymbol{\varepsilon}_L \neq \mathbf{0}$, $\Delta\boldsymbol{\varepsilon}_P = \mathbf{0}$,
- a clearance-plastic mechanism (sublimit state): $\Delta\boldsymbol{\varepsilon}_L \neq \mathbf{0}$, $\Delta\boldsymbol{\varepsilon}_P \neq \mathbf{0}$,
- a plastic mechanism (limit load state): $\Delta\boldsymbol{\varepsilon}_L = \mathbf{0}$, $\Delta\boldsymbol{\varepsilon}_P \neq \mathbf{0}$.

To identify the proper type of a structure one has to check all relations of the mathematical model. If these relations are satisfied then the type of structure is correctly assumed, in the light of the solution uniqueness. So, the problem of structure type identification is solved by means of the trial-and-error method. Usually, a multiple modification of boundary conditions of the structural elements is needed. The boundary conditions for particular members are defined by "hinge vectors" *IPL* and *IPP* with components equal to 0 or 1 (*IPL* = 0 — clearance hinge, *IPP* = 0 — plastic hinge, *IPL* = 1 and *IPP* = 1 — fixed joint). To check the type of structure the following algorithm is used:

IF :	THEN :
1. $g_i^+ = 0$ and $\Delta\varepsilon_{Li} > 0$ or $g_i^- = 0$ and $\Delta\varepsilon_{Li} < 0$,	$IPL(i) = 1$,
2. $f_i^+ = 0$ and $\Delta\sigma_i > 0$ or $f_i^- = 0$ and $\Delta\sigma_i < 0$,	$IPP(i) = 0$,
3. $\sigma_i\Delta\varepsilon_{Pi} < 0$,	$IPP(i) = 1$, (14)
4. $\Delta\psi_i^+ g_i^+ \neq 0$ or $\Delta\psi_i^- g_i^- \neq 0$,	$IPL(i) = 0$,
5. $\sigma_i \neq 0$ and $\sigma_i\Delta\varepsilon_{Li} \neq 0$,	$IPL(i) = 1$,
6. $\sigma_i = 0$ and $\sigma_i\Delta\varepsilon_{Li} = 0$ and $\Delta\sigma_i\varepsilon_{Li} \leq 0$,	$IPP(i) = 0$.

In (14), "*i*" denotes the number of the node, and components of $\mathbf{f}$, $\mathbf{g}$ and $\boldsymbol{\psi}$ are divided into positive and negative groups, relating to positive and negative clearance and plastic strains.

Calculations of the structure or mechanism allow to determine the state variable vectors: $\Delta\mathbf{u}$, $\Delta\boldsymbol{\varepsilon}_L$, $\Delta\boldsymbol{\varepsilon}_E$, $\Delta\boldsymbol{\varepsilon}_P$ and $\Delta\boldsymbol{\sigma}$. The scaling procedure provides information on a current value of load or displacement multiplier.

A. New contacts appearance

In the case of appearance of new contacts at passive clearance hinges one has to find the least positive value of multiplier μ_A for which $\mathbf{g}_P + \Delta\mathbf{g}_P = 0$:

$$\mathbf{M}_P^T(\boldsymbol{\varepsilon}_L + \mu_A\Delta\boldsymbol{\varepsilon}_L) - \mathbf{l}_P = \mathbf{0}\,.$$

The problem can be formulated as: "Find min $\mu_A > 0$ such that the following linear equations are satisfied

$$\mu_A\mathbf{M}_P^T\Delta\boldsymbol{\varepsilon}_L = \mathbf{g}_P\text{''}. \tag{15}$$

In the cases where mechanisms develop the stresses are equal to zero. Then the kinematic multiplier $\mu_K = \mu_A$ and the static one $\mu_S = 0$. If new contacts appear in the kinematically stable system then $\mu_K = \mu_S = \mu_A$.

B. Loss of existing contact

The loss of existing contact can occur if at the closed clearance hinge within a given load increment the absolute value of generalized stresses decreases. The problem is then stated: "Find min $\mu_B > 0$ that satisfies the following set of linear equations

$$\boldsymbol{\sigma} + \mu_B \Delta\boldsymbol{\sigma} = \mathbf{0}\text{''}, \tag{16}$$

where $\boldsymbol{\sigma}$ is the stress vector composed of bending moments at all cross-sections with active unilateral constraints. In the case considered here both the static and kinematic multipliers are equal to each other, i.e. $\mu_K = \mu_S = \mu_B$.

C. Plastic hinge appearance

Within the given load increment at passive cross-sections with respect to yielding a bending moment can be equal to the fully plastic one, and consequently the plastic hinge can occur. So, the problem is: "Find min $\mu_C > 0$, that satisfies the set of the linear equations

$$\mathbf{N}_P^T(\boldsymbol{\sigma} + \mu_C \Delta\boldsymbol{\sigma}) - \mathbf{k}_P = \mathbf{0}\text{''}. \tag{17}$$

The static and kinematic multipliers are equal to each other: $\mu_K = \mu_S = \mu_C$.

D. Plastic unloading

Closing of a plastic hinge appears if the plastic dissipation at this hinge becomes non-positive, i.e. $\boldsymbol{\sigma}^T \Delta\boldsymbol{\varepsilon}_P < 0$. This information leads only to a proper change of the structure type.

In the last part of each increments the state variables are accumulated and updated according to the formulas:

$$\begin{aligned}
&\mathbf{p}_j = \mathbf{p}_j + \mu_S \Delta\mathbf{p}_j\,, && \boldsymbol{\varepsilon}_{Ej} = \boldsymbol{\varepsilon}_{Ej} + \mu_K \Delta\boldsymbol{\varepsilon}_{Ej}\,,\\
&\boldsymbol{\sigma}_j = \boldsymbol{\sigma}_j + \mu_S \Delta\boldsymbol{\sigma}_j\,, && \boldsymbol{\varepsilon}_{Pj} = \boldsymbol{\varepsilon}_{Pj} + \mu_K \Delta\boldsymbol{\varepsilon}_{Pj}\,,\\
&\mathbf{u}_j = \mathbf{u}_j + \mu_K \Delta\mathbf{u}_j\,, && \mathbf{g}_j = \mathbf{g}_j + \mu_K \Delta\mathbf{g}_j\,,\\
&\boldsymbol{\varepsilon}_{Lj} = \boldsymbol{\varepsilon}_{Lj} + \mu_K \Delta\boldsymbol{\varepsilon}_{Lj}\,, && \mathbf{f}_j = \mathbf{f}_j + \mu_S \Delta\mathbf{f}_j\,.
\end{aligned} \tag{18}$$

The computer program contains the procedure in which within each increment the following components of the current energy are calculated: strain energy W_ε , stress energy W_σ , elastic energy W_E, clearance distortion work W_L , plastic distortion work W_P and dissipation D. These quantities are also used to verify the correctness of calculations.

5. DESCRIPTION OF COMPUTER PROGRAMS

The computer program RC (Rotation Constraints) has been worked out in three versions adopted to various needs.

B — Basic version

The procedure allows to calculate SEP continuous beams subjected to *multiparameter loads.* It is assumed that the vertical forces can be applied at each computational node. Consecutive increments are interactively declared by the user. If the ultimate limit load is attained then the user can declare a kinematic multiplier, and the beam deforms according to a current plastic flow mechanism. Three result files are created:

- the main file which contains complete information on state variables for both the nominal and current increments,

- the "diagram" file to make $P-\Delta$ diagrams for arbitrary components of loads and displacements,
- the "energy" file where a "length" of the displacement path and corresponding particular components of the energy are recorded.

The displacement path length is evaluated according to the following. Deformations of the beam within j-step of calculations are characterized by a vertical displacement rate "length" $\Delta \mathbf{u}_j$ which is expressed as

$$\Delta u_j = (\Delta \mathbf{u}_j^T \cdot \Delta \mathbf{u}_j)^{\frac{1}{2}}, \tag{19}$$

where the right hand side is the norm of nodal displacement rate vector $\Delta \mathbf{u}_j$ evaluated for all structural nodes of the beam. The *length of the displacement path* S_U is assumed to be the sum of Δu_j up to a current calculation step number m:

$$S_U = \sum_j^m \Delta u_j\,. \tag{20}$$

C — Cyclic loading version

The procedure allows to analyse the SEP beams under *three-parameter* cyclic loads that can vary within the yield polygon. Number of cycles is declared by the user in the interactive way. The program gives information on the presence of plastic deformation rates and can be used in the shakedown analysis. The identical resulting files to those of B-version are created.

ESL — Elastic-Sublimit-Limit load version

The computer program allows us to determine the limit, sublimit and elastic regions in the load space.

The *limit* load (yield) region contains all load multipliers corresponding to the load which can be carried by the structure. It can be shown that the yield region is always convex as well as path- and distortion-independent. In other words the shape and dimensions of this region do not depend on the load history as well as the presence of gaps and initial plastic strains. The limit load region is bounded by the limit (yield) surface whose points correspond to plastic flow mechanisms. At the yield load point the normality rule for displacement rates is valid.

The *sublimit* loads correspond to clearance-plastic mechanisms. The loads associated with these mechanisms create the surface which bounds a sublimit region in the load space. The sublimit region is strongly path-dependent, and therefore in order to obtain a unique solution the proportional loading is assumed. The sublimit region can be non-convex and is dependent on gap distributions and initial plastic deformations. Note that discontinuities of the sublimit surface can occur. Obviously, the sublimit region is contained within the limit region, and the displacement rates are orthogonal to the sublimit surface.

The *elastic* region is defined as a set of points in the load (or load multiplier) space that correspond to an elastic response of the structure. The elastic region is bounded by the elastic surface. In the case of SEP structures the elastic region can be non-convex, but is path-independent and star-shaped with respect to the origin of the load space. The shape and dimensions of the elastic surface strongly depend on gap distributions and initial plastic strains. The elastic region is contained within the sublimit region which corresponds to the same gap and initial plastic strain distributions.

To determine the elastic, first sublimit and yield surfaces (polyhedrons) in the load multiplier space for *three-parameter loads* the ESL-version is used. One of the load multiplier is treated as a given parameter. The points which create the polyhedron sides are calculated assuming star-like (radial) load-paths with increasing slope in the plane of the remaining two load multipliers. The results of calculations are presented by means of a contour-line corresponding to a given value of load parameter. The "star-like" calculations may be preceded by an initial three-parameter cycle of loading. However, one has to be aware that the solution of the problem is complete if the polyhedrons are star-shaped with respect to the origin of the radial load-path.

6. EXAMPLES

6.1. Example 1

Consider a continuous beam assembled of beam elements made of an elastic-perfectly plastic (EpP) material. The elements have constant ideal I-cross sections. The beam is shown in Fig. 5a, together with the mechanical properties of the material, distribution of gaps and their values (in radians).

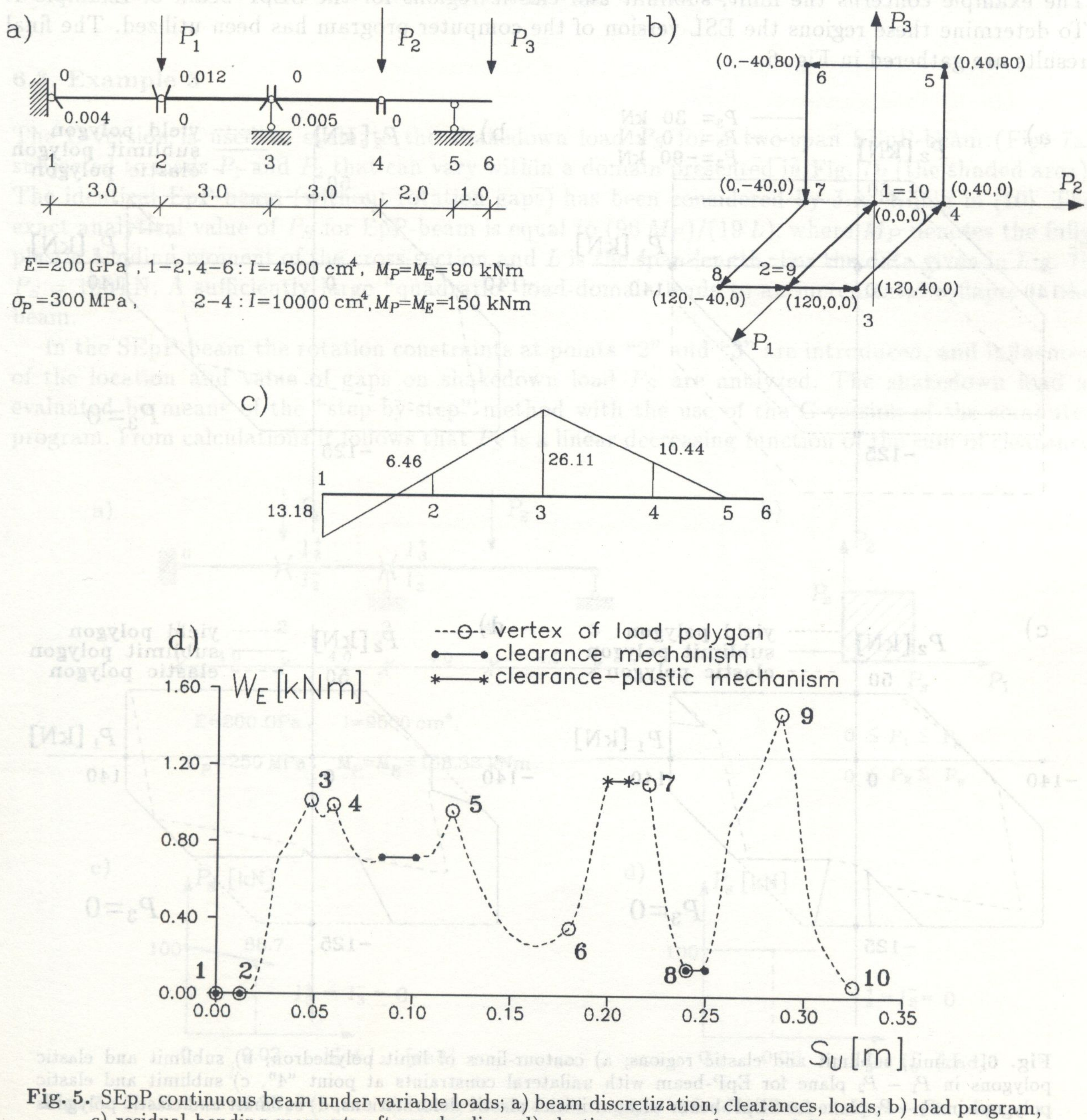

Fig. 5. SEpP continuous beam under variable loads; a) beam discretization, clearances, loads, b) load program, c) residual bending moments after unloading, d) elastic energy v-s length of displacement path

In Fig. 5a, E and σ_P denote the Young's modulus and the yield stress of the material, respectively; I is the moment of inertia of the cross sections, M_E and M_P are the yield and fully plastic bending moments, respectively. The beam is subjected to a three-parameter load program P_1, P_2, P_3 according to a closed loop in the load space (1-2-3-...-10) presented diagrammatically in Fig. 5b. Calculations have been carried out by means of the C-version.

The behaviour of the beam during the loading cycle demonstrates interesting and unexpected effects. In order to illustrate this problem the current elastic energy W_E versus the displacement-path length S_U is plotted. The $W_E(S_U)$ diagram as well as the clearance and clearance-plastic (sublimit) mechanisms are presented in Fig. 5d. Figure 5c illustrates the residual bending moments that remain after unloading.

6.2. Example 2

The example concerns the limit, sublimit and elastic regions for the SEpP-beam of Example 1. To determine these regions the ESL-version of the computer program has been utilized. The final results are gathered in Fig. 6.

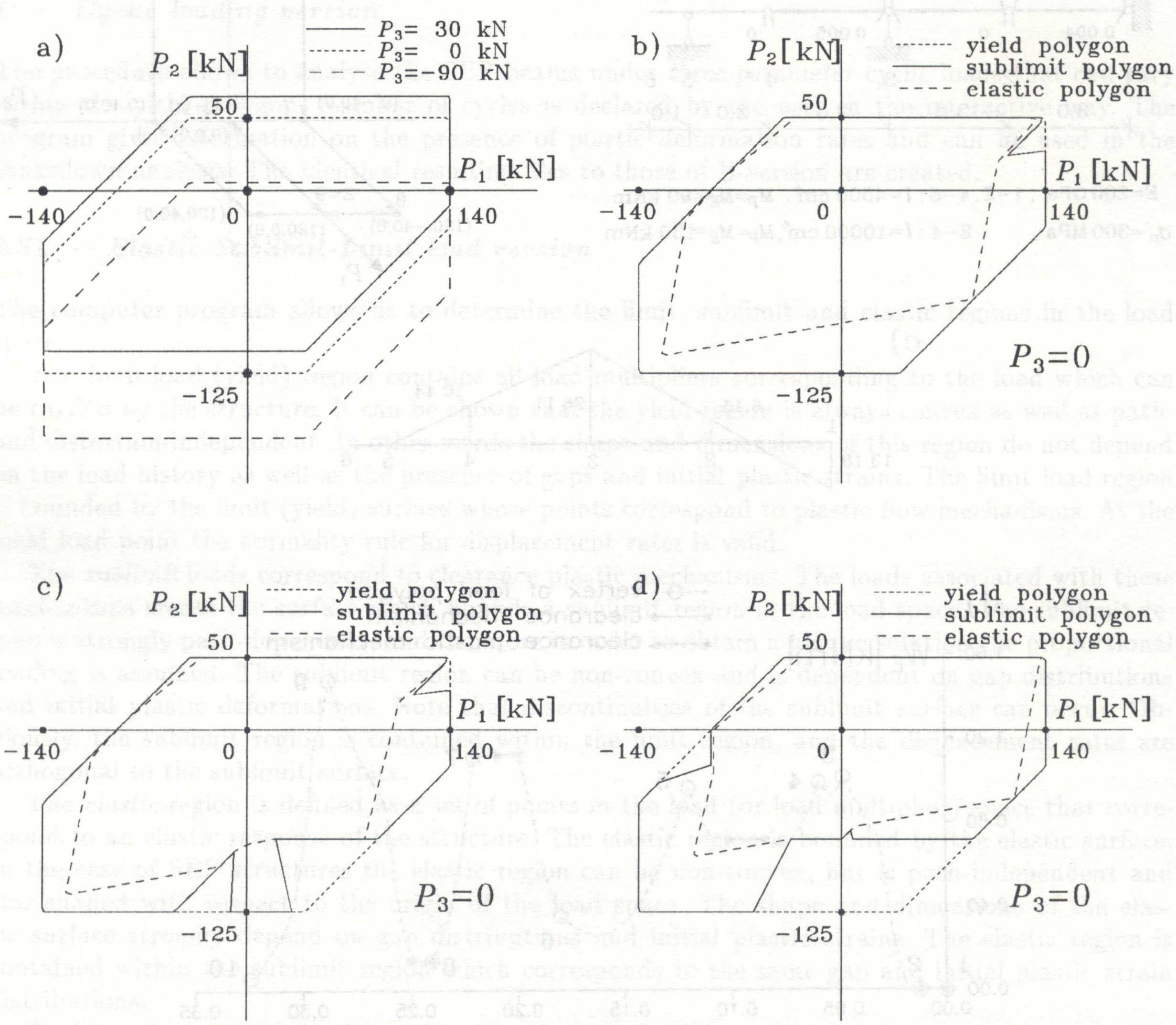

Fig. 6. Limit, sublimit and elastic regions; a) contour-lines of limit polyhedron, b) sublimit and elastic polygons in $P_1 - P_2$ plane for EpP-beam with unilateral constraints at point "4", c) sublimit and elastic polygons in $P_1 - P_2$ plane for SEpP-beam without initial plastic deformations, d) sublimit and elastic polygons in $P_1 - P_2$ plane for SEpP-beam with initial plastic deformations

In Fig. 6a the yield region (polyhedron) is presented by means of contour-lines in $P_1 - P_2$ plane, prepared for $P_3 = -90$ kN, $P_3 = 0$ and $P_3 = 30$ kN. Since the limit load does not depend on any distortions the yield polygons have been determined for the ideal structure without gaps at points "1", "2", "3" and "5". However, at point "4" the typical unilateral constraints are introduced and therefore they remain in the ideal EpP-structure. So, Fig. 6a presents the yield polygons for the structure with unilateral constraints. It is clearly seen that particular contour-lines are shifted with respect to the load-space origin and they do not exhibit the symmetry commonly noted for bilateral constraints.

Figure 6b also concerns the EpP-beam with unilateral constraints at point "4" and illustrates the sublimit and elastic polygons prepared in $P_1 - P_2$ plane with $P_3 = 0$. One can observe a non-convexity of the elastic region and discontinuity of the sublimit region.

Figures 6c and 6d are related to the case of $P_3 = 0$ and SEpP-beam with gaps of Example 1. Both figures present the sublimit and elastic polygons, but Fig. 6d is prepared after the initial load-cycle described in details in Example 1. Effects of initial plastic deformations formed during the load-cycle can be estimated when comparing Figs. 6c and 6d.

6.3. Example 3

The C-version is used to evaluate the shakedown load P_S for a two-span SEpP-beam (Fig. 7a) subjected to loads P_1 and P_2 that can vary within a domain presented in Fig. 7b (the shaded area). The identical EpP-beam (without rotation gaps) has been considered by J.A. König in [10]. The exact analytical value of P_S for EpP-beam is equal to $(96\,M_P)/(19\,L)$, where M_P denotes the fully plastic bending moment of the cross-section and L is the span-length. For the data given in Fig. 7b $P_S = 100$ kN. A sufficiently large "quadratic" load-domain leads to an incremental collapse of the beam.

In the SEpP-beam the rotation constraints at points "2" and "3" are introduced, and influences of the location and value of gaps on shakedown load P_S are analyzed. The shakedown load is evaluated by means of the "step-by-step" method with the use of the C-version of the computer program. From calculations it follows that P_S is a linear decreasing function of the sum of clearance

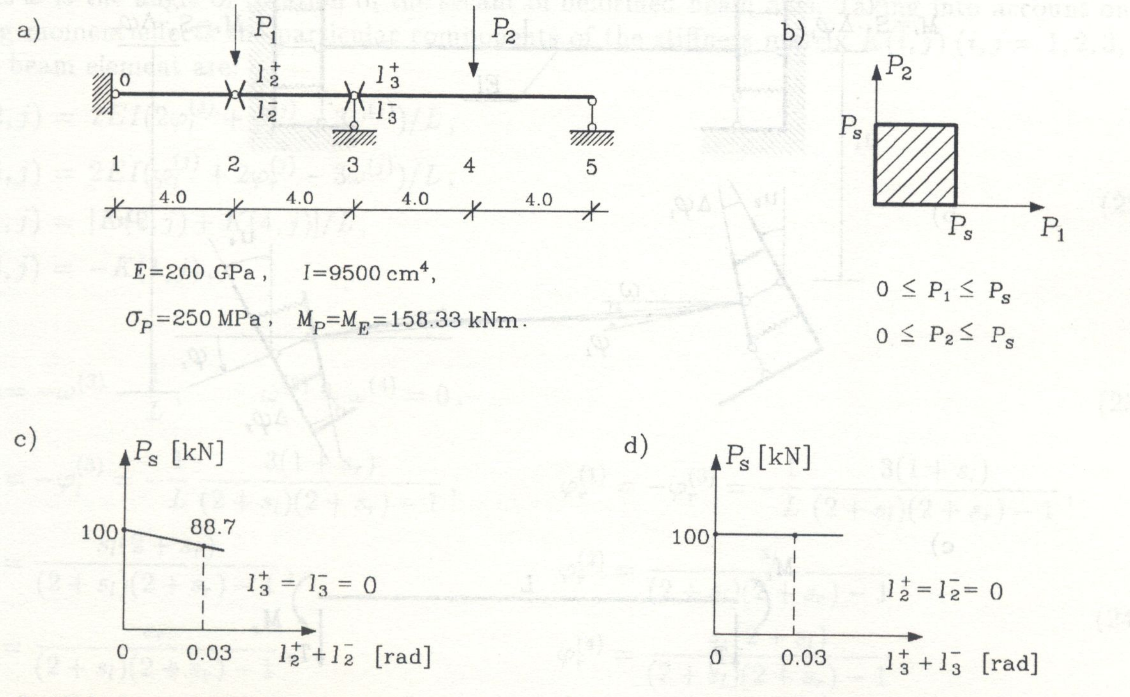

Fig. 7. Shakedown of SEpP-beam; a) beam, loads, rotation gaps, b) load domain, c) P_S v-s sum of gaps at mid-span point "2", d) P_S v-s sum of gaps at support point "3"

moduli at mid-span point "2" (i.e. $l_2^+ + l_2^-$) if the gaps at support point "3" are equal to zero (cf. Fig. 7c). In the case where rotation gaps are introduced only at point "3" the shakedown load does not depend on their values and correspond to that of EpP-beam (see Fig. 7d). The results obtained here coincides with those presented in [4].

7. Closure

The elastic-plastic beams with rotation constraints represent a relatively simple class of SEP structures. The results presented in the work show that even in this case the problem is strongly non-linear and the behaviour of slackened structures is far from that of "common" elastic-plastic structures with bilateral constraints. Particularly interesting are the energy problems in which locking, elastic and plastic strains are incorporated. It may be mentioned that several effects observed cannot be intuitively explained with the use of "classical" approach and engineering experience. It relates, for instance, to the case of proportional loading where in the absence of plastic unloading the elastic energy can be a non-monotone function of the load multiplier. This problem will be considered in a separate paper. It seems that the examples enclosed in the paper are a good illustration of the complexity of SEP-structures.

The solution of SEP structure problems, even in the simplest cases, requires the LP and QP methods. The beam element used herein does not coincide with the standard model (cf. [9,13,14]). The concept of the element with a rotational elastic compliance of the supports is similar to that proposed in [11]. The "support" elasticity allow us to avoid difficulties related to applications of the LP and QP methods. The method proposed in the work can be also used to trusses and frames where one-parameter yield and contact conditions are introduced. It should be added that all the computational methods to solve the problems of unilateral constraints are very sensitive to computer round-off errors. It can be also observed in the procedure proposed. One should be aware of the fact that the strain states can be non-uniquely determined whereas the uniqueness with respect to the stress states is guaranteed.

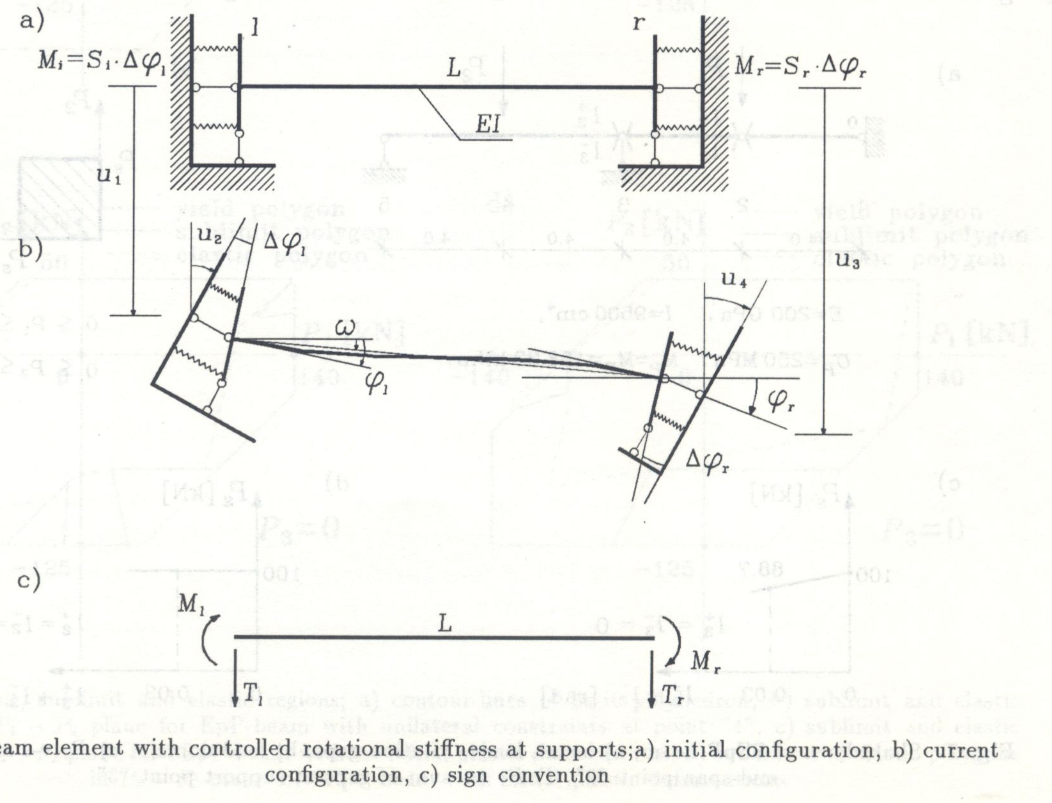

Fig. 8. Beam element with controlled rotational stiffness at supports;a) initial configuration, b) current configuration, c) sign convention

The computer programs presented are mainly directed to qualitative research of mechanical deformable systems. Since the problem of SEP-structures is very complicated and still insufficiently recognized the programs can be used to formulate and verify new hypotheses. The authors believe that qualitatively new features of SEP-structures shown in the work are of significance for both the theory of deformable systems and the engineering practice.

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APPENDIX

Beam Element

The elastic beam structure and "articulated" system (mechanism) is calculated with the use of the concept of beam element with controlled rotational stiffness at both supports. The mechanical model of such an element is shown in Fig. 8.

In order to determine the stiffness matrix of the elastic beam element we recall the well-known relations:

$$\begin{aligned} M_l &= 2EI(2\varphi_l + \varphi_r - 3\omega)/L\,, \\ M_r &= 2EI(\varphi_l + 2\varphi_r - 3\omega)/L\,, \\ T_l &= -T_r = (M_l + M_r)/L\,, \end{aligned} \tag{21}$$

where M_l, M_r, T_l, T_r denote bending moments and shear forces at the left and right ends, respectively, E is the Young's modulus, I denotes the moment of inertia of the cross-section and L is the element length. The angles of rotation at both ends of the element are denoted by φ_l, φ_r, whereas ω is the angle of rotation of the secant of deformed beam axis. Taking into account only bending moment effects the particular components of the stiffness matrix $K(i,j)$ $(i,j = 1,2,3,4)$ for the beam element are:

$$\begin{aligned} K(2,j) &= 2EI(2\varphi_l^{(j)} + \varphi_r^{(j)} - 3\omega^{(j)})/L\,, \\ K(4,j) &= 2EI(\varphi_l^{(j)} + 2\varphi_r^{(j)} - 3\omega^{(j)})/L\,, \\ K(1,j) &= [K(2,j) + K(4,j)]/L\,, \\ K(3,j) &= -K(1,j)\,, \end{aligned} \tag{22}$$

where

$$\omega^{(1)} = -\omega^{(3)} - \frac{1}{L}\,, \qquad \omega^{(2)} = \omega^{(4)} = 0\,, \tag{23}$$

$$\begin{aligned} &\varphi_l^{(1)} = -\varphi_l^{(3)} = -\frac{1}{L}\frac{3(1+s_r)}{(2+s_l)(2+s_r)-1}\,, && \varphi_r^{(1)} = -\varphi_r^{(3)} = -\frac{1}{L}\frac{3(1+s_l)}{(2+s_l)(2+s_r)-1}\,, \\ &\varphi_l^{(2)} = \frac{s_l(2+s_r)}{(2+s_l)(2+s_r)-1}\,, && \varphi_r^{(2)} = \frac{s_l}{(2+s_l)(2+s_r)-1}\,, \\ &\varphi_l^{(4)} = \frac{s_r}{(2+s_l)(2+s_r)-1}\,, && \varphi_r^{(4)} = \frac{s_r(2+s_l)}{(2+s_l)(2+s_r)-1}\,, \\ &s_l = \frac{S_l L}{2EI}\,, \qquad s_r = \frac{S_r L}{2EI}\,, && s_l\,, s_r \in \langle 0, \infty)\,. \end{aligned} \tag{24}$$

In (24), S_l, S_r, s_l, s_r are dimension [kNm] and dimensionless rotational stiffness of elastic springs at the left and right ends, respectively. The fully fixed end is obtained assuming that s_l or s_r is equal to 1.0E+15 and for the hinge 1.0E-15. Calculations are carried out introducing the double precision. The numerical results are then of satisfactory exactness.

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