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Matrix wavefront reduction by alternating directions of node renumbering

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The paper presents a heuristic method of node renumbering for wavefront reduction of the coefficient matrix of a linear system of equilibrium equations obtained in Finite Element (FEM) or in Finite Difference (FDM) methods for regular rectangular domains. From among all the node renumbering techniques for the Banachiewicz-Cholesky triangular decomposition of an assembled matrix with a compact (the least sparse possible) profile, the method presented herein assures the best reduction of matrix wavefront and time of decomposition.

1. Introduction

In the paper [7] a heuristic method of alternating directions of element renumbering for frontal solution of systems of linear equilibrium equations in FEM was presented. It was found that the method assures the best reduction of the mean and root-mean-square wavefront as well as of the solution time of the system for regular domains. Theoretical formulae for wavefront reduction were derived and corresponding numerical tests confirming the theory were presented.

In this paper we describe a corresponding method of alternating directions of node renumbering and its influence on the solution of a linear system of algebraic equations by the Banachiewicz-Cholesky decomposition of fully assembled matrices obtained in FEM and FDM for regular domains. We present an analysis of such characteristics of the triangular factor of the matrix, as wavefront and fill-in, which have the greatest impact on the efficiency of the solution process. The comparison of the presented method with other methods of node renumbering shows that it gives better results than any other method preserving the compact profile of the matrix.

2. Survey and comparison of node renumbering methods

It is well known that the efficiency of solution of a system of linear algebraic equations obtained in FEM or FDM analysis with preliminary assembly of the coefficient matrix depends upon the node numbering of the discrete model mesh. The existing methods of evaluating the best possible node renumbering can be divided into two groups:

1. Methods with a compact profile of the matrix, where its non-zero elements are located in the closest possible vicinity of the main diagonal in the form of a band or a sky-line. We can include here such methods, as Reverse Cuthill-McKee (RCM) method [4], Sloan and Randolph method [10], method of analysis of geometric configuration of the discrete model [3], and method of decomposition of the node adjacency graph, associated with the mesh, with respect to a path of nodes [8].

2. Methods with a sparse profile of the matrix, where its non-zero elements are scattered in a special, sophisticated way, ensuring that the number of zero elements becoming non-zero during

the process of matrix decomposition into triangular factors is the least possible. We can include into this group such methods, as minimum degree [4], refined quotient tree [4], parallel dissections [4], and George's nested dissections methods [4]. All these methods are based on decompositions of the node adjacency graph and defining such a renumbering of the vertices of the graph (nodes of the mesh) which results in the minimum fill-in of the matrix. During the solution of the system of equations only the non-zero elements of the matrix and those of its fill-in are stored and operated upon.

It was proved theoretically [4], that the nested dissections method is the most effective one. For example, for a square $n \cdot n$ mesh of nodes with one unknown at each node, it enables the decomposition of the corresponding matrix into triangular factors by the Banachiewicz-Cholesky method using only approximately $\frac{829}{84} n^3$ operations, whereas row-by-row or column-by-column numbering of nodes requires approximately $\frac{1}{2} n^4$ operations. The George's method reaches the minimum order of the number of required arithmetic operations, which, as it follows from the computational complexity of the problem [6], in the above example is n^3.

Nevertheless, it is striking that, despite all its advantages, the nested dissections method is hardly ever used in finite element method systems, the more so as its effectivity increases with the number of unknowns. It seems that the following factors are responsible for this situation. Firstly, all finite element method systems from their very beginning were oriented at the use of frontal [5], front-skyline [2] or matrix partitioning methods [1]. George's method, however, requires a sparse matrix storage which is a completly different approach. Secondly, the nested dissections method requires a considerable number of preliminary operations that must be performed before matrix assembling in order to decompose the graph associated with the mesh and define the numbering of nodes [4]. Finally, George's method results in a considerable increase in efficiency only for matrices of large dimensions.

In the case of regular rectangular domains, considered later in this paper, all methods of group one, except for the RCM method, give identical numbering of nodes.

3. MATRIX WAVEFRONT FOR RECTANGULAR DOMAIN FOR EXISTING NODE NUMBERING METHODS

For a given symmetric square matrix $A = [A_{ij}]$ of dimension $n \cdot n$ let us define after [4]

$$\begin{aligned} f_i(A) &= \min\{j : A_{ij} \neq 0\}\,, \\ \beta_i(A) &= i - f_i(A)\,. \end{aligned} \tag{1}$$

The number $\beta_i(A)$ is called the i-th bandwidth of the matrix.

The wavefront of matrix A for column i, or the number of active rows in column i, is defined as

$$w_i(A) = \left| \left\{ k > i : \bigvee_{l \le i} A_{kl} \neq 0 \right\} \right| \tag{2}$$

The maximum ($w_{\max}(A)$), mean ($w_{\text{mean}}(A)$) and root-mean-square ($w_{\text{mnsq}}(A)$) wavefronts of matrix A are given by

$$w_{\max}(A) = \max_{i=1,2,\ldots,n} w_i(A)\,, \tag{3}$$

$$w_{\text{mean}}(A) = \frac{1}{n} \sum_{i=1}^{n} w_i(A)\,, \tag{4}$$

$$w_{\text{mnsq}}(A) = \sqrt{\frac{1}{n} \sum_{i=1}^{n} w_i^2(A)}\,. \tag{5}$$

The profile $P(A)$ of matrix A is defined as the following subset of matrix elements

$$P(A) = \{A_{ij} : f_i(A) \le j < i\} . \tag{6}$$

The number of elements of $P(A)$ will be called the magnitude of the profile.
The following equalities hold

$$|P(A)| = \sum_{i=1}^{n} \beta_i(A) = \sum_{i=1}^{n} w_i(A) . \tag{7}$$

If $Nonz(A)$ denotes the number of non-zero elements of the profile $P(A)$, the fill-in $Fill(A)$ of the profile of A is given by

$$Fill(A) = Nonz(L) - Nonz(A) , \tag{8}$$

where L is the lower triangular factor of A resulting from the Banachiewicz-Cholesky or Gauss decomposition

$$A = L\,L^T , \tag{9}$$

$$A = L\,U . \tag{10}$$

Let us consider now a rectangular domain discretised by FDM or FEM with the help of a rectangular mesh with $n \cdot l$ nodes ($l \ge n$). If we apply the column-by-column numbering of nodes as shown in Fig. 1, we can derive the following values for maximum, mean and root-mean-square wavefront of the matrix of coefficients of the system of linear equations obtained by FEM (4-node elements) or FDM (9-node differential schemes) with one unknown at each node

$$\begin{aligned} w_{\max} &= n+1 , \\ w_{\text{mean}} &= \frac{n^2 l - n^2 + nl - l}{nl} , \\ w_{\text{mnsq}} &= \sqrt{\frac{(3l-4)n^3 + 6(l-1)n^2 + (13-3l)n - 3l - 6}{3nl}} . \end{aligned} \tag{11}$$

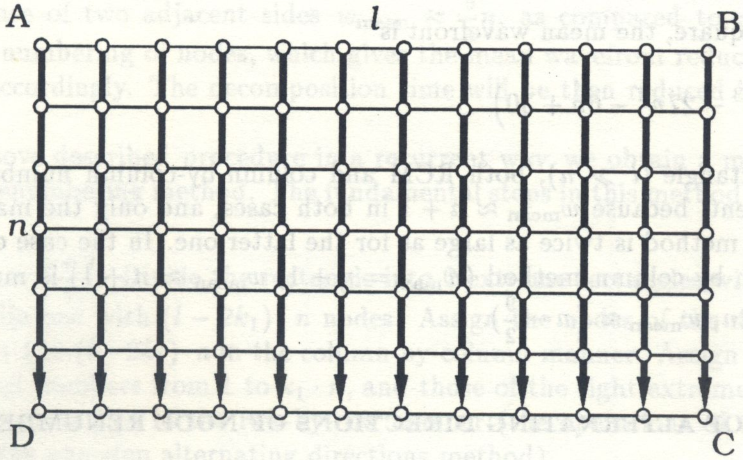

Fig. 1. Mesh for a rectangle with $n \cdot l$ nodes — nodes numbered column-by-column

In the case of a square ($l = n$), we have

$$\begin{aligned} w_{\text{mean}} &= \frac{n^2-1}{n} = n - \frac{1}{n} , \\ w_{\text{mnsq}} &= \sqrt{\frac{3n^4 + 2n^3 - 9n^2 + 10n - 6}{3n^2}} . \end{aligned} \tag{12}$$

If supernodes are located on one of the longer sides of the rectangle (AB or DC in Fig. 1), which means that supernodes are assigned the last numbers and the matrix decomposition will not be performed with respect to their corresponding rows and columns of the matrix, the values of maximum and mean wavefront are given by

$$\begin{aligned} w_{\max} &= n + l - 1 , \\ w_{\text{mean}} &= \frac{1}{2nl}\left(2n^2 l - 2n^2 + nl^2 - nl + 2n - l^2 - l\right) . \end{aligned} \tag{13}$$

In the case of a square, these formulae reduce to

$$\begin{aligned} w_{\max} &= 2n - 1 , \\ w_{\text{mean}} &= \frac{3n^3 - 4n^2 + n}{2n^2} \end{aligned} \tag{14}$$

If supernodes are located on two adjacent sides of the rectangle (e.g. AB and BC in Fig. 1), we have

$$\begin{aligned} w_{\max} &= n + l - 1 , \\ w_{\text{mean}} &= \frac{1}{2nl}\left(2n^2 l - 3n^2 + nl^2 - 3nl + 5n - l^2 + l - 2\right) . \end{aligned} \tag{15}$$

For a square, they become

$$\begin{aligned} w_{\max} &= 2n - 1 , \\ w_{\text{mean}} &= \frac{1}{2n^2}\left(3n^3 - 7n^2 + 5n - 2\right) . \end{aligned} \tag{16}$$

For the RCM method of node renumbering [4], in the case without supernodes, we obtain

$$\begin{aligned} w_{\max} &= 2n , \\ w_{\text{mean}} &= \frac{1}{6nl}\left(6n^2 l + 6nl - 6l + 2n^3 - 33n^2 + n + 30\right) . \end{aligned} \tag{17}$$

In the case of a square, the mean wavefront is

$$w_{\text{mean}} = \frac{1}{6n^2}\left(8n^3 - 27n^2 - 6n + 30\right) . \tag{18}$$

For a slender rectangle ($l \gg n$), both RCM and column-by-column numbering methods are, in principle, equivalent, because $w_{\text{mean}} \approx n + 1$ in both cases, and only the maximumn wavefront $w_{\max}$ for the former method is twice as large as for the latter one. In the case of a square domain, however, the column-by-column method ($w_{\max} = n + 1$, $w_{\text{mean}} \approx n + 1$) is much superior to the RCM one ($w_{\max} = 2n$, $w_{\text{mean}} \approx \frac{4}{3}n - \frac{9}{2}$).

4. Description of alternating directions of node renumbering method

Let us now change the column-by-column numbering of nodes in the following manner. First, we number row-by-row the $k < \frac{n}{2}$ columns of nodes, next we number column-by-column the $l - 2k$ following columns, and, finally, we number row-by-row the last k columns of nodes. We can derive then the following values for maximum and mean wavefront

$$\begin{aligned} w_{\max} &= n + k , \\ w_{\text{mean}} &= \frac{1}{nl}\left[2k^2(n-1) + k(-n^2 - n + 2) + n^2 l + nl - l - n\right] . \end{aligned} \tag{19}$$

The minimum value of w_{mean} will be reached for

$$k = \frac{n+2}{4}, \tag{20}$$

the same as in the case of alternating directions of element renumbering [9].

For a square, for k given by (20), we have for large n

$$w_{\mathrm{mean}} \approx \frac{7}{8}n + \frac{5}{8}, \tag{21}$$

which results in 12.5 per cent lower mean wavefront and over 23 per cent time reduction for matrix decomposition into triangular factors. This reduction is obtained at the cost of local increase in the maximum wavefront (cf Fig. 3) of up to 25 per cent.

The evaluation of the root-mean-square wavefront w_{mnsq} is much more complicated, but it leads to the value of k, which for $n \leq 100$ differs only slighty from the one given by (20). Therefore, this formula may be used without much loss of generality.

Applying the described method to a rectangle with supernodes on its longer side we have, accordingly

$$\begin{aligned} w_{\mathrm{max}} &= n + l - k + 2, \\ w_{\mathrm{mean}} &= \frac{1}{2nl}\left[2k^2(n-2) + k(-2n^2 + 2n + 4) + 2n^2 l + nl^2 - nl - 2n + l^2 - l\right], \\ k &= \frac{n+1}{2}, \end{aligned} \tag{22}$$

and with supernodes on its two adjacent sides

$$\begin{aligned} w_{\mathrm{max}} &= n + l - 1, \\ w_{\mathrm{mean}} &= \frac{1}{2nl}\left[k^2(n-2) + k(-n^2 + 2n) + 2n^2 l - 2n^2 + nl^2 - 3nl + 2n - l^2 + l\right], \\ k &= \frac{n}{2}. \end{aligned} \tag{23}$$

For a square ($l = n$), for large n, we have, in the case of supernodes on its longer side, $w_{\mathrm{mean}} \approx \frac{11}{8}n$, and in the case of two adjacent sides $w_{\mathrm{mean}} \approx \frac{5}{4}n$, as compared to $w_{\mathrm{mean}} \approx \frac{3}{2}n$ for the column-by-column numbering of nodes, which gives the mean wavefront reduction of 16.7 per cent and 8.3 per cent, accordingly. The decomposition time will be then reduced by 30.5 per cent and 16 per cent.

Applying the above described procedure in a recurrent way, we obtain a multi-step alternating directions of node renumbering method. The fundamental steps in this method may be summarized as follows:

Step 1. Assume $k_1 = \frac{n+2}{4}$. Divide the rectangle into 2 extreme rectangles with $k_1 \cdot n$ nodes each, and a middle one with $(l - 2k_1) \cdot n$ nodes. Assign the nodes of the middle part numbers from $k_1 \cdot n + 1$ to $(l - 2k_1) \cdot n$ in the column-by-column manner. Assign the nodes of the left extreme part numbers from 1 to $k_1 \cdot n$, and those of the right extreme part numbers from $(l - 2k_1)n + 1$ to $l \cdot n$ in the row-by-row manner (the operations of this step correspond exactly to the one-step alternating directions method).

Step 2. Assume $k_2 = \frac{k_1 + 1}{2}$. In the extreme parts of the two extreme rectangles from step 1, with $k_2 \cdot k_1$ nodes each, switch the numbering method from row-by-row to column-by-column one.

Step i. Assume $k_i = \frac{k_{i-1}}{2}$. In the extreme parts of the four extreme rectangles from step $i-1$, with $k_i \cdot k_{i-1}$ nodes each, switch the numbering method from row-by-row to column-by-column, or from column-by-column to row-by-row, depending upon the direction of numbering used in step $i - 1$. Repeat step i until k_i becomes less or equal one.

An example of application of the multi-step method of alternating directions of node renumbering is shown in Fig. 2 for $n = 13$, $k_1 = 4$, $k_2 = 3$ and $k_3 = 2$. Figure 3 presents a comparison of the wavefront change for the one-step alternating directions ($n = 13$ i $k_1 = 4$) and for the column-by-column numbering methods.

For a square, for large n, reduction of the matrix decomposition time achieved when using the multi-step alternating directions method may reach 27 per cent. Tables 1 and 2 present values of wavefront, fill-in of the matrix, and matrix decomposition time in the case of a square mesh $n = 17$ for column-by-column and alternating directions node renumbering methods: Table 1 — for the matrix obtained by FEM using 4-node elements, and Table 2 — for the one obtained by FDM using 5-node differential schemes and the Dirichlet boundary condition.

As it could have been expected, the maximum reduction is achieved for the first step of the multi-step method; the succesive steps give much smaller corrections. Therefore, the application of the one-step method is often sufficient in practical cases.

Table 1

Node numbering method	$w_{\max}$	w_{mean}	w_{mnsq}	$Fill(A)$	$\frac{Fill(A)}{Nonz(A)} \cdot 100\%$	Decomposition time [%]
column-by-column	18	16.94	17.25	3840	285.5	100
alternating directions $k_1 = 5$	22	15.39	16.02	3392	252.2	86.2
alternating directions $k_1 = 5$, $k_2 = 3$	22	15.31	15.96	3360	249.8	85.6
alternating directions $k_1 = 5$, $k_2 = 3$, $k_3 = 2$	22	15.27	15.92	3356	249.5	85.2

Table 2

Node numbering method	$w_{\max}$	w_{mean}	w_{mnsq}	$Fill(A)$	$\frac{Fill(A)}{Nonz(A)} \cdot 100\%$	Decomposition time [%]
column-by-column	15	13.24	13.95	2744	371.3	100
alternating directions $k_1 = 5$	18	12.20	12.98	2408	325.8	86.6
alternating directions $k_1 = 5$, $k_2 = 3$	18	12.14	12.92	2393	323.8	85.8
alternating directions $k_1 = 5$, $k_2 = 3$, $k_3 = 2$	18	12.13	12.91	2393	323.8	85.6

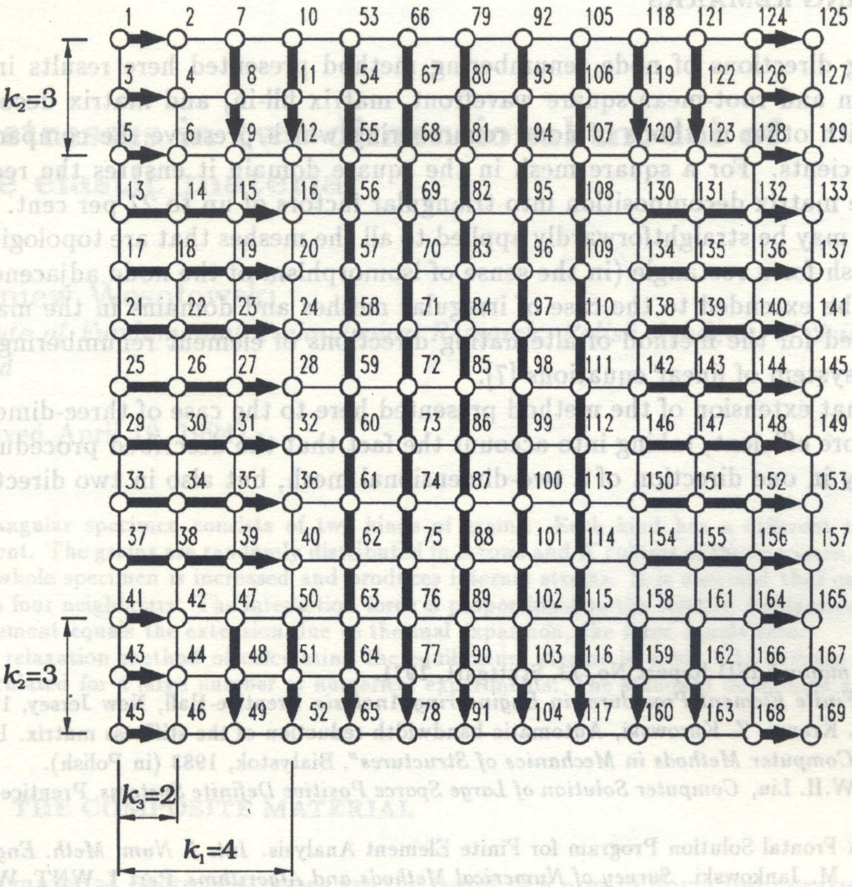

Fig. 2. Mesh for a square with 13 · 13 nodes. Three-step alternating directions node renumbering method ($n = 13$, $k_1 = 4$, $k_2 = 3$, $k_3 = 2$). Arrows show directions of node numbering

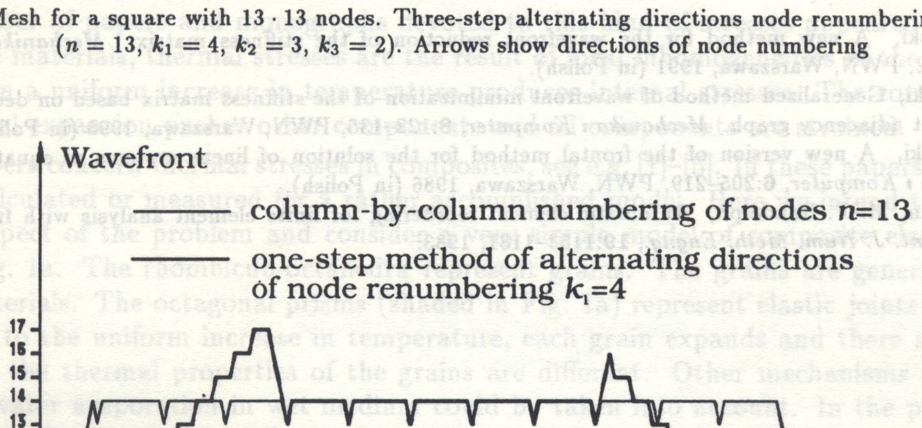

Fig. 3. Change of frontwidth for a 13 · 13 mesh in a square for column-by-column and one-step alternating directions of node renumbering methods ($k_1 = 4$)

5. Concluding remarks

The alternating directions of node renumbering method presented here results in maximum reduction of mean and root-mean-square wavefront, matrix fill-in, and matrix decomposition time as compared with other methods of node renumbering which preserve the compact profile of the matrix of coefficients. For a square mesh in the square domain it ensures the reduction of time required for the matrix decomposition into triangular factors of up to 27 per cent.

The method may be straightforwardly applied to all the meshes that are topologically equivalent to a regular mesh for a rectangle (in the sense of isomorphism of the node adjacency graphs).

Its use may be extended to the case of irregular meshes and domains in the manner similar to the one presented for the method of alternating directions of element renumbering for the frontal solution of the system of linear equations [7].

It appears that extension of the method presented here to the case of three-dimensional meshes may be even more efficient, taking into account the fact that the described procedure may be then applied not only in one direction of a two-dimensional mesh, but also in two directions of a space one.

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