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Problems of the equilibrium of a rigid body and mechanical systems

Sanh Do
Department of Applied Mechanics, Hanoi University of Technology, Vietnam

Khoa Dang Do
Department of Mechanical Engineering, University of Texas at Austin, USA

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In this article one of the greatest generalized methods for establishing the equilibrium equations of a rigid body and the set of rigid bodies is proposed. It is related to six equations of moments of force about six the edges of a reference tetragon. It is possible to obtain different alternatives by substituting the force moment-equation for the force project-equation. Four different forms of equilibrium are established. It is important writing equilibrium equations of bodies possible to apply the special software as Mathcad, Maple.

Keywords: equilibrium of a rigid body, equilibrium of mechanical system, matrix method, frame tetragon, generalized tetragon frame of axis

1. Introduction

As it is known, there are two methods for solving the equilibrium problem of a rigid body and mechanical systems. One is based on the Poisson theorem (the geometrical method) and the other- on the principle of virtual work (the analytical method).

In the present article, the matrix expression for two mentioned methods is introduced, with which it is easy to apply such software as Maple, Mathcad, etc., for solving these problems.

2. Equations of equilibrium of a rigid body

Let us consider a free body in equilibrium under the action of a set of forces $(\vec{F}_1, \vec{F}_2, \ldots, \vec{F}_N)$. Based on the Poinsot theorem [1], the set of forces under consideration is reduced into the given point O of reference to a resultant force $\vec{R}' = \sum \vec{F}_k$, $\vec{R}'$, $\vec{R}'$ and a resultant couple of moment

$$\vec{M}_o = \sum \vec{m}_o(\vec{F}_k) = \sum_{k=1}^{N} \vec{r}_k \times \vec{F}_k,$$

where $\vec{r}_k$ denotes the position vector ,which runs from point O to any point on the line of action of the force $\vec{F}_k$.

As it is known, a body is in equilibrium, if and only if the resultant force and the resultant couple of moment are both zero, i.e.

$$\vec{R}' = \sum_{k=1}^{N} \vec{F}_k = 0, \qquad \vec{M}_o = \sum_{k=1}^{N} \vec{r}_k \times \vec{F}_k = 0. \tag{1}$$

These two vector equations may be written as follows:

$$\vec{R}' = \sum_{k=1}^{N} \vec{F}_k = 0 \longleftrightarrow \begin{cases} \sum\limits_{k=1}^{N} F_{kx} = 0, \\ \sum\limits_{k=1}^{N} F_{ky} = 0, \\ \sum\limits_{k=1}^{N} F_{kz} = 0, \end{cases} \qquad \vec{M}_o = \sum_{k=1}^{N} \vec{m}_o(\vec{F}_k) \longleftrightarrow \begin{cases} \sum\limits_{k=1}^{N} \bar{m}_{ox}(\vec{F}_k) = 0, \\ \sum\limits_{k=1}^{N} \bar{m}_{oy}(\vec{F}_k) = 0, \\ \sum\limits_{k=1}^{N} \bar{m}_{oz}(\vec{F}_k) = 0. \end{cases} \tag{2}$$

The six equations (2) are both necessary and sufficient conditions to create complete equilibrium. The reference axes may be chosen arbitrarily for convenience, the only restriction is that a right-handed coordinate system must be used with vector notation .The six scalar relations of equations (2) are independent conditions, since any of them may be valid by itself. The number of moment equations about coordinate axes is of three only [7–9]. However, the projection equations of forces on the coordinate axes might replace a part or a whole moment of equations of forces . In connection with this we have the following theorem.

Theorem. *A body is in equilibrium under the action of a set of forces, if and only if the following equations are fulfilled:*

$$\sum_{k=1}^{N} \vec{m}_{\Delta_i}(\vec{F}_k) = 0, \qquad i = 1, \ldots, 6, \tag{3}$$

where $\Delta_i,\ i = 1, \ldots, 6$ *are consecutive edges of a reference tetragon (Fig. 1).*

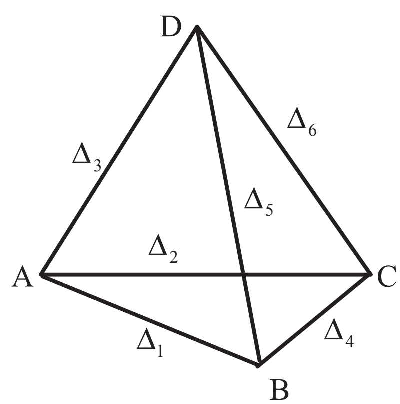

Fig. 1. The tetragon frame of axis

In order to prove this, it is necessary to show that the set of equations (3) is equivalent to the sic equations (2).

Indeed, if the relations (2)are valid, then the resultant force is equal to zero by the first three equations of (2). Next, the resultant couple of moment is also equal to zero. This is based on the last three equations (2) and the theory of the variation of the resultant couple of moment. On contrary, if the set of equations (3) is valid, then the resultant force of the set of forces is equal to zero, because if it is not satisfactory then the resultant force must lie the same time on six edges of tetragon. This is impossible.

Note 1. The equations (3) are still valid for the case when apex A tends to infinity and the edges at apex B are perpendicular to each other (Fig. 2).

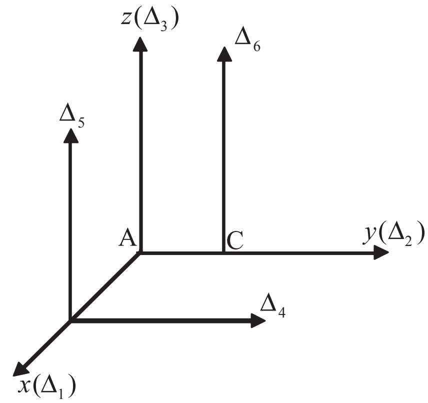

Fig. 2. The generalized tetragon frame of axis

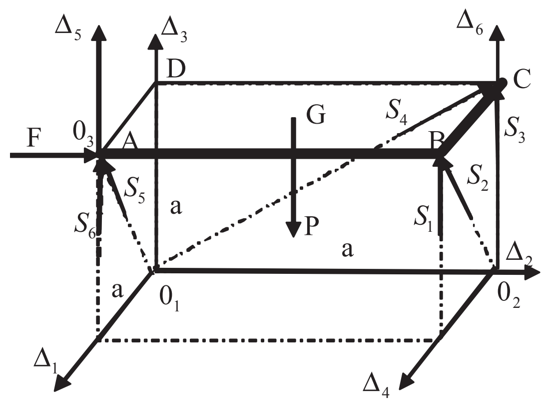

Fig. 3. Example illustrating the equilibrium equations (3)

Example 1. A uniform steel square plate (Fig. 3) with a weight P is suspended in the horizontal plane by six mass less rods, which form a cube. Compute the reaction forces in the rods.

Let us consider the plate $ABCD$ in equilibrium under the action of its weight, the applied force F and the reaction forces of rods, which are assumed to be in compression state. Denote the reaction forces by S_1, S_2, S_3, S_4, S_5, S_6. The axes are chosen as shown in Fig. 3. The equations of equilibrium of the plate are written as follows:

$$\sum \bar{m}_{\Delta_1}(\vec{F}) = -Fa + S_1 a + 0.5a\sqrt{2}S_2 + S_3 a - 0.5aP = 0, \tag{4a}$$

$$\sum \bar{m}_{\Delta_2}(\vec{F}) = -S_1 a - S_6 a + 0.5aP = 0, \tag{4b}$$

$$\sum \bar{m}_{\Delta_3}(\vec{F}) = Fa - 0.5a\sqrt{2}S_2 = 0, \tag{4c}$$

$$\sum \bar{m}_{\Delta_4}(\vec{F}) = -Fa - 0.5a\sqrt{2}S_4 - 0.5a\sqrt{2}S_5 - S_6 a + 0.5aP = 0, \tag{4d}$$

$$\sum \bar{m}_{\Delta_5}(\vec{F}) = -0.5a\sqrt{2}S_2 - 0.5a\sqrt{2}S_4 = 0, \tag{4e}$$

$$\sum \bar{m}_{\Delta_6}(\vec{F}) = Fa + 0.5a\sqrt{2}S_5 = 0. \tag{4f}$$

The reaction forces in the rods will equal:

$$S_1 = F, \quad S_2 = F\sqrt{2}, \quad S_3 = 0.5P - F, \quad S_4 = -F\sqrt{2}, \quad S_5 = -F\sqrt{2}, \quad S_6 = F + 0.5P.$$

Thus, the rods 1 and 2 (and 6) are in tension state, the rods 4, 5 are in the compression state (the rod 3 is in tension if $F < 0.5P$).

Note 2. The equations of equilibrium are written in different forms. Which are:

First alternative

The form of six moment equations about six axes in which three axes lie in the coordinate plane xy and are un-concurrent and the rest is perpendicular to this coordinate plane (i.e. parallel to the z-axis)and incompliant and un-coplanar. This alternative is one of the four alternatives, which is named the first alternative. The different alternatives can be constructed by substituting the projection equations for moment equations. Such way we obtain the

Second alternative

The form of one force-projection equation and five moment equations. For this purpose one of the moment equations about the coordinate axis (the 4-axis,5-axis and 6-axix) is substituted by the force projection equation on the axis, which lies in coordinate plane being perpendicular to the axis of the taken moment (the projection equation of forces on the z-axis is substituted for the moment equation about the 4-axis, the projection equation of all forces on the x-axis – for the moment equation about the 6-axis, the projection equation of forces on the y-axis – for the moment equation about the 5-axis).

Third alternative

The form of two projection equations and four moment equations. This form is constructed by substituting two projection equations for two moment equations about two axes in three axes: 4-axis, 5-axis and 6-axis. Two moment equations about two axes are substituted by two of the projection equations on two axes being intersections of the coordinate planes including these two axes with the rest coordinate plane. For example, the projection equations on the x-coordinate axis and y-coordinate axis are substituted for the moment equations about 6-axis and 5-axis respectively.

Fourth alternative

The form of the three projection equations of forces on three coordinate axes is substituted for three moment equations about these axes.

3. The matrix method expressing the equations of equilibrium

The moment equations of forces for equilibrium in almost all of the books on engineering mechanics [7–9]are written in the form of equations of moments about axes, either in the analytical expression of the vector product. However, it is easy to create these equations by the wave matrix method [6, 10, 11]. This method allows to take advantages of such software as Mathcad, Matlab, Maple in calculation.

Some calculations involving matrix operations [6, 10, 11] are briefly provided below.

3.1. The matrix expression of vector

In 3-dimensional space a vector $\vec{a}(a_x, a_y, a_z)$ is identified by three projection components on reference coordinate system, i.e. (a_x, a_y, a_z) .In connection with this, let us introduce a (3×1) column matrix $\mathbf{a}$ constructed by components, which are projections on three coordinate axes of vector $\vec{a}$, i.e.

$$\mathbf{a}^T = [a_x, a_y, a_z],$$

where notation T located at the right top of the letter denotes matrix transposition and the bold letter designates a matrix.

By means of the expression of the vector as a matrix, some calculations of the vector might be substituted by the ones of the matrix.

a) vector addition. The sum of two vectors $\vec{a}(a_x, a_y, a_z)$, $\vec{b}(b_x, b_y, b_z)$ is a vector $\vec{c}(c_x, c_y, c_z)$, where $c_x = a_x + b_x$, $c_y = a_y + b_y$, $c_z = a_z + b_z$. Therefore the vector addition may be substituted by the matrix addition, i.e. $\mathbf{c} = \mathbf{a} + \mathbf{b}$,

b) the scalar product. The scalar product of two vectors $\vec{a}$, $\vec{b}$ reduces to the algebraic multiplication of the corresponding projections on the coordinate axes and their addition, i.e.

$$\mathbf{a}^T\mathbf{b} = \mathbf{b}^T\mathbf{a},$$

c) the vector product. Consider two vectors $\vec{a}$, $\vec{b}$. We apply these vectors to the same origin of a reference coordinate system. The vector product of two vectors is defined as a free vector $\vec{c}(c_x, c_y, c_z)$, where $c_x = a_y b_z - a_z b_y$, $c_y = a_z b_x - a_x b_z$, $c_z = a_x b_y - a_y b_x$. In order to express the vector product in matrix form, let us apply the wave matrix, which is a (3×3) asymmetric matrix of the form:

$$\tilde{\mathbf{a}} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}.$$

We have:

$$\mathbf{c} = \tilde{\mathbf{a}}\mathbf{b} = -\tilde{\mathbf{b}}\mathbf{a}.$$

3.2. The equilibrium equations of a rigid body in matrix form

The equilibrium equations (2) may be written as follows:

$$\sum \mathbf{F_k} = \mathbf{0}, \qquad \sum \tilde{\mathbf{r}}_\mathbf{k}.\mathbf{F_k} = \mathbf{0}, \tag{5}$$

where

$$\mathbf{F}_k^T = \begin{bmatrix} F_{kx} & F_{ky} & F_{kz} \end{bmatrix}, \qquad \tilde{\mathbf{r}}_k = \begin{bmatrix} 0 & -z_k & y_k \\ z_k & 0 & -x_k \\ -y_k & x_k & 0 \end{bmatrix}.$$

Note:

1. The above mentioned forms of the equilibrium equations can be written in the matrix form. However, only the last three forms are easy to use because in these forms, the projection components of forces on the coordinate axes are identical, although the coordinates of the points of action of forces are different, they can be calculated easily by means of the formula of the coordinate transformation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ z_o \\ 1 \end{bmatrix}, \tag{6}$$

where x_o, y_o, z_o and x, y, z are the coordinate components of point M in the coordinate system having the origin at the point O and O' respectively, a, b, c are the coordinate components of O in respect to the coordinate system at O', which takes positive values, if the translation displacement of the coordinate system is along positive direction of the axis and takes minus values in contrary case.

2. Based on the mentioned note, that the moment equation of the $\vec{F}$ about axes Δ being parallel to the z-coordinate axis can be calculated by the following formula

$$\vec{m}_{\Delta}(\vec{F}) = \begin{bmatrix} -y & x & 0 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ 0 \end{bmatrix}, \tag{7}$$

where F_x, F_y are the projection components of the force $\vec{F}$ on the x,y-axes respectively of the right-handed rectangular coordinate system $Oxyz$, which has the z-axis to be coincided to the Δ-axis.

By (7) the moment equation of force $\vec{F}$ about Δ' being parallel to the Δ-axis, will be computed by the formula:

$$\vec{m}_{\Delta'}(\vec{F}) = \begin{bmatrix} -(y+b) & x+a & 0 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ 0 \end{bmatrix}. \tag{8}$$

It is important that when determining the moment equations about parallel axes to apply directly the formula (8). In case when drawing the moments about the axes, which are neither identical to the coordinate axes nor parallel to the coordinate axes nor parallel to the z-axis it is necessary to apply the rotation of coordinate axes [2, 6, 7].

3. From the above mentioned alternatives it is easy to determine the forms of equilibrium equations of a body in two dimensions.

Example 2. An uniform square plate of weight P and length of edge a is supported by two bearings A, B and rests smoothly at the middle point E of the edge BC. The plate is subjected by the force $\vec{F}$ at H ($\|DH\| = h$) is in parallel direction to the edge AD and inclines to the horizontal plane at an angle α. Determine the reaction forces at A, B and E (Fig. 4).

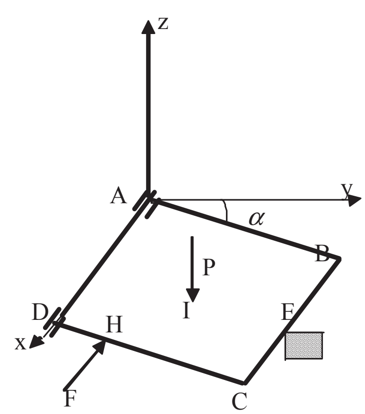

Fig. 4. Example illustrating the equilibrium of spatial body

Let us choose the rectangular coordinate axes $Axyz$ as shown in Fig. 4. Consider the plate is in equilibrium under action of the applied forces: $\vec{F}$ at $H(a, h\cos\alpha, h\sin\alpha)$, $\vec{P}(0, 0, -P)$ at $I(0.5a, 0.5a\cos\alpha, 0.5a\sin\alpha)$ and the reaction forces: $\vec{R}_A(X_A, Y_A, Z_A)$ at $A(0, 0, 0)$, $\vec{R}_B(0, Y_B, Z_B)$ at $B(a, 0, 0)$, $\vec{N}(0, N\sin\alpha, N\cos\alpha)$ at $E(0.5a, 0.5a\cos\alpha, -0.5a\sin\alpha)$.

Writing the equilibrium equations corresponding to the *fourth alternative*, we get:

$$\begin{bmatrix} X_A \\ Y_A \\ Z_A \end{bmatrix} + \begin{bmatrix} 0 \\ Y_B \\ Z_B \end{bmatrix} + \begin{bmatrix} 0 \\ N\sin\alpha \\ N\cos\alpha \end{bmatrix} + \begin{bmatrix} -F \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -P \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{aligned} &\begin{bmatrix} 0 & -h\sin\alpha & h\cos\alpha \\ h\sin\alpha & 0 & -a \\ -h\cos\alpha & a & 0 \end{bmatrix} \begin{bmatrix} -F \\ 0 \\ 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & -0.5a\sin\alpha & 0.5a\cos\alpha \\ 0.5a\sin\alpha & 0 & -0.5a \\ -0.5a\cos\alpha & 0.5a & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -P \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & a & 0 \end{bmatrix} \begin{bmatrix} 0 \\ Y_B \\ Z_B \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & a\sin\alpha & 0.5a\cos\alpha \\ a\sin\alpha & 0 & -0.5a \\ -0.5a\cos\alpha & 0.5a & 0 \end{bmatrix} \begin{bmatrix} 0 \\ N\sin\alpha \\ N\cos\alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

We have then:

$$\begin{aligned} X_A - F &= 0, \\ Y_A + Y_B + N\sin\alpha &= 0, \\ Z_A + Z_B + N\cos\alpha - P &= 0, \\ -0.5a\cos\alpha P + aN &= 0, \\ -h\sin\alpha F - 0.5aP - 0.5a\cos\alpha N - aZ_B &= 0, \\ h\cos\alpha F + 0.5a\sin\alpha N + aY_B &= 0. \end{aligned}$$

Solving these equations we obtain:

$$\begin{aligned} X_A &= F, \\ N &= 0.5\cos\alpha P, \\ Y_B &= -(0,25P + \frac{h}{a}\cos\alpha F), \\ Z_B &= \frac{h}{a}\sin\alpha - (0.5 + 0,25\cos^2\alpha)P, \\ Z_A &= P(1,5 - 0.5\cos^2\alpha)P - \frac{h}{a}\sin\alpha. \end{aligned}$$

Example 3. The vertical mast supports force F and is constrained by two fixed cables BC and BD by a ball-socket connection at A (see [6], pp.130.) Calculate the tensions T_1 in the cable BD, T_2 in the cable BC and the reaction force at A.

Let us choose the rectangular coordinate axes $Axyz$ shown as on Fig. 5. Consider the mast to be equilibrium under action of the applied force $\vec{F}(0, F, 0)$ at the action point $G(0, 0.5)$, the tensions $\vec{T_1}(T_{1x}, T_{1y}, T_{1z})$ at $D(4, 4, 2)$, $\vec{T_2}(T_{2x}, 0, T_{2z})$ at $C(5, 0, 0)$ and the reaction force $\vec{R}_A(X_A, Y_A, Z_A)$ at

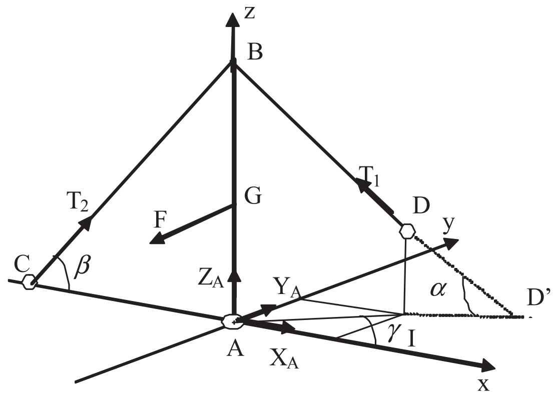

Fig. 5. Equilibrium of the vertical mast

$A(0,0,0)$. It is easy to calculate:

$$\sin\alpha=\frac{\sqrt{6}}{3},\quad \cos\alpha=\frac{\sqrt{3}}{3},\quad \sin\gamma=0.5\sqrt{2},\quad \cos\gamma=0.5\sqrt{2},\quad \sin\beta=0.4\sqrt{5},\quad \cos\beta=0.2\sqrt{5},$$

$$T_{1x}=T_1\cos\alpha\cos\gamma=0.5\frac{\sqrt{6}}{3}T_1,\quad T_{1y}=T_1\cos\alpha\sin\gamma=0.5\frac{\sqrt{6}}{3}T_1,\quad T_{1z}=T_1\sin\alpha=\frac{\sqrt{6}}{3}T_1,$$

$$T_{2x}=T_2\cos\beta=0.2\sqrt{5}T_2,\qquad T_{2y}=0,\qquad T_{2z}=T_2\sin\beta=0.4\sqrt{5}T_2.$$

Writing the equilibrium equations (the *fourth alternative*), we get:

$$\begin{bmatrix} X_A\\ Y_A\\ Z_A\end{bmatrix}+\begin{bmatrix} -T_1\cos\alpha\cos\gamma\\ -T_1\cos\alpha\sin\gamma\\ T_1\sin\alpha\end{bmatrix}+\begin{bmatrix} T_2\cos\beta\\ 0\\ T_2\sin\beta\end{bmatrix}+\begin{bmatrix} 0\\ -F\\ 0\end{bmatrix}=\begin{bmatrix} 0\\ 0\\ 0\end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 5\\ 0 & 0 & -5\\ -5 & 5 & 0\end{bmatrix}\begin{bmatrix} -T_1\cos\alpha\cos\gamma\\ -T_1\cos\alpha\sin\gamma\\ T_1\sin\alpha\end{bmatrix}+\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 5\\ 0 & -5 & 0\end{bmatrix}\begin{bmatrix} T_2\cos\beta\\ 0\\ T_2\sin\beta\end{bmatrix}$$
$$+\begin{bmatrix} 0 & -5 & 0\\ -5 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix} 0\\ -F\\ 0\end{bmatrix}=\begin{bmatrix} 0\\ 0\\ 0\end{bmatrix}.$$

By writing the matrix equations in the development form, we obtain:

$$\begin{aligned}
X_A-\frac{0,5\sqrt{6}}{3}T_1+0,2\sqrt{5}T_2&=0,\\
Y_A-\frac{0,5\sqrt{6}}{3}T_1-F&=0,\\
Z_A+\frac{\sqrt{6}}{3}T_1+0,4\sqrt{5}T_2&=0,\\
\frac{\sqrt{6}}{3}T_1+2\sqrt{5}T_2&=0,\\
-\frac{5\sqrt{6}}{3}T_1+2\sqrt{5}T_2&=0,\\
\frac{2,5\sqrt{6}}{3}T_1-\frac{2,5\sqrt{6}}{3}T_1&=0.
\end{aligned}$$

Solving these equations (the last equation is fulfilled identically), we obtain:

$$X_A = -4\,\mathrm{kN}, \qquad Y_A = 2\,\mathrm{kN}, \qquad Z_A = 0\,\mathrm{kN},$$
$$T_1 = -2\sqrt{6}\,\mathrm{kN} \approx -4,9\,\mathrm{kN}, \qquad T_2 = 2\sqrt{5}\,\mathrm{kN} \approx 4,47\,\mathrm{kN}.$$

Component X_A of the reaction force at A and the tension T_1 in the cable are opposite to the direction chosen in Fig. 5.

Note that it is possible that you will have to apply the second alternative for solving this problem, i.e. the system of five moment equations and one projection equation. For this purpose let us replace the equations of projection of forces on the x-y axes by the moment equations about Cz' and Dz'' being parallel to the z-axis, and having origin at $C(-5,0,0)$ and $D(4,4,2)$ respectively. As it is above mentioned, the coordinate components of the forces are unchanged.

In case the coordinate system $Axyz$ is translated to C in formula (7) we take: $a = 5$, $b = 0$, $c = 0$, but in the case the coordinate system $Axyz$ is translated to D, then $a = -4$, $b = -4$, $c = -2$. In other words, the system of moment equations consists of the moment equations about 3 xyz-axes, the moment equations about the Dz'' and Cz', which are parallel to the z-axis having the origins at D and C.

Therefore the equilibrium equations of the mast now will be:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 5 & -5 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_A \\ Y_A \\ Z_A \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 5 & 0 \\ 0 & 0 & -5 & 0 \\ -5 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -5 & 10 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -T_1\cos\alpha\cos\gamma \\ -T_1\cos\alpha\sin\gamma \\ T_1\sin\alpha \\ 1 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & -5 & 0 & 0 \\ 5 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} T_2\cos\beta \\ 0 \\ T_2\sin\beta \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & -5 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 5 & -5 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -F \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

It is easy to see that the fourth and fifth rows of generalized wave matrices and are deduced from the third row by taking into account the displacement of the origin of the axis (C and D) in the translation transformation (7).

Example 4. The homogenous rectangular door $ABCD$ of the length $AB = a\sqrt{3}$, width $AD = a$ and weight P rotates about the vertical axis AB. The door is supported by means of the ball-and-socket hinge at A and the bottom bearing B. The door is positioned at the angle of 120° with respect to the door mould . Force Q acts at D and is parallel to the edge EA of the door frame. The door is in equilibrium by the cable EC. Compute the tension in the cable and the reaction forces at A and B (see Fig. 6).

Let us choose the rectangular coordinate system Axyz in which the z-axis is coincided with the rotational axis, the x-axis is perpendicular to plan $ABCD$ and the y-axis is along the edge AD. At the given position, the sizes are defined as in the figure. The plate $ABCD$ is in equilibrium under action of the following forces: $\vec{R}_A(X_A, Y_A, Z_A)$ with $A(0,0,0)$, $\vec{R}_B(X_B, Y_B, 0)$ with $B(0,0,a\sqrt{3})$, $\vec{T}(-T\,c1\,c2, c1\,s2\,T, s1\,T)$ with $E(0,-a,0)$, $\vec{P}(0,0,-P)$ with $G'(0.5\,ac3, 0.5\,as3, 0)$, $\vec{Q}(0,Q,0)$ with $D(ac3, as3, 0)$, where: $c1 \equiv \cos\alpha$, $s1 \equiv \sin\alpha$, $c2 \equiv \cos\beta$, $s2 \equiv \sin\beta$, $c3 \equiv \cos\gamma$, $s3 \equiv \sin\gamma$, $\alpha = 45\,\deg$, $\beta = 60°$, $\gamma = 30°$.

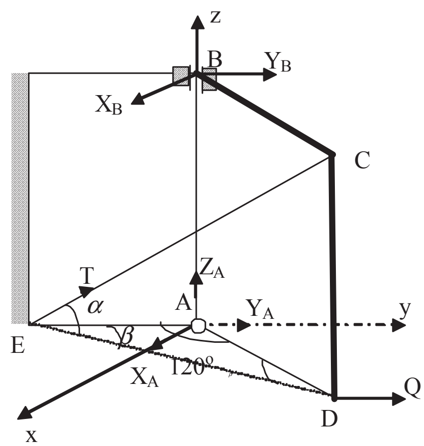

Fig. 6. Equilibrium of the door of a dam

Let us apply the equilibrium equations in the third form consisting of four moment equations about the axes x, y, z, the axis having the origin at $E(\Delta_4)$, and two projection equations on the y, z axes, that are:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_A \\ Y_A \\ Z_A \end{bmatrix} + \begin{bmatrix} 0 & -a\sqrt{3} & 0 \\ a\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 \\ -a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_B \\ Y_B \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -a \\ 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Tc\alpha c\beta \\ Tc\alpha s\beta \\ Ts\alpha \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & as\,\gamma \\ 0 & 0 & -ac\,\gamma \\ -as\,\gamma & ac\gamma & 0 \\ -a(1+s\,\gamma) & ac\,\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ Q \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0.5as\,\gamma \\ 0 & 0 & -0.5ac\,\gamma \\ -0.5as\,\gamma & 0,5ac\,\gamma & 0 \\ -a(1+s\,\gamma) & 0.5ac\,\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -P \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The equilibrium equations in development are of the form as below:

$$\begin{aligned} -a\sqrt{3}Y_B - a\sin\alpha T - 0.5a\sin\gamma P &= 0, \\ a\sqrt{3}X_B + 0,5a\cos\gamma P &= 0, \\ a\cos\alpha\cos\beta T + a\cos\gamma Q &= 0, \\ -aX_A - aX_B + a\cos\gamma Q &= 0, \\ Y_A + Y_b + T\cos\alpha\cos\beta + Q &= 0, \\ Z_A + \sin\alpha T - P &= 0. \end{aligned}$$

Solving these equations we obtain:

$$X_A = 0.25(P + 2Q\sqrt{3}), \qquad Y_A = 0.5(P\frac{\sqrt{3}}{6} + Q), \qquad Z_A = P + Q\sqrt{3},$$

$$X_B = -0.25P, \qquad Y_B = Q - \frac{0.5\sqrt{3}}{6}P, \qquad T = -Q\sqrt{6}.$$

The forces with results of minus signs have opposite direction. The matrix method allows to use easily such software as Mathcad, Maple to solve the problems of the equilibrium body.

4. The equilibrium equations of a body in the of forces in plane

Let us consider a body to be in the equilibrium state of forces in plane. This is the particular case of the spatial forces. In this case the resultant vector of force lies in the same plane of forces, but the resultant vector of the couple of forces is perpendicular to the plane of forces. Therefore it is possible to apply all obtained results from the problem of equilibrium of spatial forces to this problem. For this aim let us choose a coordinate system of reference as follows: the xy- plane includes the forces, and therefore the z-axis is perpendicular to this plane. Using this way we directly get three alternatives from the fourth alternative of the equilibrium equations of the spatial forces:

First alternative

Three equations of the zero sum of the moments of all forces about three points, not on the same straight line. These are three moment equations about three axes, which are same perpendicular to the plane of forces. Three equations about three axes in same the plane of forces are identified to zero.

Second alternative

This form consists one projection equation on the axis and two equations of moments of forces about two; the points must not lie perpendicular to the projection axis. This alternative is corresponding to the *third alternative*.

Third alternative

An alternative set of equilibrium equations is two of the projection equations and one of the moment equations, for example, two of the projection equations on x-and y-axes and the moment equation about the point of intersection of two axes. This alternative is corresponding to the fourth alternative in the case of spatial forces.

Example 5. Consider the beam in equilibrium under action of the forces shown as in Fig. 7. Compute the reaction forces at A and C.

Let us choose the rectangular coordinate system Axy, where the axis Ax is in horizontal direction. We write the equilibrium equations of the beam under consideration in third alternative, which are of the form:

$$\sum \begin{bmatrix} F_{kx} \\ F_{ky} \end{bmatrix} = 0, \qquad \sum \begin{bmatrix} -y_k & x_k \end{bmatrix} \begin{bmatrix} F_{kx} \\ F_{ky} \end{bmatrix} = 0. \tag{9}$$

For the aim of writing the equations (9) let us give an account of the projections and the point of action of the forces, which are: $\vec{R}_A(X,Y)$, $A(0,0)$, $\vec{N}_C(-Ns1, Nc1)$, $C(3ac1, 3as1)$, $\vec{P}(0,-P)$,

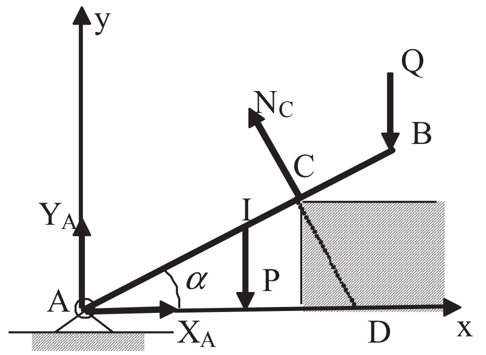

Fig. 7. Matrix method applying for an equilibrium body in the plane

$I(2ac1, 2as1)$, $\vec{Q}(0, -Q)$, $D(4ac1, 4as1)$. Equations (9) are written as follows:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -3as1 & 3ac1 \end{bmatrix} \begin{bmatrix} -Ns1 \\ Nc1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -4as1 & 4ac1 \end{bmatrix} \begin{bmatrix} 0 \\ -Q \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2as1 & 2ac1 \end{bmatrix} \begin{bmatrix} 0 \\ -P \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{9a}$$

where $c1 \equiv \cos\alpha$, $s1 \equiv \sin\alpha$.

Equations of equilibrium of the beam under consideration now are of the form:

$$\begin{aligned} X - Ns1 &= 0, \\ Y + Nc1 - P - Q &= 0, \\ 3aN - 2ac1\,P - 4ac1\,Q &= 0. \end{aligned} \tag{9b}$$

In order to write the equations of the equilibrium in different alternatives, let us take into account (7).

Second alternative

Two moment equations about A and B and one projection equation on the x-axis, i.e. substitute the moment equation of forces about $D\,(3a/c1, 0)$ for the projection equation on the y-axe, that is:

$$\begin{bmatrix} 1 & 0 \\ 0 & -\frac{3a}{c1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -3as1 & -3a\left(c1 + \frac{1}{c1}\right) \\ -3as1 & -3ac1 \end{bmatrix} \begin{bmatrix} -Nc1 \\ Ns1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -4as1 & a\left(4c1 - \frac{3}{c1}\right) \\ -4as1 & 4ac1 \end{bmatrix} \begin{bmatrix} 0 \\ -Q \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2as1 & a\left(2c1 - \frac{3}{c1}\right) \\ -2as1 & 2ac1 \end{bmatrix} \begin{bmatrix} 0 \\ -P \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

In accordance with *second alternative*, the equations of equilibrium of the system are of the form:

$$\begin{aligned} X - s1\,N &= 0, \\ -\frac{3a}{c1}Y - a\left(4c1 - \frac{3}{c1}\right)Q - a\left(2c1 - \frac{3}{c1}\right)P &= 0, \\ 3aN - 4ac1\,Q - 2ac1\,P &= 0. \end{aligned} \tag{9c}$$

Third alternative

These are three equations of moments about the points A, D and E, where $E\,(0, 3a/s1)$, E is the point of intersection between the axis Ay and DC. In this case the coordinate system is translated from A to E. Therefore in (7) we take $a = 0$ and $b = -3a/s1$. Therefore the equations of equilibrium corresponding to this alternative are established, substituting the moment equation about E for the projection equation on the x-axis, which are:

$$\begin{aligned} &\begin{bmatrix} \frac{3a}{s1} & 0 \\ 0 & -\frac{3a}{c1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} 3a\left(\frac{1}{s1} - s1\right) & 3ac1 \\ -3as1 & 3a\left(c1 - \frac{1}{c1}\right) \\ -3as1 & -3ac1 \end{bmatrix} \begin{bmatrix} -Ns1 \\ Nc1 \end{bmatrix} \\ &+ \begin{bmatrix} a\left(\frac{3}{s1} - 4s1\right) & 4ac1 \\ -4as1 & a\left(4c1 - \frac{3}{c1}\right) \\ -4as1 & 4ac1 \end{bmatrix} \begin{bmatrix} 0 \\ -Q \end{bmatrix} \\ &+ \begin{bmatrix} a\left(\frac{3}{s1} - 2s1\right) & 2ac1 \\ 2as1 & a\left(2c1 - \frac{3}{c1}\right) \\ -2as1 & 2as1 \end{bmatrix} \begin{bmatrix} 0 \\ -P \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

The equations of equilibrium by the *third alternative* of the system under consideration are of the form:

$$\begin{aligned} \frac{3a}{s1}X - 4ac1Q - 2ac1P &= 0, \\ -\frac{3a}{c1}Y - a\left(4c1 - \frac{3}{c1}\right)Q - a\left(2c1 - \frac{3}{c1}\right)P &= 0, \\ 3aN - 4ac1Q - 2ac1P &= 0. \end{aligned} \tag{9d}$$

Example 6. Two spans of a bridge are connected by the hinges at A, B and C. Each span has a weight of P, the load Q acts at D. All the dimensions are shown in Fig. 8. Compute the reaction forces at the hinges.

Firstly, let us consider the two spans of the bridge in whole as one body, which is in equilibrium under the action of the system of forces: the reaction forces at A, and C, the applied forces: the weight of the bridge spans and load Q.

Secondly, we choose the coordinate system‘s origin at A, the x-axis is horizontal and pointing right, the y-axis is vertical and pointing up. Let us define the forces and their points of action. These are $\vec{R}_A(X_A, Y_A)$, $A(0, 0)$, $\vec{R}_C(X_C, Y_C)$, $C(6a, 0)$, $\vec{P}_1(0, -P)$, $I_1(a, 2a)$, $\vec{P}_2(0, -P)$, $I_2(5a, 2a)$, $\vec{Q}(0, -Q)$, $D(2a, 3a)$.

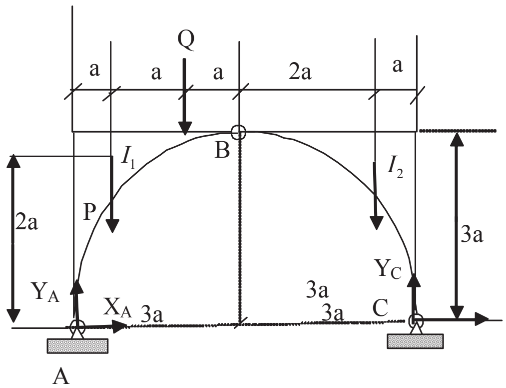

Fig. 8. Matrix method applying equilibrium of bodies system in plane

Write the equilibrium equations of the system in the *second alternative*, which consists of two moment equations about A and C and one projection equation on the x-axis, which are:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -6a \end{bmatrix} \begin{bmatrix} X_A \\ Y_A \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 6a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_C \\ Y_C \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & a \\ 0 & 5a \end{bmatrix} \begin{bmatrix} 0 \\ -P \end{bmatrix}$$
$$+ \begin{bmatrix} 1 & 0 \\ 0 & 2a \\ 0 & -4a \end{bmatrix} \begin{bmatrix} 0 \\ -Q \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -5a \\ 0 & -a \end{bmatrix} \begin{bmatrix} 0 \\ -P \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{10}$$

The equations (10) are of the form:

$$\begin{aligned} X_A + X_C &= 0, \\ 6aY_C - aP - 2aQ - 5aP &= 0, \\ -6aY_A + 5aP + 4aQ + aP &= 0. \end{aligned} \tag{10a}$$

As a next step let us consider the right span of the bridge, which is in equilibrium under the action of the forces: $\vec{R}_B(X_B, Y_B)$, $B(0,0)$, $\vec{P}_2(0,-P)$, $I_2(2a,-a)$, $\vec{R}_C(X_C, Y_C)$, $C(3a,-3a)$, where the chosen coordinate system is the old one translating to B. When the old frame is translated to C the coordinates of the points of application will be: $B(-3a, 3a)$, $I_2(-a, 2a)$, $C(0,0)$.

Equations of equilibrium of the right span of the bridge are of the form:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -3a & -3a \end{bmatrix} \begin{bmatrix} X_B \\ Y_B \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3a & 3a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_C \\ Y_C \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ a & 2a \\ -2a & -a \end{bmatrix} \begin{bmatrix} 0 \\ -P \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{10b}$$

By (10b), we obtain the system of equations:

$$\begin{aligned} X_B + X_C &= 0, \\ 3X_C + 3aY_C - 2aP &= 0, \\ -3aX_B - 3aY_B + aP &= 0. \end{aligned} \tag{10c}$$

Solving the systems (10b) and (10c), we get:

$$X_A = \frac{1}{3}(P+Q), \qquad X_B = \frac{1}{3}(P+Q), \qquad X_C = -\frac{1}{3}(P+Q),$$
$$Y_A = P + \frac{2}{3}Q, \qquad Y_B = -\frac{1}{3}Q, \qquad Y_C = P + \frac{1}{3}Q.$$

5. Equilibrium condition for holonomic systems with redundant coordinates

As it is known, the sufficient and necessary condition for a mechanical system being equilibrium in the inertia reference frame is, that the generalized forces corresponding to the Lagrange coordinate (independent coordinates or holonomic coordinates) is equal to zero.

However, in some cases it is not convenient to use the independent coordinates, for example, in the cases of closing loops. In such a case it will be better to choose the redundant coordinates. The method of the redundant coordinates plays an important part in investigating complex mechanisms, which are treated to be constrained mechanical systems. By this method, the determination of the position of the closed- loop systems is easy. On the other hand, the use of redundant coordinates will encounter troubles in the kinematical problems. For example, when applying the Principle of Compatibility for constrained systems [2, 3], it is necessary to express the dependent virtual displacements in terms of the independent ones. This serves in the calculation of generalized forces of constraints and therefore it is necessary to compute the matrix of the coefficients in the established expressions between the displacements. Such a method has been constructed in [4, 5].

Let us consider the holonomic system of n degrees of freedom. Its position is defined by the m coordinates ($m > n$). In such a case, there are some relations between the chosen coordinates, which are:

$$f_\alpha(q_1, q_2, \ldots, q_m) = 0, \qquad \alpha = 1, \ldots, s, \tag{11}$$

where the f_α is functionally independent. The number of degrees of freedom of the system under consideration will be equal to $k = m - n$.

The principle of the virtual work reduces to the form:

$$\sum \delta A(\vec{F}_k) = \sum_{j=1}^{m} Q_j \delta q_j = 0, \tag{12}$$

where $Q_j (j = 1, \ldots, m)$ are the generalized forces corresponding to redundant coordinates.

By (11) the virtual displacements δq_j, $j = 1, \ldots, m$ are not independent, but they have to satisfy the following relations:

$$\sum_{j=1}^{m} \frac{\partial f_\alpha}{\partial q_j} \delta q_j = 0 \tag{13}$$

Assume, that by (13) the displacements can be expressed in terms of the independent displacements, which are denoted by $\delta\sigma_r$, $r = 1, \ldots, n$, $n = m - s$. In this way, we have:

$$\delta q_j = \sum_{r=1}^{n} d_{jr} \delta\sigma_r, \qquad j = 1, \ldots, m. \tag{14}$$

In the matrix form, the relations (14) are written as follows:

$$\boldsymbol{\delta}\mathbf{q} = \mathbf{D}\boldsymbol{\delta\sigma}, \tag{15}$$

where $\boldsymbol{\delta}\mathbf{q}$, $\boldsymbol{\delta\sigma}$ are the matrices of the form:

$$\boldsymbol{\delta}\mathbf{q}^T = \begin{bmatrix} \delta q_1 & \delta q_2 & \ldots & \delta q_m \end{bmatrix}, \qquad \boldsymbol{\delta\sigma}^T = \begin{bmatrix} \delta\sigma_1 & \delta\sigma_2 & \ldots & \delta\sigma_n \end{bmatrix},$$

but $\mathbf{D}$ is the matrix of the form:

$$D = \begin{bmatrix} d_{11} & d_{12} & \ldots & d_{1m} \\ d_{21} & d_{22} & \ldots & d_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ d_{n1} & d_{m2} & \ldots & d_{nm} \end{bmatrix}. \tag{16}$$

By placing (14) into (12) we obtain:

$$\sum \delta A(\vec{F}) = \sum_{r=1}^{n} \left[\sum_{j=1}^{m} Q_j d_{jr} \right] \delta\sigma_r = 0.$$

By virtue of independence of $\delta\sigma_r$, $r = 1, \ldots, n$, we get:

$$\sum_{j=1}^{m} Q_j d_{jr} = 0, \qquad r = 1, \ldots, n. \tag{17}$$

In the matrix form, the relations (17) take the following form:

$$\mathbf{DQ} = \mathbf{0}, \tag{18}$$

which express the equilibrium condition of the system written in redundant coordinates, where $\mathbf{Q}$ is the matrix of the generalized forces corresponding redundant coordinates,

$$\mathbf{Q}^T = \left[\begin{array}{cccc} Q_1 & Q_2 & \ldots & Q_m \end{array} \right].$$

Example 7. A compression machine shown as in Fig. 9. Compute the magnitude of force Q, which is made by the compressing P at the given position, shown in the Fig. 9.

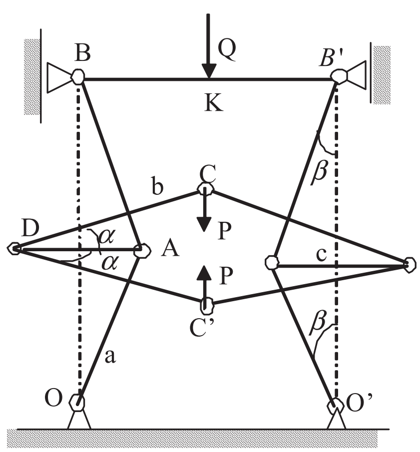

Fig. 9. Equilibrium of a compression

The machine under consideration has one degree of freedom. It is easy to see that the determination of the position of the system by one generalized coordinate is not convenient. Let us choose the coordinates y, α, β to be the generalized coordinates, where y is the ordinate of the press K, while α and β are showed in the Fig. 9 and are the redundant coordinates. There are two constraint equations, which are:

$$f_1 \equiv y - 2a\cos\beta, \qquad f_2 \equiv a\sin\beta - b\cos\beta - c = 0. \tag{19}$$

The constraints are ideal, the sum of the virtual work of the applied forces P and Q are of the form:

$$\sum \delta A(\vec{F}_k) = P\delta y_C - Q\delta y_K, \tag{20}$$

where y_C and y_K denote the points of acting of the forces P and Q. There are the following relations:

$$y_K = a\cos\beta - b\sin\alpha, \qquad y_C = a\cos\beta + b\sin\alpha. \tag{21}$$

By varying both members of the obtained relations, we have:

$$\delta y_K = -a\sin\beta\delta\beta - a\cos\alpha\delta\alpha, \qquad \delta y_C = -a\sin\beta\delta\beta + a\cos\alpha\delta\alpha. \tag{22}$$

By placing these variances into the expression of the virtual work of applied forces, we calculate the generalized forces corresponding to coordinates

$$Q_y = -Q, \qquad Q_\alpha = -2Pb\cos\alpha, \qquad Q_\beta = 0. \tag{23}$$

Let us choose the coordinate y to be the independent coordinate. By (22) the dependent virtual displacements $\delta\alpha$, $\delta\beta$ are expressed in terms of independent virtual displacement δy, that are:

$$\delta\alpha = -0.5\frac{\cot\beta}{b\sin\alpha}\delta y, \qquad \delta\beta = -0.5\frac{1}{a\sin\beta}\delta y.$$

Matrix $\mathbf{D}$ of dimensions (1×3) appearing in (18) now takes the form:

$$\mathbf{D} = \left[\begin{array}{ccc} 1 & -\frac{\cot\beta}{b\sin\alpha} & -\frac{1}{2a\sin\beta} \end{array}\right]. \tag{24}$$

The equilibrium condition of the system under consideration will be written as follows:

$$\mathbf{DQ} = \left[\begin{array}{ccc} 1 & -\frac{\cot an\beta}{b\sin\alpha} & -\frac{1}{2a\sin\beta} \end{array}\right] \left[\begin{array}{c} -Q \\ -2P\cos\alpha \\ 0 \end{array}\right] = 0. \tag{25}$$

From (25) we compute the compression force Q, that is:

$$Q = P\cot\alpha\cot\beta.$$

Example 8. The quick-return mechanism of the horizontal planar machine is shown in Fig. 10.

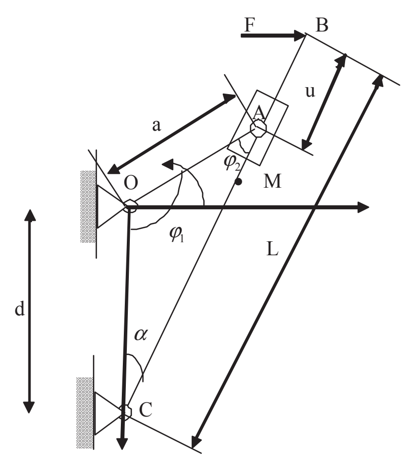

Fig. 10. Equilibrium of a quick-return mechanism

The crank OA is of length a, the rod CB is of length L and the distance between two axes O and C is equal d. At the position under consideration the crank OA inclines to the vertical angle φ and is acted by the couple of moment M of the counter-clockwise direction, while the rod CB is acted by the force at B in horizontal direction.

The system under consideration has one degree of freedom with ideal holonomic constraints under the action of applied force F, and couple of moment M.

As it is known, the condition of the equilibrium is defined by the equation (18), which is established by means of the matrices $\mathbf{Q}$ and $\mathbf{D}$. In connection with this, we choose the coordinate to be φ, α and u, where the redundant coordinates φ, α are the position angles of the cranks OA and CB respectively, while u defines the position of the slider A with respect to point B of link CB.

First, let us compute the virtual work of the applied forces,that are:

$$\sum \delta A = M\delta\varphi + FL\cos\alpha\,\delta\alpha. \tag{26}$$

The generalized forces corresponding to the generalized coordinates are:

$$Q_\varphi = M, \qquad Q_\alpha = FL\cos\alpha, \qquad Q_u = 0. \tag{27}$$

In order to compute matrix $\mathbf{D}$ in the formulas (18) it is necessary to express the dependent virtual displacements in terms of independent virtual displacements. However, by stationary constraints, it is possible to express the dependent generalized velocities in terms of the independent ones. For this aim we apply the method of transmission matrix [10, 11]. First, we choose the link frames, which are denoted by vectors:

$\vec{x}_o$ – the fundamental frame, directing the vertical to down direction, i.e., directing from O to C,
$\vec{x}_1$ – the body frame of the link OA, directing from O to A,
$\vec{x}_3$ – the body frame of the link CB having the origin at B, directing from C to B.

As a next step, let us introduce the parameters defining the position of links in the body frames. That are:

φ_1 – the position parameter of the crank OA in the fundamental frame $\varphi_1 \equiv \varphi$),
φ_2 – the position parameter of the slider A in the body frame $\vec{x}_1$,
u – the position parameter of the link CB in the body frame $\vec{x}_2$.

From now on let us introduce the following symbols:

$$\cos\varphi_i \equiv ci, \qquad \sin\varphi_i \equiv si, \qquad \cos(\varphi_i+\varphi_j) \equiv cij, \qquad \sin(\varphi_i+\varphi_j) \equiv sij.$$

The transmission matrices are of the form [10, 11]:

$$\mathbf{T_1} = \begin{bmatrix} c1 & -s1 & 0 \\ s1 & c1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{T_2} = \begin{bmatrix} c2 & -s2 & r \\ s2 & c2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{T_3} = \begin{bmatrix} 1 & 0 & u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\mathbf{T_1'} = \begin{bmatrix} -s1 & -c1 & 0 \\ c1 & -s1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{T_2'} = \begin{bmatrix} -s2 & -c2 & 0 \\ c2 & -s2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{T_3'} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By the condition of point C fixed in the fundamental frame, we obtain the constraint equations. Taking into account that the coordinates of point C in the fundamental frame ($\vec{x}_0$) and the body frame ($\vec{x}_2$) are $(d, 0)$ and $(L, 0)$ respectively. The constraint equations are of the form:

$$\mathbf{T_1 T_2 T_3} \begin{bmatrix} L \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c1 & -s1 & 0 \\ s1 & c1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c2 & -s2 & a \\ s2 & c2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & u \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} d \\ 0 \\ 1 \end{bmatrix}. \tag{28}$$

Deriving both members of the equation (28), we obtain

$$\mathbf{T_1'T_2T_3}\begin{bmatrix} L\dot{\varphi}_1 \\ 0 \\ 0 \end{bmatrix} + \mathbf{T_1T_2'T_3}\begin{bmatrix} L\dot{\varphi}_2 \\ 0 \\ 0 \end{bmatrix} + \mathbf{T_1T_2T_3'}\begin{bmatrix} L\dot{u} \\ 0 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} (L-u)s12(\dot{\varphi}_1+\dot{\varphi}_2) - as1\dot{\varphi}_1 + c12\dot{u} \\ -(L-u)c12(\dot{\varphi}_1+\dot{\varphi}_2) + ac1\dot{\varphi}_1 + s12\dot{u} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The constraint equations are of the form:

$$\begin{aligned} (L-u)s12(\dot{\varphi}_1+\dot{\varphi}_2) - rs1\dot{\varphi}_1 + c12\dot{u} = 0, \\ -(L-u)c12(\dot{\varphi}_1+\dot{\varphi}_2) + rc1\dot{\varphi}_1 + s12\dot{u} = 0. \end{aligned} \tag{29}$$

By these equations we compute:

$$\dot{\varphi}_2 = -\dot{\varphi}_2 + \frac{rc2}{L-u}\dot{\varphi}_1, \qquad \dot{u} = -rs2\dot{\varphi}_1 = -rs2\omega_1. \tag{30}$$

The absolute angular velocity of the link CB will be:

$$\omega_2 = \dot{\varphi}_1 + \dot{\varphi}_2 = \frac{rc2}{L-u}\dot{\varphi}_1 = \frac{rc2}{L-u}\omega_1, \tag{31}$$

where ω_1 is the absolute angular velocity the crank OA.

The matrix $\mathbf{D}$ in (18) now is of the form

$$\mathbf{D} = \begin{bmatrix} 1 & \frac{rc2}{L-u} & -rs2 \end{bmatrix}. \tag{32}$$

The equilibrium condition (18) will be:

$$\mathbf{DQ} = \begin{bmatrix} 1 & \frac{rc2}{L-u} & -rs2 \end{bmatrix}\begin{bmatrix} M \\ FL\cos\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} M + \frac{FLr\cos\varphi_2\cos\alpha}{L-u} \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}. \tag{33}$$

Taking into account that

$$\varphi_2 = \pi - \varphi_1 - \alpha \to \cos\varphi_2 = -\cos(\varphi_1+\alpha),\ \sin\varphi_2 = \sin(\varphi_1+\alpha) \tag{34}$$

we calculate the magnitude of couple of moment M acting on the crank OA, that keeps the system under consideration in equilibrium, that is:

$$M = -\frac{FLr\cos\alpha\cos(\varphi_1+\alpha)}{L-u}.$$

The minus sign signifies that the couple of moment M has the clockwise direction.

6. Conclusions

The presented method is one of greatest generality for investigating the problem of the equilibrium of a rigid body and a set of rigid bodies. It is a simple geometrical method. It is necessary to know the coordinates of the forces and of the action points of the forces. In comparison with the analytical method, we only need to determine the matrix $\mathbf{D}$ by (16) and the generalized forces. It is important that this method allows to apply useful special software such as Mathcad, Maple for solving the problem of statics.

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