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A time-step insensitive recurrent approach to analyze non-stationary random responses

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The recurrent approach constructed via the stochastic central difference (SCD) is a very fast method for analyzing non-stationary random responses. However, the computational results depend to a great extent upon the discrete time-step size. A new recurrent approach is proposed in this report. It is based on the theory of linear differential equations. Theoretical analysis shows that this algorithm is unconditionally stable for all damped systems. Two examples show that the proposed approach is not sensitive to the time-step.

Keywords: non-stationary response, linear differential equations, variance, finite difference, stochastic central difference.

1. Introduction

Random phenomena exist throughout the physical world. At first these apparent flukes caught the interest of only a few physicists, including Einstein [1]. However, after a seminal conference organized by Crandall in 1958, related studies blossomed in the engineering community. Non-stationary random response analysis was just one branch of random phenomena research that arose as a consequence. Since that conference, many approaches have been introduced or developed in order to analyze the non-stationary random response [2, 3]. Most of them are time-consuming and are not feasible for projects that require handling thousands of degrees of freedom. Such large-scale projects occur indeed frequently in engineering, and can be solved efficiently with the finite element method or the finite difference method provide that only the deterministic aspect is interested. As modern design developing, the non- deterministic aspect must be considered also. Therefore, finding efficient approaches to non-stationary response analysis has been enticing researchers from relevant communities for decades.

Originally, the finite difference method was used to solve a deterministic ordinary differential equation (ODE) numerically. It was introduced into the random field in 1986 by Prof. To [4], and is called the stochastic central difference (SCD) method. Analogies to the deterministic case, the stochastic Houbolt method [5] and the stochastic Newmark algorithms, were also proposed and

investigated [6, 7]. From a computational efficiency point of view, these finite difference approaches have advantageous features compared to other methods.

However, when applying the SCD method, the time-step must be chosen with care [4–6, 8, 9]. Prof. To *et al.* have already suggested some tricks, such as adaptive time schemes [10] and time co-ordinate transformation [11], to relax the strict requirements of choosing the time-step. Despite several efforts, the choice of time-step is still unsatisfactory. For example, the time-step in a recent paper by Chen *et al.* is still determined by empirical formulas [12].

In this report, a recurrent approach for analyzing non-stationary responses is presented. This approach is based on the theory of linear differential equations. Its computational efficiency is comparable to that of the SCD method, but it is not sensitive to the time-step when the time-step is smaller than a certain threshold. This is instantiated by two numerical examples.

2. Recurrent relationship

The governing equation of a linear time invariant system excited by the modulated white noise is

$$\{\dot{\mathbf{y}}\} = [\mathbf{A}]\,\{\mathbf{y}\} + [\mathbf{B}(t)]\,\{\mathbf{w}(t)\}\,. \tag{1}$$

Here $\{\mathbf{y}\}$ is the n-dimension response vector. $\{\mathbf{w}(\mathbf{t})\}$ is a m-dimension excitation vector. The entry of $\{\mathbf{w}(t)\}$ is assumed as the white noise with the unitary power, and independent of each other. $[\mathbf{B}(t)]$ is an $n \times m$ modulating matrix, whose entries are deterministic functions of the time t. $[\mathbf{A}]$ is an $n \times n$ constant matrix determined by the system only.

For an individual realization of the random excitation process, Eq. (1) is a deterministic ODE set with constant coefficients. The relationship between the responses at instant t_1 and t_2 $(\Delta t = t_2 - t_1 \geq 0)$ can be determined in light of the theory of linear ODEs. Denoting that $[\mathbf{H}] = \exp([\mathbf{A}]\,\Delta t)$, $[\mathbf{H}(t)] = \exp([\mathbf{A}]\,t)$, we have

$$\{\mathbf{y}(t_2)\} = [\mathbf{H}]\,\{\mathbf{y}(t_1)\} + \int_{t_1}^{t_2} [\mathbf{H}(t_2-\tau)]\,[\mathbf{B}(\tau)]\{\mathbf{w}(\tau)\,\mathrm{d}\tau\}\,. \tag{2}$$

Based on Eq. (2), the mean square response $[R_y(t_2)] = E\left[\{y(t_2)\}\,\{y(t_2)\}^T\right]$ is as follows

$$\begin{aligned}[R_y(t_2)] = {} & [\mathbf{H}]\,[R_y(t_1)]\,[\mathbf{H}]^T + [\mathbf{H}]\int_{t_1}^{t_2} [R_{yw}(t_1,\tau)]\,[\mathbf{B}(\tau)]^T[\mathbf{H}(t_2-\tau)]^T\mathrm{d}\tau \\ & + \int_{t_1}^{t_2} [\mathbf{H}(t_2-\tau)][\mathbf{B}(\tau)]\,[R_{wy}(\tau,t_1)]\,\mathrm{d}\tau[\mathbf{H}]^T \\ & + \int_{t_1}^{t_2}\int_{t_1}^{t_2} [\mathbf{H}(t_2-\tau_1)][\mathbf{B}(\tau_1)]\,[R_w(\tau_1,\tau_2)]\,[\mathbf{B}(\tau_2)]^T[\mathbf{H}(t_2-\tau_2)]^T\mathrm{d}\tau_1\mathrm{d}\tau_2\,.\end{aligned} \tag{3}$$

Here $R_{yw}(t_1,t_2) = R_{wy}(t_2,t_1)$ is the cross correlation between $y(t_1)$ and $w(t_2)$. Obviously, $\{y(t_1)\}$ is determined by only the excitation $w(t)$ before t_1. In light of the property of the white noise, $R_{yw}(t_1,\tau)$ for $\tau \in (t_1,t_2]$ equals to zero. Thereby, both the second term and third term in Eq. (3) vanish.

According to the assumption of the independence among the entries of $w(t)$, $[R_w(\tau_1,\tau_2)]$ can be written as

$$[R_w(\tau_1,\tau_2)] = E\left[\{w(\tau_1)\}\{w(\tau_2)\}^T\right] = 2\pi\delta(\tau_1-\tau_2)I_{m\times m}\,, \tag{4}$$

where $I_{m\times m}$ is an identity matrix, and $\delta(\tau)$ is the Dirac delta function. Thus, the fourth term at the right-hand side of Eq. (3) is

$$\begin{aligned}&\int_{t_1}^{t_2}\int_{t_1}^{t_2} [\mathbf{H}(t_2-\tau_1)][\mathbf{B}(\tau_1)]\,[R_w(\tau_1,\tau_2)]\,[\mathbf{B}(\tau_2)]^T[\mathbf{H}(t_2-\tau_2)]^T\mathrm{d}\tau_1\mathrm{d}\tau_2 \\ &\quad = \int_{t_1}^{t_2}\int_{t_1}^{t_2} [\mathbf{H}(t_2-\tau_1)][\mathbf{B}(\tau_1)]2\pi\delta(\tau_1-\tau_2)I_{m\times m}[\mathbf{B}(\tau_2)]^T[\mathbf{H}(t_2-\tau_2)]^T\mathrm{d}\tau_1\mathrm{d}\tau_2\,.\end{aligned} \tag{5}$$

In light of the sampling property of the Dirac function

$$\int_{t_1}^{t_2}\int_{t_1}^{t_2}[\mathbf{H}(t_2-\tau_1)][\mathbf{B}(\tau_1)]\,[R_w(\tau_1,\tau_2)]\,[\mathbf{B}(\tau_2)]^T[\mathbf{H}(t_2-\tau_2)]^T\mathrm{d}\tau_1\mathrm{d}\tau_2 = 2\pi\int_{t_1}^{t_2}[\mathbf{H}(t_2-\tau)][\mathbf{B}(\tau)][\mathbf{B}(\tau)]^T[\mathbf{H}(t_2-\tau)]^T\mathrm{d}\tau\,. \tag{6}$$

Substituting Eq. (6) into Eq. (3) leads to

$$[R_y(t_2)] = [\mathbf{H}]\,[R_y(t_1)]\,[\mathbf{H}]^T + 2\pi\int_{t_1}^{t_2}[\mathbf{H}(t_2-\tau)][\mathbf{B}(\tau)][\mathbf{B}(\tau)]^T[\mathbf{H}(t_2-\tau)]^T\mathrm{d}\tau\,. \tag{7}$$

It is worth noting that this second-order moment relationship does not have anything to do with the complete information about the probability structure of the excitation. This equation does not contain moments higher than the second order. That is to say, Eq. (7) is self-closed.

However, these properties can not be extended automatically to analyze the stochastic response of a nonlinear system. Generally, the nonlinear deterministic system can be linearized with sufficient precision in a short time interval. We may wish to retain the above properties in the nonlinear case, but this is not possible. For a nonlinear system excited by the stochastic processes, the linearization is usually based on the ensemble equivalence, such as one calculated by the statistical linearization method. This class of equivalence often involves the moments at t_1 or t_2 whose orders are greater than the second order. Unfortunately, the moment closure scheme necessitates the morphology assumption of the response probability distribution [13].

3. Algorithm implementation

Since Eq. (7) is deterministic, we can apply deterministic discrete methods to it. The second term at the right-hand side is an integral. If $[\mathbf{B}(t)]$ is a simple function, such as an exponential or triangular one, then this integral can be obtained explicitly. Sometimes, $[\mathbf{B}(t)]$ is very complicated, particularly in cases where $[\mathbf{B}(t)]$ is estimated from recorded data. That is why a numerical method is more appropriate. In this report, the trapezoidal scheme is used. It can be verified that

$$[R_y(t_2)] \approx [\mathbf{H}]\left([R_y(t_1)] + \pi\Delta t[\mathbf{B}(t_1)][\mathbf{B}(t_1)]^T\right)[\mathbf{H}]^T + \pi\Delta t[\mathbf{B}(t_2)][\mathbf{B}(t_2)]^T\,. \tag{8}$$

If the error of $[\mathbf{H}]$ can be ignored, the computational error of Eq. (8) comes from the trapezoidal integration only.

The remaining issue is computing $[\mathbf{H}] = \exp([\mathbf{A}]\Delta t)$, the matrix exponential. This is a classic problem in linear system theory [14, 15]. In 1978, Van Loan *et al.* listed nineteen ways of computing the exponential of a matrix [14]. New approaches are still being developed [15]. In the vibration community, $[\mathbf{H}]$ is conventionally computed from the eigenvalues and eigenvectors of $[\mathbf{A}]$ by the complex modal theory. Computing $[\mathbf{H}]$ by the precise time-integration has been receiving much attention recently [16]. In the ensuing numerical examples, we employ the built-in MATLAB function *expm* directly, which uses the Padé approximation with scaling and squaring [14].

Provided with the accurate value of $[\mathbf{H}]$, the initial variance matrix $[R_y(0)]$, and the matrix $[\mathbf{H}]$, the variance at any instant can be computed by Eq. (8).

The vibration equation of an n-degrees-of-freedom system is

$$[\mathbf{M}]\{\ddot{\mathbf{x}}\} + [\mathbf{C}]\{\dot{\mathbf{x}}\} + [\mathbf{K}]\{\mathbf{x}\} = [\mathbf{G}(t)]\{\mathbf{f}(\mathbf{t})\}\,, \tag{9}$$

where $[\mathbf{M}]$, $[\mathbf{C}]$ and $[\mathbf{K}]$ are the mass, damp and stiffness matrix respectively. $[\mathbf{M}]$ and $[\mathbf{K}]$ can be obtained by the finite element method. $[\mathbf{C}]$ is often taken as the proportional damping matrix, but no limitation is imposed here. $\{\mathbf{f}(\mathbf{t})\}$ is the white noise vector, whose components are independent of each other, and of the unitary power. $[\mathbf{G}(t)]$ stands for the envelope of the stochastic excitation.

After introducing the state vector $\{\mathbf{y}\} = \left\{\mathbf{x}^T, \dot{\mathbf{x}}^T\right\}^T$, Eq. (9) can be reformulated as:

$$\{\dot{\mathbf{y}}\} = \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}\{\mathbf{y}\} + \begin{bmatrix} \mathbf{0}_{n\times n} \\ \mathbf{M}^{-1}\mathbf{G} \end{bmatrix}\{\mathbf{w}(t)\}\,. \tag{10}$$

Correspondingly, the matrices in Eq. (1) are as follows:

$$\{\mathbf{A}\} = \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}\{\mathbf{y}\}\,, \qquad \{\mathbf{B}\} = \begin{bmatrix} \mathbf{0}_{n\times n} \\ \mathbf{M}^{-1}\mathbf{G} \end{bmatrix}, \qquad \{\mathbf{w}(t)\} = \{\mathbf{f}(t)\}\ .$$

4. Stability analysis

Assuming that $[\mathbf{H}]$ is accurate, and the error matrix of $[\mathbf{R}_y(t)]$ at $t = t_1$ is $[\varepsilon]_1$. According to Eq. (8), we have

$$[\mathbf{R}_y(t_2)] + [\varepsilon]_2 = [\mathbf{H}]\Big([\mathbf{R}_y(t_1)] + [\varepsilon]_1 + \pi\Delta t[\mathbf{B}(t_1)][\mathbf{B}(t_1)]^T\Big)[\mathbf{H}]^T + \pi\Delta t[\mathbf{B}(t_2)][\mathbf{B}(t_2)]^T\,,$$

where $[\varepsilon]_2$ is the error matrix of $[\mathbf{R}_y(t_2)]$. The above equation can be simplified as follows

$$[\varepsilon]_2 = [\mathbf{H}][\varepsilon]_1[\mathbf{H}]^T\,. \tag{11}$$

Regarding the Frobenius matrix norm, we have:

$$\big\|[\varepsilon]_2\big\|_2 = \big\|[\mathbf{H}][\varepsilon]_1[\mathbf{H}]^T\big\|_2 \leqslant \big\|\mathbf{H}\big\|_2^2\big\|[\varepsilon]_1\big\|_2 = \sigma_1^2\big\|[\varepsilon]_1\big\|_2\,. \tag{12}$$

Here σ_1 is the greatest singular value of $[\mathbf{H}]$, which is the pivotal index to the algorithm stability. It can be argued that $\sigma_1 < 1$ as follows.

For a damped system, it can be proved that $2n$ singular values of $[\mathbf{H}]$ are n-pairs of conjugate complex numbers (here, we ignore the infrequent case of coincident modal frequencies). The $2n$ values are $\exp(-\lambda_i\Delta t)$ and $\exp(-\lambda_i^*\Delta t)$ $(i = 1, 2, \ldots, n)$, where λ_i is the complex modal frequency of the ith order. All stable linear systems have positive damping. This means $\lambda_i > 0$, that is to say, $\exp(-\lambda_i\Delta t) \leqslant 1$. Thus $\sigma_1 = \max\big[\big|\exp(-\lambda_i\Delta t)\big|\big] < 1$.

Eq. (12) shows that the error matrix norm should not be amplified. Thus the recurrent formula Eq. (8) is unconditionally stable. If the computational error of $[\mathbf{H}]$ is so significant that the approximate value of σ_1 is greater than 1, then Eq. (8) may be unstable. Fortunately many accurate approaches to compute $[\mathbf{H}] = \exp([\mathbf{A}]\Delta t)$ have been constructed [14, 15].

5. Numerical examples

5.1. Example 1

The first example is based on a one degree-of-freedom system as follows

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \exp(\beta t)w(t)\,. \tag{13}$$

Here ζ and ω_n are the system damping ratio and natural frequency respectively. $\beta < 0$ is the index describing the attenuating rate of the white noise envelope.

Here four cases of the damping ratio $\zeta = 0.01$, 0.01, 0.025, 0.1 and two cases of $\beta = 0$, -0.1 are examined. The natural frequency is fixed at $\omega_n = 2\pi$ rad/s, as a result, the period of the damping free system is 1 s.

In order to pinpoint the insensitivity to the time-step, three cases of $\Delta t = 0.001$, 0.01, 0.1 are compared, where the greatest time-step $\Delta t = 0.1$ is one-tenth of the system's natural period. The largest time-step is 100 times greater than the smallest.

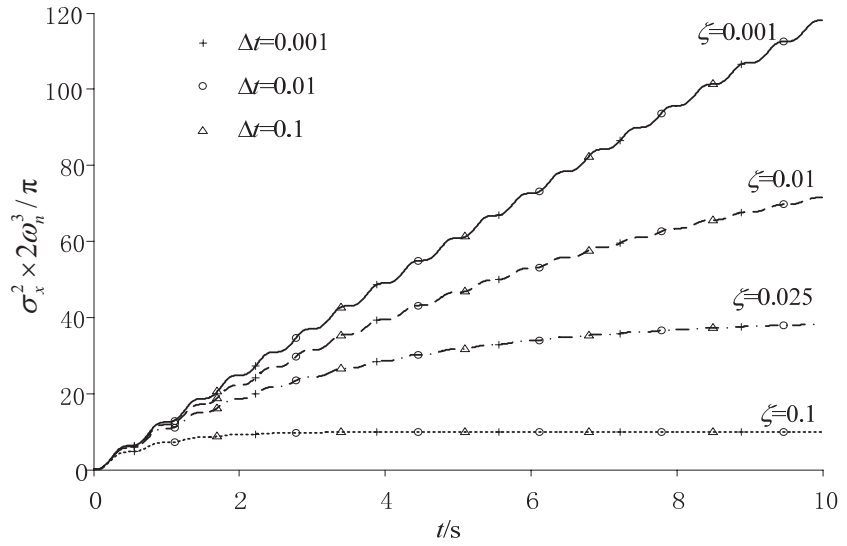


Fig. 1. Computational results of the example 1 with $\beta = 0$

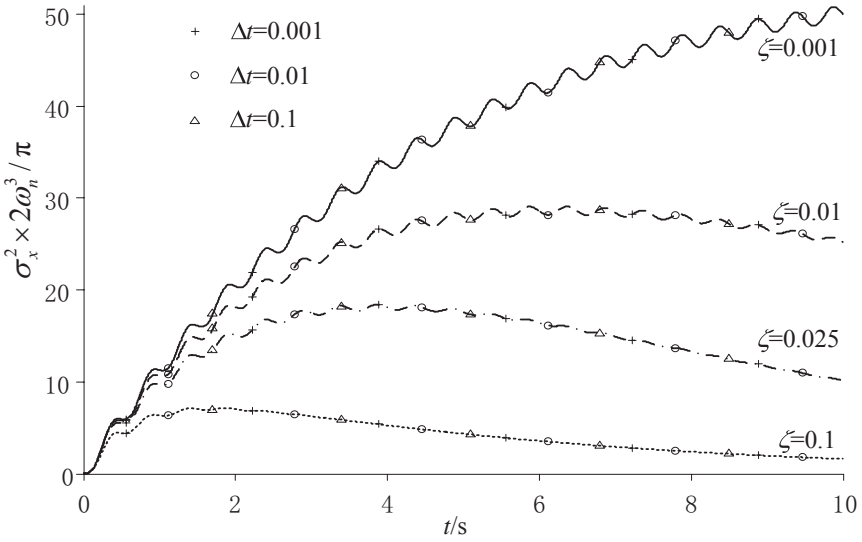


Fig. 2. Computational results of the example 1 with $\beta = -0.1$

The computational results are presented in Figs. 1 and 2, where the exact solutions can be found [17]. In Fig. 1, $\beta = 0$ is a common example when the non-stationary random response is being discussed in textbooks. The computational results from all three cases of the time-steps coincide with the exact solutions, although the time-step sizes are dramatically different. Some results picked from the plots are listed in Tables 1–4 , where E_r is the relative error, and defined as $E_r = \left(\sigma_x^2 - \tilde{\sigma}_x^2\right)/\sigma_x^2$. Here σ_x^2, $\tilde{\sigma}_x^2$ is the exact solution and approximate solution by Eq. (8), respectively.

Inspecting these tables, we find firstly that the difference between the exact solutions and the computational results by Eq. (8) are minor, while the time-steps span changes from 0.001 to 0.1. This indicates that the algorithm by Eq. (8) is insensitive to a changing time-step. In contrast, the performance of the conventional SCD method depends significantly on the time-step.

Secondly, as in any discrete approach, the time-step affects the computational accuracy. When the time-step decreases to one-tenth, the accuracy improves by up to 100 times.

Thirdly, the relative error does not accelerate as the time component increases. One contributive factor is the growing trend of the response variance, which is the denominator of the relative error.

Table 1. Relative error for $\zeta = 0.001$

t	$2\sigma_x^2\omega_n^3/\pi$	E_r ($\Delta t = 0.001$)	E_r ($\Delta t = 0.01$)	E_r ($\Delta t = 0.1$)
1.0	12.4877	0.6547E-11	0.6550E-9	0.7825E-7
2.0	24.8196	0.6523 E-11	0.6509E-9	0.7783 E-7
3.0	36.9974	0.6469 E-11	0.6468E-9	0.7741 E-7
4.0	49.0231	0.6422 E-11	0.6427E-9	0.7699 E-7
5.0	60.8987	0.6368 E-11	0.6387E-9	0.7657 E-7
6.0	72.6259	0.6253 E-11	0.6346 E-9	0.7616 E-7
7.0	84.2067	0.6128 E-11	0.6305E-9	0.7575 E-7
8.0	95.6429	0.6021 E-11	0.6265E-9	0.7534 E-7
9.0	106.9363	0.5927 E-11	0.6225E-9	0.7493 E-7
10.0	118.0887	0.5836 E-11	0.6185E-9	0.7452 E-7

Table 2. Relative error for $\zeta = 0.01$

t	$2\sigma_x^2\omega_n^3/\pi$	E_r ($\Delta t = 0.001$)	E_r ($\Delta t = 0.01$)	E_r ($\Delta t = 0.1$)
1.0	11.8094	0.6175 E-9	0.6187 E-7	0.7452 E-5
2.0	22.2242	0.5786 E-9	0.5800 E-7	0.7054 E-5
3.0	31.4091	0.5415 E-9	0.5429 E-7	0.6673 E-5
4.0	39.5093	0.5061 E-9	0.5076 E-7	0.6311 E-5
5.0	46.6529	0.4724 E-9	0.4740 E-7	0.5965 E-5
6.0	52.9529	0.4404 E-9	0.4420 E-7	0.5637 E-5
7.0	58.5088	0.4100 E-9	0.4116 E-7	0.5325 E-5
8.0	63.4087	0.3812 E-9	0.3829 E-7	0.5030 E-5
9.0	67.7299	0.3538 E-9	0.3557 E-7	0.4751 E-5
10.0	71.5408	0.3281 E-9	0.3301 E-7	0.4488 E-5

Table 3. Relative error for $\zeta = 0.025$

t	$2\sigma_x^2\omega_n^3/\pi$	E_r ($\Delta t = 0.001$)	E_r ($\Delta t = 0.01$)	E_r ($\Delta t = 0.1$)
1.0	10.7868	0.3501 E-8	0.3508 E-6	0.4289 E-4
2.0	18.6647	0.2956 E-8	0.2963 E-6	0.3729 E-4
3.0	24.4181	0.2476 E-8	0.2483 E-6	0.3235 E-4
4.0	28.6201	0.2057 E-8	0.2064 E-6	0.2805 E-4
5.0	31.6889	0.1696 E-8	0.1703 E-6	0.2435 E-4
6.0	31.6889	0.1696 E-8	0.1703 E-6	0.2435 E-4
7.0	35.5670	0.1129 E-8	0.1136 E-6	0.1852 E-4
8.0	36.7624	0.0912 E-8	0.0918 E-6	0.1629 E-4
9.0	37.6355	0.0732 E-8	0.0738 E-6	0.1444 E-4
10.0	38.2731	0.0584 E-8	0.0591 E-6	0.1292 E-4

Another factor is that, for a damped system, the size of $[\mathbf{H}(t)]$ will attenuate as the time component increases. Therefore, the absolute contribution to the non-stationary response from the integral term in Eq. (8) will fade. This will weaken the error effect of the trapezoidal numerical integral on the relative error. This is why the error of slightly damped case attenuates more slowly than that of the heavily damped.

Table 4. Relative error for $\zeta = 0.1$

t	$2\sigma_x^2\omega_n^3/\pi$	E_r ($\Delta t = 0.001$)	E_r ($\Delta t = 0.01$)	E_r ($\Delta t = 0.1$)
1.0	7.1719	0.3314 E-7	0.3326 E-5	0.4505 E-3
2.0	9.2001	0.1472 E-7	0.1483 E-5	0.2616 E-3
3.0	9.7738	0.0592 E-7	0.0602 E-5	0.1713 E-3
4.0	9.9360	0.0221 E-7	0.0231 E-5	0.1332 E-3
5.0	9.9819	0.0078 E-7	0.0088 E-5	0.1186 E-3
6.0	9.9949	0.0027 E-7	0.0037 E-5	0.1133 E-3
7.0	9.9986	0.0009 E-7	0.0019 E-5	0.1115 E-3
8.0	9.9996	0.0003 E-7	0.0013 E-5	0.1109 E-3
9.0	9.9999	0.0001 E-7	0.0011 E-5	0.1107 E-3
10.0	10.0000	0.0000 E-7	0.0011 E-5	0.1106 E-3

In Fig. 2, $\beta = -0.1$. Similar to Fig. 1, the computational results cannot be distinguished from the exact solutions in all three time-step cases. Due to the exponentially attenuated excitation envelope, the trend of the variance response first goes up, and then decreases down.

5.2. Example 2

The second example was examined in [6]. This is a system with two degrees of freedom. The parameters in Eq. (9) are

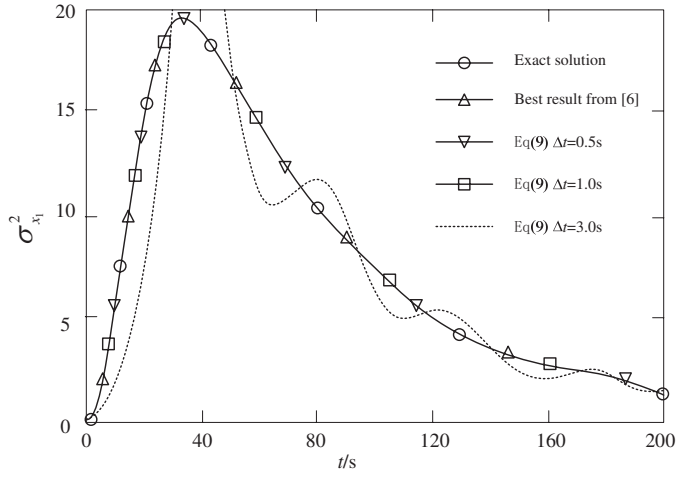
$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \mathbf{C} = 0.009875\,\mathbf{M} + 0.0009875\,\mathbf{K},$$

$$[\mathbf{G}] = \begin{bmatrix} 4\exp(-0.05t) - 4\exp(-0.10t) \\ 0 \end{bmatrix}.$$

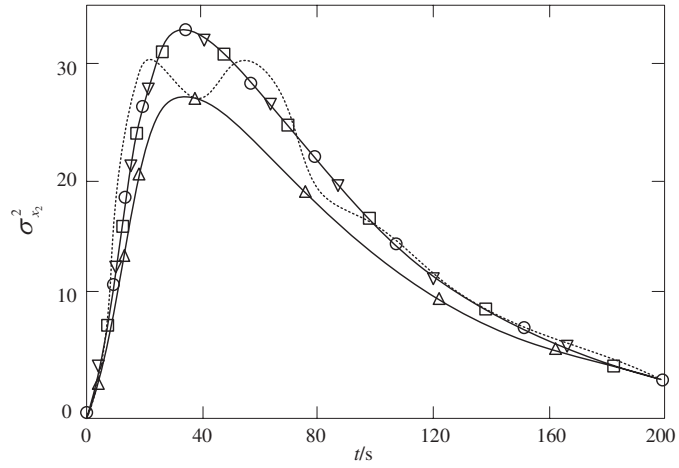
The stochastic Newmark finite differential method was tested by this example [6]. After massive computations and comparisons, Prof. To concluded that the most significant factor controlling the accuracy is the time-step size. Only the computational results for the Newmark parameter $\beta = 1/10$ ($\Delta t = 1.05\,\mathrm{s}$) from [6] are shown in Fig. 3, along with the exact solution. Most results from [6] deviate significantly from the exact solutions. The selected ones, shown in Fig. 3, are the best from the match aspect of $\sigma_{x_1}^2$ between the computational and the exact results. Examining Fig. 3(b), and Fig. 3(c), we find that $\sigma_{x_2}^2$ and $R_{x_1x_2}$ are quite imprecise, where $R_{x_1x_2}$ stands for the cross correlation between x_1 and x_2.

The proposed algorithm Eq. (8) is examined at three time-step sizes. These time-steps are $\Delta t = 0.5\,\mathrm{s}$, $1\,\mathrm{s}$ and $3\,\mathrm{s}$, respectively. One of them, $\Delta t = 1\,\mathrm{s}$, is close to the case of $\Delta t = 1.05\,\mathrm{s}$ adopted in [6], the results of which are shown in Fig. 3 selectively. In the cases of $\Delta t = 0.5\,\mathrm{s}$ and $1\,\mathrm{s}$, Fig. 3 shows that all the computational results of $\sigma_{x_1}^2$, $\sigma_{x_2}^2$ and $R_{x_1x_2}$ follow the exact solutions without deviation. This demonstrates again that Eq. (8) is not sensitive to the time-step size.

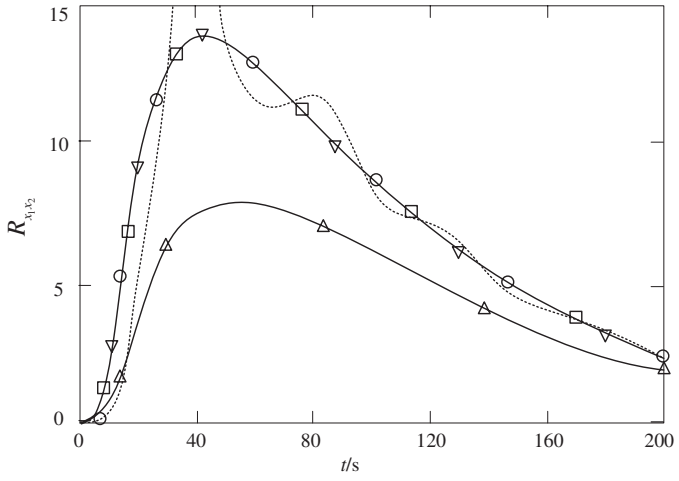
For $\Delta t = 3\,\mathrm{s}$, however, the computational results contain significant error and fluctuate dramatically. In signal processing theory, a step of this size has violated the sampling theorem. The second damping-free natural frequency of this example is $0.2575\,\mathrm{Hz}$. As a result, the corresponding period is $3.8832\,\mathrm{s}$. Thus, the Nyquist sampling frequency for the second principal vibration is $0.51510\,\mathrm{Hz}$, corresponding to the critical sampling interval of $1.9416\,\mathrm{s}$. Therefore, a large time-step as $\Delta t = 3\,\mathrm{s}$ will inevitably cause significant error.



(a) variance of x_1



(b) variance of x_2



(c) Cross correlation between x_1 and x_2

Fig. 3. Computational results of the example 2

6. Conclusive remarks

A recurrent approach for analyzing non-stationary random response is proposed. It is based on the theory of linear differential equations. This approach only requires the information contained in the second-order moment of the excitation. Its computational efficacy is commensurate with other finite difference based approaches, but it is not sensitive to the discrete time-step. Theoretical analysis also shows that this algorithm is unconditionally stable for damped systems. These performances are instantiated by two numerical examples.

However, the current work is limited to the case of linear structures excited by the modulated evolutionary white noise. For the more general cases, such as non-linear issues or color excitation, further research is needed.

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