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Bilevel limit analysis of self-hardening rod systems under moving load*

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This paper considers results of an analysis of self-hardening systems (SHS), i.e. load-carrying systems with improved strength and rigidity. The indicated structural features can be only found if geometrical nonlinearity is taken into consideration. Material deforming diagrams can be non-monotonic and non-smooth, and constraints can be unilateral, with gaps. Furthermore, optimisation of a mathematical model of a rod structure as a discrete mechanical system withstanding dead (constant) and/or moving loads is proposed. This model is formulated using bilevel mathematical programming. The limit parameters of standard loads and actions are found in the low-level optimisation. An extreme energy principle is proposed to obtain the limit parameters of these actions. On the upper level, the parameters of moving load are maximized. A positive influence of equilibrium or quasi-equilibrium constant load with the possible preloading of SHS is shown. A set of criteria for the stability of plastic yielding of structures, including non-smooth and non-convex problems of optimisation is given. The paper presents an exemplary application of the proposed method which takes into account the self-hardening effect.

Keywords: self-hardening systems, bilevel programming, limit analysis, constant equilibrium and moving load.

1. Introduction

The issue of preventing failures of load-carrying systems, including building structures and bridges, is closely connected with the analysis of construction failures of either sudden or gradual nature. This paper considers issues of creating load-carrying systems the failures of which would occur gradually under one-path monotonic or variable loadings, which prevents a disastrous failure. Due to geometry and topology of certain classes, such systems with improved strength, rigidity and safety, and therefore are called self-hardening (earlier – geometrically hardening [2, 3, 5]) systems (SHS) [4].

The great sensitivity of carrying capacity, in terms of its geometric and topological parameters, was found and presented in [2]. The analogous influence of prestressing on carrying capacity was proved in the experiment presented in [4] for the elastic strut-framed column.

In this paper, the problem of external actions on the structure with increased reliability is formulated in the low-level optimisation. The upper-level optimisation involves searching for parameters of criteria of the considered systems different from the ones used in low-level optimisation.

In such approach, the definition of "limit analysis" includes the serviceability limit state, i.e., the conditions constraining excessive deformations. So, the design engineer may now simultaneously consider two possible conditions of failure [3].

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Furthermore, mathematical models and methods of limit analysis for the structures are stated. Load carrying capacity of the systems with regard to inelastic deformations and large displacements are also considered. Material deforming diagrams can be non-monotonic and non-smooth [6]. The solution of arising optimisation problems identifies the nature of failures of the structures. Additionally, the non-uniqueness of problem solutions is investigated.

Systems can be called self-hardening if the conditions of plastic yield stability of structures are satisfied. These are formal attributes (class criteria) for such systems. With some extra conditions, these criteria may also be applied to elastic systems, which have not arrived at the state of limit equilibrium. A set of criteria for the stability of plastic yielding of structures, including non-smooth and non-convex problems of optimisation, was given in [3].

As shown in this study, constant equilibrium, quasi-equilibrium load or preloading of structures have a positive effect on the behaviour of the SHS systems. Then, we have a problem of upper-level optimisation of limit analysis. The upper-level optimisation deals with the parameters of a moving load on a structure. Another approach with the cost and preloading criteria on the upper-level optimisation was presented in [5].

For practical implementation of such systems in the design, we must have a current software package that enables a reliable analysis of the geometrical and physical nonlinearity of the systems. Here we used the numerical FEA system ABAQUS [1] and the analytical/symbolic system Wolfram Mathematica [15].

2. Problem statement

2.1. Governing conditions

In the analysis of SHS systems, let us consider a structure under constant (dead) load $\mathbf{F}_c$ and variable (moving or live) load $\mu\mathbf{F}_v(\mathbf{y})$,

$$\mathbf{F} = \mathbf{F}_c + \mu\mathbf{F}_v(\mathbf{y}), \tag{1}$$

where moving loads $\mu\mathbf{F}_v(\mathbf{y})$ belong to domain Ω_F:

$$\mu\mathbf{F}_v(\mathbf{y}) \in \Omega_F(\mathbf{F}_j(\mathbf{y}), j \in J), \tag{2}$$

set Ω_F is specified by characteristics of action cycles, μ is a parameter of the moving load, and $\mathbf{y}$ is a vector or scalar of the coordinate of moving loads.

Notations [3]:
$\mathbf{u}$, $\mathbf{F} \in \mathbf{R}^n$ – vectors of generalised displacements and external forces (loads) of a discrete system of a structure (n – the number of degree of freedom);
$\mathbf{q}, \mathbf{e}, \mathbf{p}, \mathbf{d}, \mathbf{S} \in \mathbf{R}^m$ – vectors of full, elastic and plastic generalised strains as well as vectors of given distortions and internal forces (m – dimension of internal forces and strain vectors; the total number of braces);
$\boldsymbol{\lambda}$, $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$, $\boldsymbol{\xi}$, $\mathbf{K} \in \mathbf{R}^z$ – vectors of generalized plastic multipliers, functions of yielding and plastic constants for $[1{:}z]$ yielding regimes (z – the number of yielding regimes);
$\mathbf{F}_j \in \mathbf{R}^n$ – vectors of generalized independent j-th loadings, $j \in J$ (J – set of independent actions);
$\mathbf{T} \in \mathbf{R}^z$ – vectors of weight multipliers, corresponding to safety factors $\boldsymbol{\varphi}$ $(\mathbf{S}, \boldsymbol{\lambda}, \mathbf{K}) \in \mathbf{R}^z$;
Ω_F – domains (sets) of forces $\mathbf{F}$; indices e, r and p relate to elastic, residual and initial (prestressed) state parameters.

The conditions of the system state include the geometric and equilibrium equations

$$\mathbf{q} = \mathbf{e}(\mathbf{t}) + \mathbf{p} + \mathbf{d}, \tag{3}$$

$$\boldsymbol{\gamma}(\mathbf{u}) = \mathbf{q}, \tag{4}$$

$$\mathbf{A}_n(\mathbf{u})\mathbf{S} = \mathbf{F}_c + \mu\mathbf{F}_v(\mathbf{y}), \tag{5}$$

nonlinear physical relationship for large deformations

$$\mathbf{e} = \boldsymbol{\xi}(\mathbf{S}), \tag{6}$$

as well as conditions of yielding in the form of inequality

$$\boldsymbol{\varphi}(\cdot) \le 0. \tag{7}$$

Linearised dependencies (7) are written as

$$\boldsymbol{\varphi}(\cdot) = \mathbf{N}^{\mathrm{T}}\mathbf{S} - \mathbf{H}\boldsymbol{\lambda} - \mathbf{K}, \tag{8}$$

where $\mathbf{N}$ – matrix of gradients of $\boldsymbol{\varphi}$, $\mathbf{H}$ – matrix of hardening, and $\mathbf{K}$ – vector of yielding constants.
In case of the associated law of yielding, the generalised plastic deformations are as follows:

$$\mathbf{p} = \sum_{l\in L} \mathbf{N}_l \boldsymbol{\lambda}_l. \tag{9}$$

Also the complementary slackness conditions are fulfilled

$$\varphi_l^{\mathrm{T}} \boldsymbol{\lambda}_l = 0, \tag{10}$$

$$\boldsymbol{\lambda} \ge 0, \qquad l \in L. \tag{11}$$

If generalised elastic strains are connected with the internal forces by Hooke's law, we have

$$\mathbf{e} = \mathbf{D}\mathbf{S}, \tag{12}$$

where $\mathbf{D}$ – is an m-order block diagonal matrix of elasticity.
The criterion of yield state stability of plastic mechanism is

$$\Psi(\mathbf{u},\boldsymbol{\lambda}) = 2^{-1}\boldsymbol{\lambda}^{\mathrm{T}}\mathbf{B}\boldsymbol{\lambda} - \boldsymbol{\lambda}^{\mathrm{T}}\mathbf{C}_N(\boldsymbol{\gamma}(\mathbf{u}) - \mathbf{d}) - \mathbf{u}^{\mathrm{T}}\mathbf{F} + 2^{-1}\boldsymbol{\gamma}(\mathbf{u})^{\mathrm{T}}\mathbf{C}\boldsymbol{\gamma}(\mathbf{u}) - \boldsymbol{\gamma}(\mathbf{u})^{\mathrm{T}}\mathbf{C}\mathbf{d} \to \min \tag{13}$$

for some (smooth and convex) function Ψ, where

$$\mathbf{B} = \mathbf{H} + \mathbf{N}^{T}\mathbf{C}\mathbf{N} \tag{14}$$

for

$$\boldsymbol{\lambda} \ge \mathbf{0}. \tag{15}$$

In the compact form the problem will be as follows:

$$\Psi(\mathbf{u},\boldsymbol{\lambda}) \to \min, \qquad \boldsymbol{\lambda} \ge \mathbf{0}. \tag{16}$$

In the case of non-smooth dependences, $\mathbf{S} = \boldsymbol{\chi}^{-}(\mathbf{e})$ (for systems with unilateral or unsafe ties, etc.) and also for the non-associated law of yielding, the formulation of the problem will be:

$$\Psi(\mathbf{u},\boldsymbol{\lambda},\mathbf{S},\mathbf{e}) \to \min_{\mathbf{u},\,\boldsymbol{\lambda}\ge 0}, \tag{17}$$

$$\Psi(\mathbf{u},\boldsymbol{\lambda},\mathbf{S},\mathbf{e}) = \Psi_{\boldsymbol{\xi}}(\boldsymbol{\lambda}) - \boldsymbol{\lambda}^{\mathrm{T}}(\boldsymbol{\varphi}_p(\mathbf{S}) - \mathbf{K}) + \mathbf{S}^{\mathrm{T}}\gamma(\mathbf{u}) - \mathbf{u}^{\mathrm{T}}\mathbf{F}, \tag{18}$$

$$\boldsymbol{\gamma}(\mathbf{u},\mathbf{S}) \coloneqq -(\partial\Psi/\partial\mathbf{S}) + \boldsymbol{\gamma}(\mathbf{u}) - \mathbf{e}(\mathbf{t})\mathbf{p}\mathbf{d} = \mathbf{0}, \tag{19}$$

$$\boldsymbol{\chi}(\mathbf{S},\mathbf{e}) \coloneqq \mathbf{S} - \boldsymbol{\chi}^{-}(\mathbf{e}) = \mathbf{0}, \qquad \boldsymbol{\lambda} \ge \mathbf{0}, \tag{20}$$

where

$$\Psi_{\xi}(\boldsymbol{\lambda}) = \int_0^l \xi(\beta)^{\mathrm{T}}\delta\beta. \tag{21}$$

Equations (13) and (17) are obtained generally by algorithmic procedure. It is the criterion for the class of effective SHS structures proposed.

The displacements and/or plastic strains of the system are also usually restricted by

$$\mathbf{u}^- \le \mathbf{u} \le \mathbf{u}^+, \tag{22}$$

$$\mathbf{p}^- = \sum_{l\in L} \mathbf{N}_l \boldsymbol{\lambda}_l \le \mathbf{p}^+, \tag{23}$$

where $\mathbf{u}^-$, $\mathbf{u}^+ \in \mathbf{R}^n$, $\mathbf{p}^-$, $\mathbf{p}^+ \in \mathbf{R}^m$ – vectors of low and upper limits of corresponding values in the conditions of rigidity Eqs. (22) and (23).

3. Problem of optimum locations of moving loads (low level)

The problem of bilevel limit analysis can be formulated as follows. On the low level, at the system adaptation limit state, the power of safety factors $\boldsymbol{\varphi}$ $(\mathbf{S}(\mathbf{y}),\ \boldsymbol{\lambda},\ \mathbf{K})$ of elements for moving loads $\mu\mathbf{F}_v$ for the fixed load parameter μ, $\mu =$ const., must be minimised,

$$\mathbf{T}^{\mathrm{T}}\varphi(\mathbf{S}(\mathbf{y}),\boldsymbol{\lambda},\mathbf{K}) \to \min, \tag{24}$$

$$\mathbf{q} = \boldsymbol{\gamma}(\mathbf{u}), \tag{25}$$

$$\mathbf{A}_n(\mathbf{u})\mathbf{S} = \mathbf{F}_c + \mu\mathbf{F}_v(\mathbf{y}), \tag{26}$$

$$\mathbf{q} = \mathbf{e} + \mathbf{p} + \mathbf{d}, \tag{27}$$

$$\mathbf{e} = \boldsymbol{\kappa}^{-1}(\mathbf{S}) := \zeta(\mathbf{S}), \tag{28}$$

$$\mathbf{p} = \partial\boldsymbol{\psi}\cdot\boldsymbol{\lambda}, \tag{29}$$

$$\boldsymbol{\varphi}(\mathbf{S},\boldsymbol{\lambda},\mathbf{K}) := \boldsymbol{\varphi}_p(\mathbf{S}) - \boldsymbol{\xi}(\boldsymbol{\lambda}) - \mathbf{K} \le 0, \tag{30}$$

$$\boldsymbol{\lambda} \ge \mathbf{0}, \tag{31}$$

$$\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\lambda} = \mathbf{0}, \tag{32}$$

$$\mathbf{F}_v \in \Omega_F(\mathbf{F}_j,\ j \in J), \tag{33}$$

$$\mathbf{u}^- \le \mathbf{u} \le \mathbf{u}^+, \tag{34}$$

$$\mathbf{p}^- \le \partial\boldsymbol{\psi}\cdot\boldsymbol{\lambda} \le \mathbf{p}^+ \tag{35}$$

$$\det\mathbf{M}_k(\mathbf{S}) \ge \boldsymbol{\varepsilon}_s, \qquad k \in \mathbf{K}_a. \tag{36}$$

Inequality (36) corresponds to the earlier conditions (17).

Then, to determine the parameters of the ultimate actions on the structure, with the improved bearing capacity, we propose the following energetic principle:

Of all the statically admissible residual forces, plastic multipliers and corresponding plastic strains, satisfying the conditions of general stability and rigidity of the system, their actual values are for which the power of the safety factors is minimum.

The energetic principle for the large displacements analysis (24)–(36) is a problem of nonlinear mathematical programming. We can notice that the solution of the shakedown problem is conditioned by Eq. (17) and if it is not satisfied, the solution may not exist. Then, this problem must be solved without these conditions, but it is necessary to consider the obtained residual forces as the prestressing forces, which are to be created in the structure before its loading [3].

The monotonically increasing loading is a particular case of a cyclic loading for $|J| = 1$; restriction (17) is now not necessary.

As noted in [3], "singular" (instantly-movable or instantly-rigid) constructions [3], whose prestressing state is stable, are always considered as self-hardening systems, regardless of the direction of loads acting on them. The similar conclusion would hold true both for "tensegrity systems" and for their combination with geometrically "neutral" or all strengthening elements.

4. Equilibrium and quasi-equilibrium loads

Constant (dead) load on the structure is always present, but sometimes it gives additional preloading providing stabilisation to the system. In any case, it is recommended to take such constant load with preloading, which has to be "equilibrium" for the basic mechanism of the failure of the system.

Determination of the equilibrium load is known in the theory of geometrically variable, suspension and cable-stayed structures [11]. Such a load does not cause any displacements in the structures in the absence of deformation of the structural elements. For arbitrary constructions, the "equilibrium" load does not cause the system's kinematic displacements in the state of limit equilibrium. Accordingly, the non-equilibrium load is equal to the difference between the arbitrary and equilibrium loads.

Here, in contrast to the known approaches [11], matrix formulas are proposed for calculating the equilibrium load (see [4]).

In the equation of equilibrium

$$\mathbf{AS} = \mathbf{F} \tag{37}$$

we take some m-subvector $\mathbf{F}_0$ of the n-vector of the equilibrium load $\mathbf{F}$ given, then Eq. (37) can be written in the form of two equations as follows:

$$\begin{bmatrix} \mathbf{A}_0 \\ \mathbf{A}_1 \end{bmatrix} \mathbf{S} = \begin{bmatrix} \mathbf{F}_0 \\ \mathbf{F}_1 \end{bmatrix}, \tag{38}$$

where $\mathbf{A}_0$, $\mathbf{A}_1$ are the $m \times m$- and $s \times m$-submatrices of matrix $\mathbf{A}$, respectively, and submatrix $\mathbf{A}_0$ is nonsingular.

From Eq. (38), we find the internal force vector $\mathbf{S}$ for the equilibrium load:

$$\mathbf{S} = \mathbf{A}_0^{-1}\mathbf{F}_0. \tag{39}$$

Its substitution in the second equation gives subvector $\mathbf{F}_1$ of the equilibrium load vector $\mathbf{F}$

$$\mathbf{F}_1 = \mathbf{A}_1\mathbf{S} = \mathbf{d}_{01}\mathbf{F}_0, \tag{40}$$

where $\mathbf{d}_{01}$ is the $s \times m$-matrix,

$$\mathbf{d}_{01} = \mathbf{A}_1\mathbf{A}_0^{-1}. \tag{41}$$

Then finally, equilibrium load vector $\mathbf{F}$ takes the form

$$\begin{bmatrix} \mathbf{F}_0 \\ \mathbf{d}_{01}\mathbf{F}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{E} \\ \mathbf{d}_{01} \end{bmatrix} \mathbf{F}_0. \tag{42}$$

5. Problems of finding optimum parameters of moving load (upper level)

Finally, on the second level, we maximize parameter μ of the moving load $\mu\mathbf{F}_v$,

$$\mu^* : \mu \to \max. \tag{43}$$

The limit analysis problems (24)–(36) and (43) belong to the theory of bilevel mathematical programming.

To solve this problem, we use the following iterative design procedure. At the low level, the most unfavourable location of moving load $\mathbf{F}_v$ for given parameter μ is found. The influence of equilibrium or quasi-equilibrium load is taken into account here.

Then, at the upper level, the maximum load parameter μ^* is found. The general flowchart of the bilevel limit analysis design procedure is shown in Fig. 1.

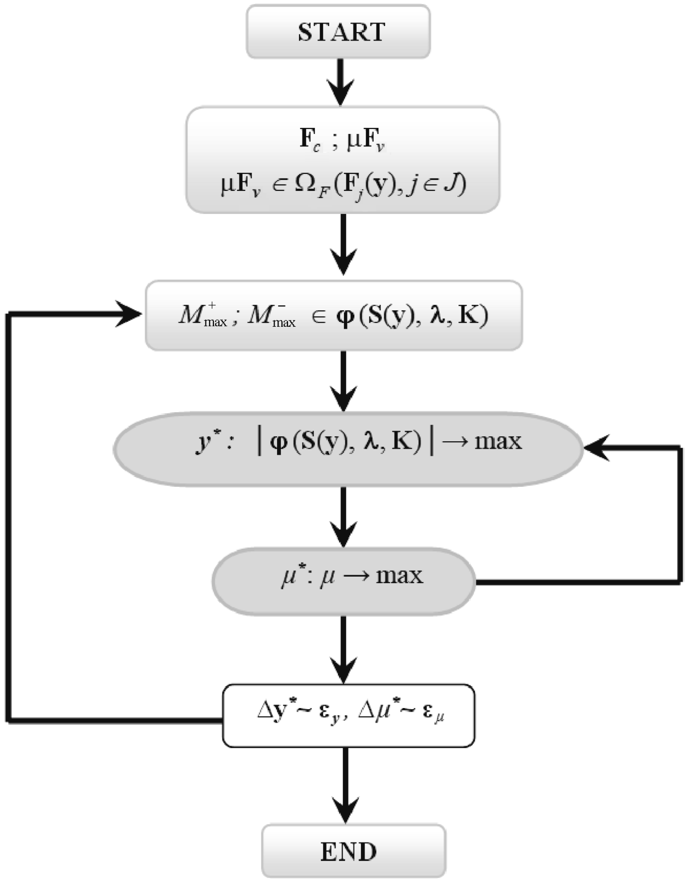

Fig. 1. A general flowchart of the bilevel limit analysis design procedure.

Please note that criteria (24) and (43) may be taken into account simultaneously, in the general nonlinear optimisation problem [3, 10]. However, its division into two levels reduces the dimension of the individual tasks and represents a kind of decomposition approach to its solution. The theory of bilevel optimisation is currently undergoing intensive development. Necessary conditions of optimisation were given in [8, 9, 12, 13], its application will be analysed in future.

6. Numerical analysis of SHS systems

6.1. Strut-framed beam

SHS systems are often used in engineering practice. Examples are beams with a rod system with three cases of posts' position (Fig. 2a,b,c). The strut-framed beam is simply supported at ends. In this study, beams with cross section of 0.3×0.5 m, $L = 12$ m, $h = 2$ m, and $a = L/3$ were connected with bars of a truss.

Analysis of this type of construction [2] has shown that, depending on the geometry of the rod system, when the load capacity is estimated, the system may be self-hardening. Calculations were carried out for the rod system, in the plastic state. All calculations were made using physically and geometrically nonlinear analysis. The numerical solution of the problem was found with the finite element method (FEM), using program ABAQUS/Standard [1] including nonlinear analysis (Nlgeom).

As a result, the limit load capacity of the system for the different truss cases, for $F_3 = 500$ kN, was estimated and the behaviour of the rod system over the limit load equilibrium was observed.

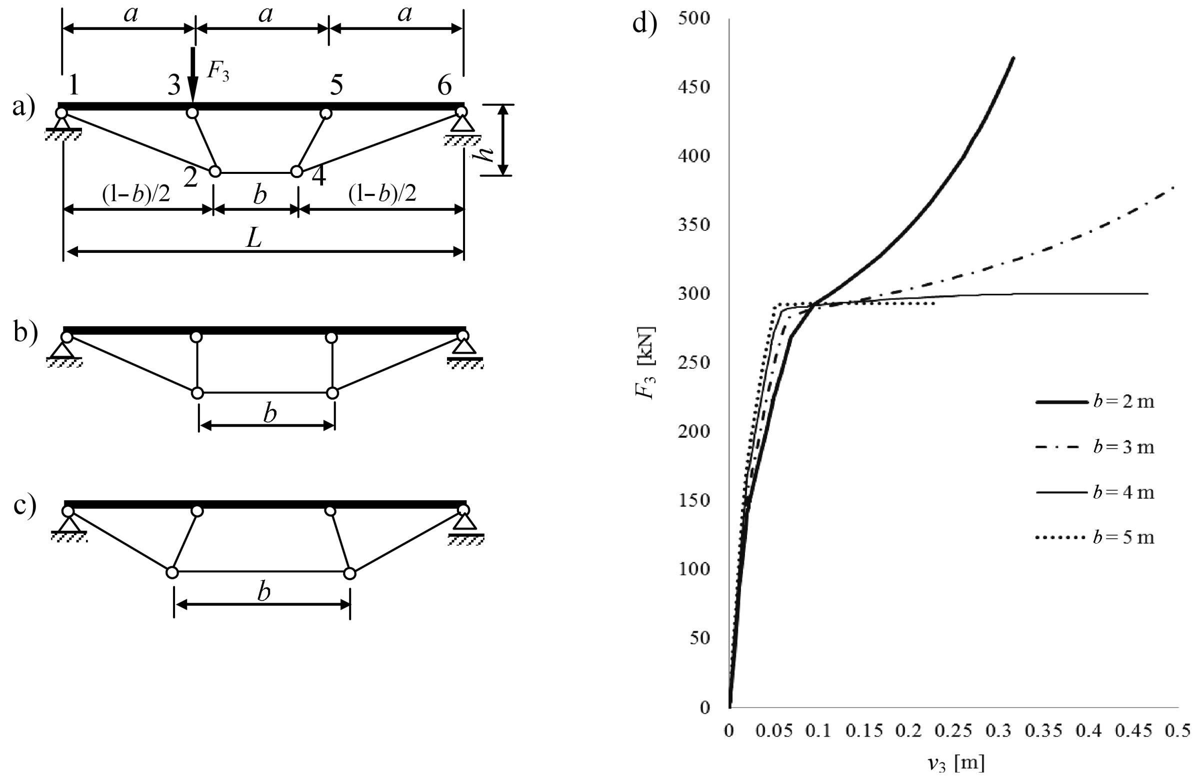

Fig. 2. Load F_3 versus displacement v_3 for the SHS system: a) $b < L/3$; b) $b = L/3$; c) $b > L/3$.

The results of numerical calculations of load F_3 versus displacement v_3 are shown in a diagram in Fig. 2d.

Self-hardening effect (the upper branch of curve $F_3 - v_3$ in Fig. 2) was observed only for the truss with inclined posts ($b = 2$ and 3 m).

When the limit load capacity was obtained in the cases for $b = 4$ and 5 m, we observed the formation of large displacements and then the failure of the construction.

6.2. Examples of viaduct systems

A similar effect can be observed in engineering constructions, for example, viaducts shown in Fig. 3.

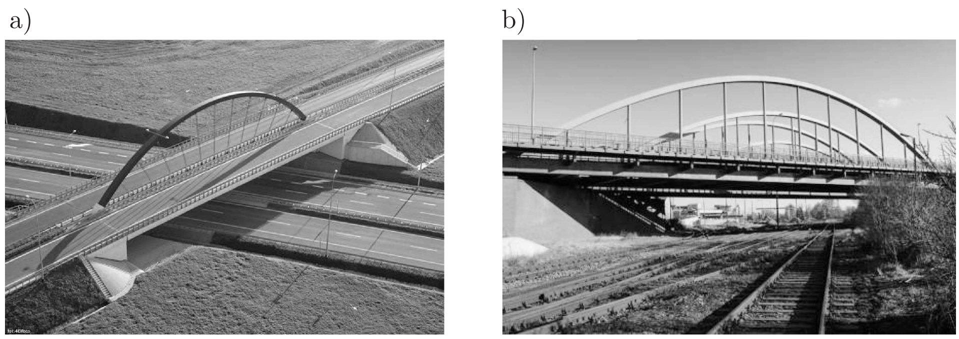

Fig. 3. View of viaducts [14]: a) WD-22 Pyrzyce, b) Gdynia.

In this article, we analysed an arch (with a system of rods) similar to a viaduct structure. The load-bearing structures of the viaducts (see Fig. 3) are modelled as systems composed of reinforced concrete beams reinforced by steel arches, steel braces (two options: inclined Fig. 3a and vertical Fig. 3b) and a concrete construction.

The numerical calculations of such systems (Figs. 4 and 5) were performed with the FEM, using program Abaqus/Standard (ABAQUS 2010) [1] with geometrically and physically nonlinear analysis. Both arches were L = 54 m long and H = 11 m high. The beam was supported at ends and loaded by force F_3 = 600 kN at node w3, where x_1 = 16.85 m – in the system with the inclined braces, and x_2 = 13.22 m – in the one with vertical braces. Figures present the relationship between load F_3 and the vertical displacement v_3 of the 3rd node with the constant force F_3 applied to the viaduct. For the structure with inclined braces, the self-hardening effect (the upper branch of curve $F_3 - v_3$ in Fig. 3) was observed accompanied by the formation of large displacements. In the structure with vertical braces (Fig. 4) – failure was observed when the limit load capacity F^0 was reached.

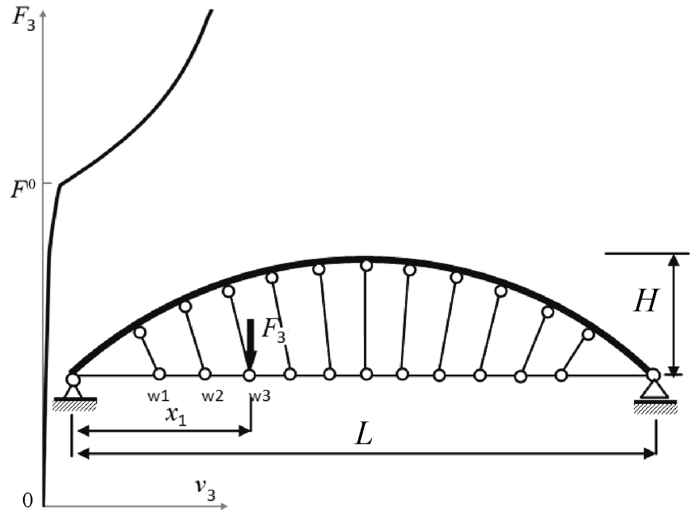

Fig. 4. Load F_3 versus displacement v_3 diagram for the system with inclined braces.

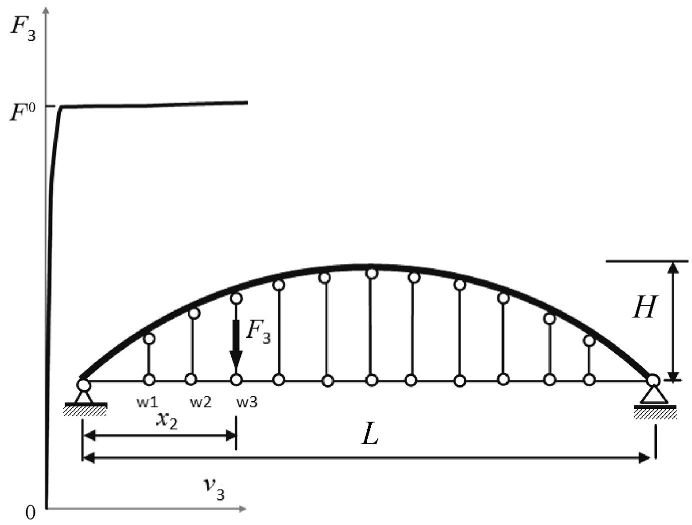

Fig. 5. Load F_3 versus displacement v_3 diagram for the system with vertical braces.

It can be shown that loading of the structure by various constant loads F_c (Fig. 6) produces a smaller vertical displacement in its limit state.

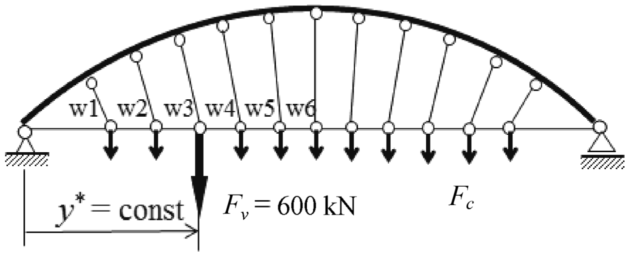

Fig. 6. The loading conditions of the viaduct system with various constant loads F_c.

Three values of dead load were considered, F_c = 100, 300 and 600 kN. The obtained results are presented in Fig. 7. For load F_c = 100 and 300 kN, there was a decrease of deflection compared to the case when F_c = 0. In the case where F_c = 600 kN, the strengthening process and then the growth of large displacements were observed.

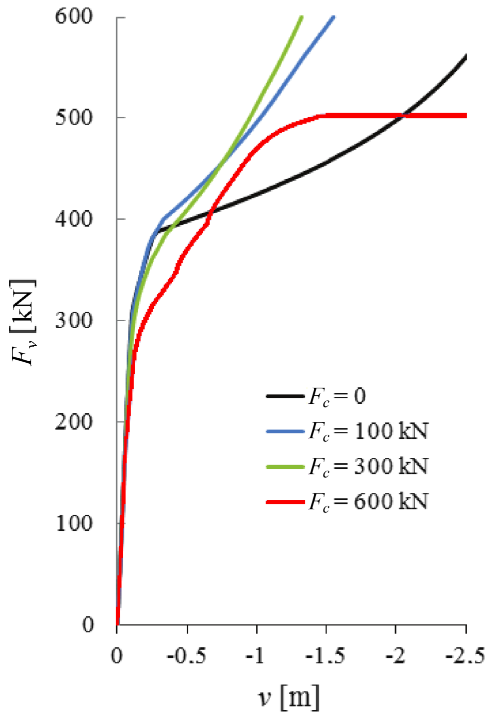

Fig. 7. Load F_v versus displacement v (node w3) for an SHS system.

Please note that a uniformly distributed constant load is used here, which serves as quasi-equilibrium load for the structure, like in practical design. Real equilibrium load F_c is very sensitive to the geometry of the system. The differences in the system geometry are more than 50% (see Fig. 8, for L = 1.0 m, F_c = 40 kN), but the final result, taking into account the authentic equilibrium load, is greater by about only 5% like for the quasi-equilibrium one.

Then, we loaded the viaduct system with inclined braces by constant load F_c = 100 kN and moving load F_v (Fig. 9). A concentrated force was applied to nodes from w1 to w6, which is at the half of the span length. Two moving load values were analysed, F_v = 300 and 600 kN. Vertical displacements of node w3 depending on the location of load F_v are shown in Figs. 10 and 11. Loading of the structure by the moving force 300 kN caused the displacement of approximately 16 cm (Fig. 10). While for a load of 600 kN, the obtained displacement values amounted to over 1.5 m (Fig. 11).

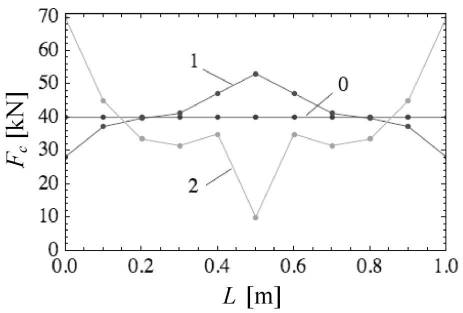

Fig. 8. Equilibrium load for the viaduct with: 0 – vertical braces; 1, 2 – different angles of braces inclination to the centre.

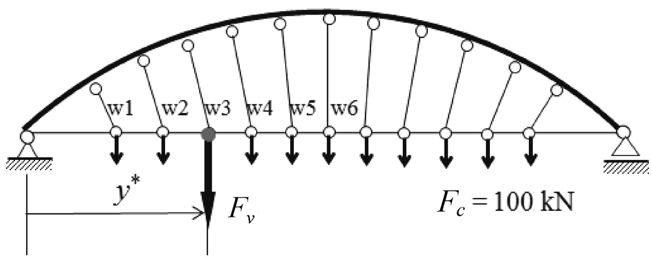

Fig. 9. Loading conditions of the viaduct system with fixed constant loads F_c = 100 kN.

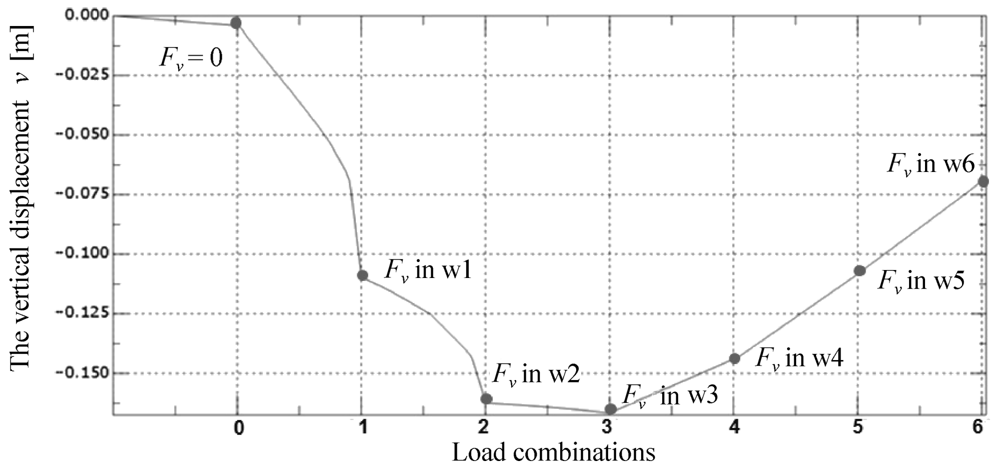

Fig. 10. Vertical displacement v versus moving load F_v = 300 kN for node w3.

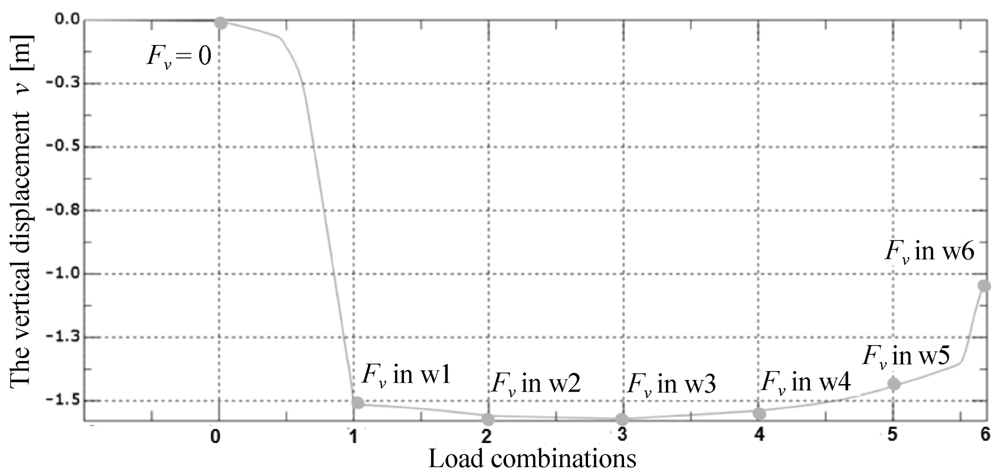

Fig. 11. Vertical displacement v versus moving load F_v = 600 kN for node w3.

The next step was to load the system with inclined braces by constant load $\mathbf{F}_c$ and moving load $\mathbf{F}_v$, to find the most unfavourable load position

$$\mathbf{F} = \mathbf{F}_c + \mu \mathbf{F}_v(\mathbf{y}),$$

μ is a parameter of the load.

In this step, we searched the maximum of moving load parameter $\mu^* : \mu \to \max$.

Constant load $\mathbf{F}_c$ was applied to each node (Fig. 12a). Moving load $\mathbf{F}_v$ was realised by gradual loading of the nodes from w1 to w11 (Fig. 12b) and then unloading them.

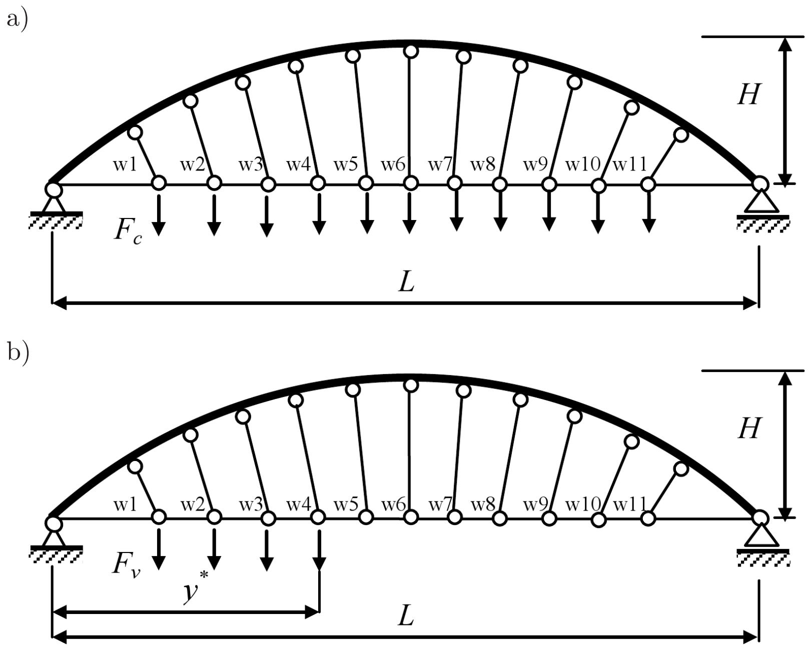

Fig. 12. Simplified scheme and load cases of the viaduct system: a) distributions of constant load $F_c = 100$ kN, b) distributions of moving load $F_v = 115$ kN.

For this system, we obtained the most unfavourable location $\mathbf{y}^*$ of moving load $\mathbf{F}_v$ applied to the construction. The value of moving load was taken from a range of load tests.

The most unfavourable location $\mathbf{y}^*$ is the one for which we obtain the minimal values of safety factors $\boldsymbol{\varphi}$ $(\mathbf{S}(\mathbf{y}), \boldsymbol{\lambda}, \mathbf{K})$ or maximum values of the internal forces (here bending moments in the arch) $|\mathbf{S}^{\max/\min}(y)|$ for the coordinate $\mathbf{y}^*$.

The envelopes of bending moments in both arches were obtained from the linear and non-linear analyses. The results of the numerical analysis (graphs of bending moments) carried out to find the coordinate $\mathbf{y}^*$ (where maximum moments in the arch were found) are presented below.

The distributions of bending moments (Figs. 13 and 14) obtained for each load situation were used to determine the envelope of the bending moments (Fig. 15) and to estimate the most unfavourable location of the moving load $\mathbf{F}_v$.

It was found that the maximum positive values of $M^+_{\max}$ moments are obtained for the load applied to node w4, while the minimum $|M_{\max}|$ for the loads applied to nodes from w1 to w5. Therefore, it is considered that the most uncomfortable location is when the load is applied to nodes from w1 to w5.

Then, in the second step of the bilevel limit analysis, we searched the maximum load parameter $\mu^* : \mu \to \max$. We assumed that the moving load is 180 kN and is implemented in six combinations:

- Combination 0 corresponds to constant load $F_c = 100$ kN.
- In combinations $1 \ldots 5$ (Fig. 16) the moving load $F_v = 180$ kN applies respectively to nodes w1 … w5.

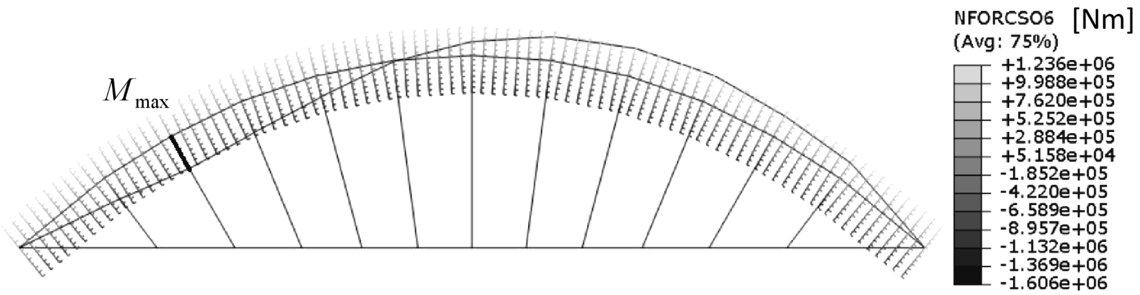

Fig. 13. Bending moments. Moving load applied from node $w1$ to node $w4$.

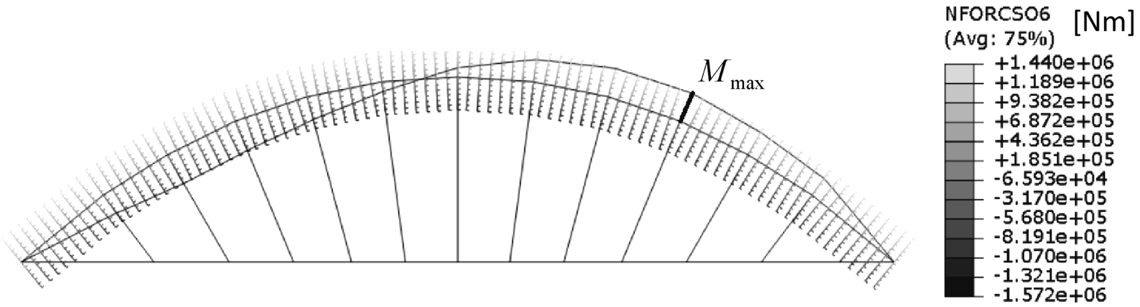

Fig. 14. Bending moments. Moving load applied from node w1 to node w5.

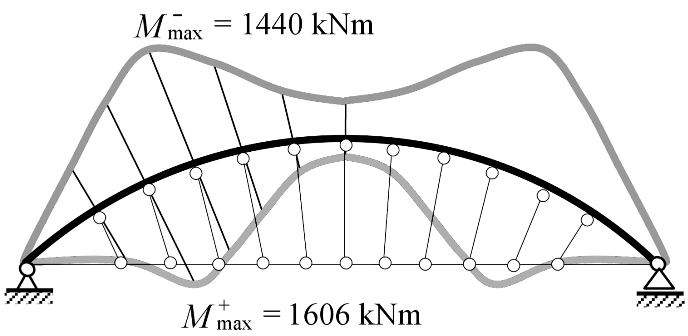

Fig. 15. The envelope of bending moments in the arch.

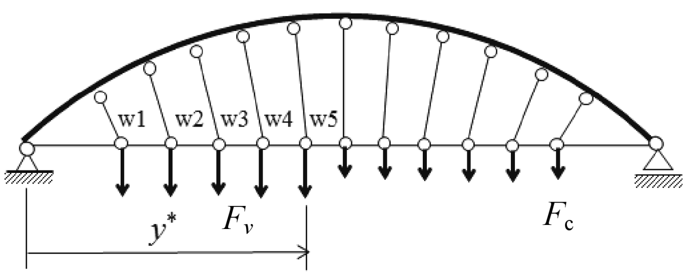

Fig. 16. The load of the viaduct system – combination 5.

Figure 17 shows the vertical displacement of nodes w1 . . . w5 depending on the location of the moving load F_v = 180 kN.

The numerical calculations show the measure of safety margin in structural design. Based on the numerical calculations for this system we obtained a parameter μ, estimated at the level of almost 1.4.

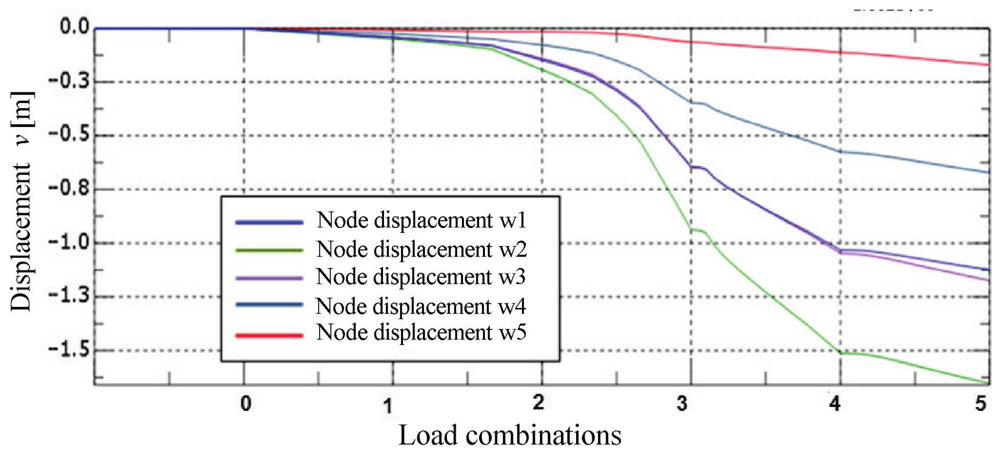

Fig. 17. The vertical displacement v versus moving load F_v diagram for five nodes.

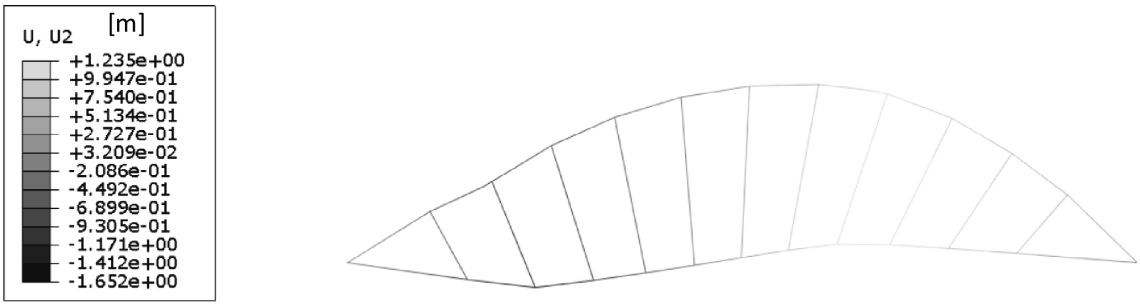

Fig. 18. System SHS in the limit state, $\mu = 1.4$.

7. Conclusions

1. The mathematical models and methods of limit analysis for the self-hardening systems (SHS) are presented. Design of this type of structures is important not only in limit load range but also in the study of elastic behaviour.

2. The optimisation problem is formulated as a bilevel mathematic programming one. To find limit parameters of load actions the extreme energy principle is suggested on the low level. On the upper level of optimisation the parameter of the moving load is maximized.

3. The obtained numerical and analytical results show that self-hardening effect and taking into account quasi-equilibrium loading are important for the design of some classes of SHS systems.

4. The numerical calculations of SHS systems presented in this paper show the level of safety margin for this class of structures.

5. The proposed approach may be used for the spatial plate and shell spatial structures of SHS systems under any various actions (accidental, seismic or fires/temperatures).

6. Necessary conditions of optimisation may be analysed in future.

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