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Supplementing the numerical solution of singular/hypersingular integral equations/inequalities with parametric inequality constraints with applications to crack problems

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Singular and hypersingular integral equations appear frequently in engineering problems. The approximate solution of these equations by using various numerical methods is well known. Here we consider the case where these equations are supplemented by inequality constraints-mainly parametric inequality constraints, but also the case of singular/hypersingular integral inequalities. The approach used here is simply to employ the computational method of quantifier elimination efficiently implemented in the computer algebra system *Mathematica* and derive the related set of necessary and sufficient conditions for the validity of the singular/hypersingular integral equation/inequality together with the related inequality constraints. The present approach is applied to singular integral equations/inequalities in the problem of periodic arrays of straight cracks under loading- and fracture-related inequality constraints by using the Lobatto-Chebyshev method. It is also applied to the hypersingular integral equation/inequality of the problem of a single straight crack under a parametric loading by using the collocation and Galerkin methods and parametric inequality constraints.

Keywords: singular integral equations/inequalities, hypersingular integral equations/inequalities, boundary integral equations/inequalities, parametric inequality constraints, numerical methods, crack problems.

1. Introduction

Singular integral equations (related to Cauchy-type principal value integrals) and hypersingular integral equations (related to Hadamard-type finite-part integrals or equivalently to derivatives of Cauchy-type integrals) are of a great practical importance in applied solid and fluid mechanics, where they are also frequently called singular BIEs (boundary integral equations) and hypersingular BIEs respectively since they are valid on the boundary of the medium. Their importance is due to the fact that these integral equations permit the solution of many related problems such as crack problems and many general elasticity problems in solid mechanics as well as many problems in fluid mechanics of ideal fluids, such as aerofoil problems. For this reason, several efficient approximate methods for their numerical solution appeared long ago.

More explicitly, with respect to singular integral equations reference can be made to the results by Kalandiya [44; 45, pp. 95–117], Erdogan and his collaborators [14–16], Ioakimidis and Theocaris [29, 30, 43, 58], Tsamasphyros and Theocaris [59, 61], Chrysakis and Tsamasphyros [7], Gerasoulis and Srivastav [19], Gerasoulis [17, 18], Golberg [22], Elliott [13] and many other researchers with significant contributions to the field. Analogously, with respect to hypersingular integral equations reference can be made to the results by Multhopp [51], Williams [63], Bueckner [5], Golecki [23–26], Ioakimidis [31–33], Kaya and Erdogan [46], Golberg [20, 21], Sladek, Sladek and Tanaka [55], Sladek and Sladek [54], Zhang, Sladek and Sladek [66], Linkov and Mogilevskaya [47], Mogilevskaya [49, 50], Guiggiani [27], Guiggiani et al. [28], Ang and Noone [3], Ang [1, 2], Ashour [4], Tweed, John and

Dunn [62] and many other researchers with significant contributions to the field. A related review by de Klerk [12] is also of interest.

The aim of this paper is to show the usefulness of a computational method called quantifier elimination and based on symbolic computations to supplement the methods of numerical solution of singular/hypersingular integral equations. This method is based on symbolic computations available in computer algebra systems, which have been of great interest in applied mechanics long ago, see, e.g., the interesting review by Pavlović [52]. Despite that, quantifier elimination is a specialized computational method with very few implementations in computer algebra systems so far. In most cases, it is based on cylindrical algebraic decomposition, an efficient algebraic algorithm originally devised by Collins [8]. Many extensions of the method appeared since its introduction. Among a very large number of related papers see, e.g., those by Collins and Hong [9] and Strzeboński [56, 57]. Still, the classical reference on quantifier elimination and cylindrical algebraic decomposition remains the interesting book edited by Caviness and Johnson [6].

The most powerful, user-friendly and modern implementation of quantifier elimination seems to be that prepared by Strzeboński and available in the popular computer algebra system *Mathematica* [64], and this is the implementation that will be exclusively used here. Details of this implementation can be found in the related tutorial [65] as well as in the book on symbolics in *Mathematica* by Trott [60, pp. 62–78].

On the other hand, a bibliography of the applications of quantifier elimination was prepared by Ratschan [53]. In computational and applied mechanics, some applications of quantifier elimination were suggested by Ioakimidis [34–40].

In this paper, we will study a new way of applying quantifier elimination to computational mechanics. This new approach consists in studying singular/hypersingular integral equations accompanied by appropriate inequality constraints. Here both these equations and the accompanying inequality constraints appear in parametric forms with one or more than one parameter. For the numerical solution of these integral equations, classical numerical methods (such as the quadrature method, the collocation method and the Galerkin method) can be used. Similarly, singular integral inequalities will also be studied in brief. Quantifier elimination can be used simultaneously with the numerical solution of the integral equation or just after the completion of this numerical solution. The present applications concern crack problems in fracture mechanics, which traditionally constitute an interesting field of application of singular/hypersingular integral equations, although such equations appear in several additional problems of computational mechanics, e.g., in fluid mechanics of ideal fluids.

This paper is organized as follows. In Sec. 2 quantifier elimination is applied to singular integral equations under parametric inequality constraints. In Sec. 3 singular integral inequalities are considered in a similar way. In Sec. 4 quantifier elimination is applied to a hypersingular integral equation again under parametric inequality constraints. Similarly, in Sec. 5 the related hypersingular integral inequality is considered. Finally, in Sec. 6 the conclusions on the obtained results are mentioned in brief, followed by a short discussion. In all cases, the applications concern singular/hypersingular integral equations/inequalities and parametric inequality constraints appearing in classical crack problems in fracture mechanics, an area where the usefulness of both singular and hypersingular integral equations was demonstrated in the past. Nevertheless, the present computational approach based on quantifier elimination is also obviously applicable to many more problems related to integral equations/inequalities, including parametric inequality constraints both in solid and fluid mechanics.

2. Application of quantifier elimination to singular integral equations under inequality constraints

In this section, we will illustrate the use of the computational method of quantifier elimination in singular integral equations with a Cauchy-type kernel accompanied by parametric inequality constraints. The singular integral equations that will be used here are those of periodic arrays of collinear and parallel cracks in an infinite plane isotropic elastic medium.

2.1. A periodic array of collinear cracks

First, we consider the classical problem of a periodic array of collinear cracks along the real axis Ox (Fig. 1). Each crack has length $2a$ and the period of the array (the distance between the midpoints of two consecutive cracks) is b (Fig. 1). The related Cauchy-type singular integral equation (singular BIE) on each crack was derived by Datsyshin and Savruk [10] and can be written in the following slightly modified and more convenient final form [29, 41]:

$$c \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} K_s(t,x)\, g(t)\, \mathrm{d}t = p(x), \qquad -1 < x < 1. \tag{1}$$

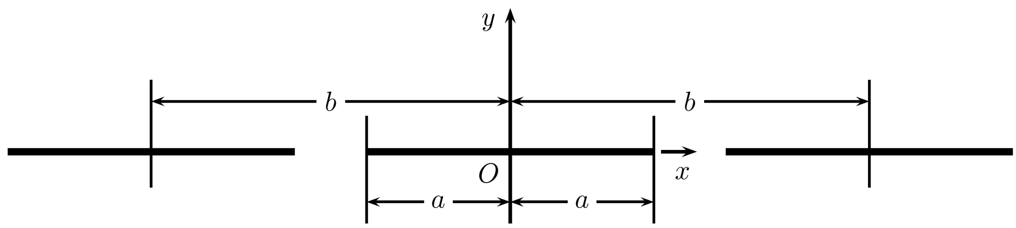

Fig. 1. A periodic array of collinear cracks.

Here the singular kernel $K_s(t,x)$ is given by [10, 29, 41]

$$K_s(t,x) = \cot[\pi c(t-x)]. \tag{2}$$

In these equations, c denotes the ratio a/b. Moreover, in the singular integral equation (1) $g(t)$ is the unknown function with $g(t)/\sqrt{1-t^2}$ being proportional to the slope $v'(t)$ of the crack opening displacement $v(t)$ and the known function $p(x)$ on the right-hand side is the normal pressure distribution (the compressive loading) applied to both edges of each crack of the periodic array. This pressure distribution $p(x)$ is assumed to be the same on all the cracks of the present periodic array of cracks. The above singular integral equation (1) is accompanied by the related condition of single-valuedness of displacements [10, 29, 41]

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} g(t)\, \mathrm{d}t = 0. \tag{3}$$

For the numerical solution of the singular integral equation (1) together with the condition (3) we can use the well-known related quadrature method and more explicitly the Lobatto-Chebyshev method based on the corresponding almost Gaussian quadrature (numerical integration) rule. For ordinary integrals, the Lobatto-Chebyshev quadrature rule with n nodes has the well-known simple form [11, p. 104]

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} g(t)\, \mathrm{d}t \approx \sum_{i=1}^{n} A_{i,n}\, g(t_{i,n}). \tag{4}$$

Here the nodes $t_{i,n}$ and the weights $A_{i,n}$ are given in [11, p. 104]

$$t_{i,n} = \cos\frac{(i-1)\pi}{n-1}, \qquad i = 1, 2, \ldots, n, \tag{5}$$

$$A_{1,n} = A_{n,n} = \frac{\pi}{2(n-1)}, \qquad A_{i,n} = \frac{\pi}{n-1}, \qquad i = 2, 3, \ldots, n-1. \tag{6}$$

The Lobatto-Chebyshev quadrature rule (4) holds also true for Cauchy-type principal value integrals of the form

$$I(x) = \fint_{-1}^{1} \frac{1}{\sqrt{1-t^2}} \frac{g(t)}{t-x} \,\mathrm{d}t, \qquad -1 < x < 1. \tag{7}$$

This happens provided that the variable x in the integral $I(x)$ takes one of the $n-1$ concrete values [29, 41, 58]

$$x_{k,n} = \cos \frac{(2k-1)\pi}{2(n-1)}, \qquad k = 1, 2, \ldots, n-1. \tag{8}$$

Under these circumstances, on the basis of Eqs. (1) and (3) the Lobatto-Chebyshev method with n nodes leads to the following well-known approximate system of n linear algebraic equations [29, 41, 58]:

$$c \sum_{i=1}^{n} A_{i,n} K_s(t_{i,n}, x_{k,n}) g_n(t_{i,n}) = p(x_{k,n}), \qquad k = 1, 2, \ldots, n-1, \tag{9}$$

$$\sum_{i=1}^{n} A_{i,n} g_n(t_{i,n}) = 0, \tag{10}$$

where $g_n(t)$ denotes an approximation to the unknown function $g(t)$. We denote the set of the above $(n-1)+1 = n$ equations by the symbol $\mathcal{E}_n$ or `eqs[n]` in *Mathematica*.

Of course, in fracture mechanics we are generally interested in the values of the stress intensity factors $K(\pm 1)$ at the crack tips $x = \pm 1$. These factors are determined (here for convenience in dimensionless form) by using the very simple equations [29, 41, 58]

$$K(1) = g(1) \qquad \text{and} \qquad K(-1) = -g(-1). \tag{11}$$

Next we also consider the following inequality constraints:

$$K(1) < K_f \qquad \text{and} \qquad K(-1) < K_f \qquad \text{with} \qquad K_f > 0. \tag{12}$$

Here K_f is an upper bound of the stress intensity factors allowing the avoidance of fracture. This bound is called the fracture toughness of the elastic material (although here we are using dimensionless length variables). The fracture toughness K_f is assumed to be a material constant. Because of Eqs. (11), the inequality constraints (12) take their final forms

$$g_n(1) < K_f \qquad \text{and} \qquad -g_n(-1) < K_f \qquad \text{with} \qquad K_f > 0, \tag{13}$$

now with the approximation $g_n(t)$ (which will be used below during quantifier elimination) to the unknown function $g(t)$.

Here we also permit the normal compressive loading $p(x)$ of the cracks of the periodic array of collinear cracks to be a variable loading but satisfying the inequality constraint

$$\forall x \in [-1, 1] \;\; p(x) \geq p_{\min} \qquad \text{with} \qquad p_{\min} > 0 \tag{14}$$

defining a lower bound $p_{\min}$ of the loading $p(x)$.

Under these conditions, we first define the assumptions

$$\mathcal{A}_0 = K_f > 0 \wedge p_{\min} > 0 \tag{15}$$

denoted by `ass0` in *Mathematica*. Here the symbol $\wedge$ denotes the logical "and". Next, on the basis of the inequality constraint (here the quantified formula) (14) we define the additional $n-1$ assumptions $\mathcal{A}_{p,n}$ concerning the normal compressive loading $p(x)$ at the collocation points $x_{k,n}$ (with $k = 1, 2, \ldots, n-1$) determined by Eqs. (8), which are

$$\mathcal{A}_{p,n} = p_{1,n} \geq p_{\min} \wedge p_{2,n} \geq p_{\min} \wedge \cdots \wedge p_{n-1,n} \geq p_{\min} \tag{16}$$

(with $p_{k,n} := p(x_{k,n})$ with $k = 1, 2, \ldots, n-1$). These assumptions $\mathcal{A}_{p,n}$ constitute a discretization of the quantified formula (14) and therefore an approximation to this formula. The set of all the above assumptions $\mathcal{A}_0$ or `ass0` and $\mathcal{A}_{p,n}$ is denoted by $\mathcal{A}_n$ or `ass[n]` in *Mathematica*, i.e., we have $\mathcal{A}_n = \mathcal{A}_0 \cup \mathcal{A}_{p,n}$. These are the assumptions that will be actually used during quantifier elimination in *Mathematica*.

Moreover, as far as the quantified variables are concerned, which will be eliminated during quantifier elimination, these are the $2n-1$ variables

$$\mathcal{V}_n = \{p_{1,n}, p_{2,n}, \ldots, p_{n-1,n}, g_{1,n}, g_{2,n}, \ldots, g_{n,n}\} \tag{17}$$

(with $g_{i,n} := g_n(t_{i,n})$ with $i = 1, 2, \ldots, n$) denoted by `var[n]` in *Mathematica*. Now we will proceed to quantifier elimination with respect to the existentially (i.e., related to the existential quantifier $\exists$, exists) quantified formula

$$\exists \mathcal{V}_n \qquad \text{such that} \qquad \mathcal{E}_n \wedge g_{1,n} < K_f \wedge -g_{n,n} < K_f, \tag{18}$$

of course under the validity of all the assumptions $\mathcal{A}_n$.

In order to perform quantifier elimination in *Mathematica* we can use either the `Resolve` command (dedicated to quantifier elimination) or the much more general `Reduce` command to generally essentially obtain the same results as far as quantifier elimination is concerned. Here we will exclusively use the `Reduce` command. The related complete *Mathematica* notebook (commands and approximate QFFs) for the present crack problem with $c = a/b = 0.45$ (i.e., with the collinear cracks very close to each other) is displayed in Appendix, paragraph A1. For this value of c, we obtained the following approximate QFFs:

$$\mathrm{QFF}_2 = K_f > 4.46607 p_{\min} \qquad \text{or} \qquad p_{\min} < 0.223911 K_f, \tag{19}$$

$$\mathrm{QFF}_3 = K_f > 2.45180 p_{\min} \qquad \text{or} \qquad p_{\min} < 0.407864 K_f, \tag{20}$$

$$\mathrm{QFF}_4 = K_f > 2.19817 p_{\min} \qquad \text{or} \qquad p_{\min} < 0.454923 K_f, \tag{21}$$

$$\mathrm{QFF}_5 = K_f > 2.13585 p_{\min} \qquad \text{or} \qquad p_{\min} < 0.468198 K_f, \tag{22}$$

$$\mathrm{QFF}_6 = K_f > 2.11937 p_{\min} \qquad \text{or} \qquad p_{\min} < 0.471839 K_f, \tag{23}$$

$$\mathrm{QFF}_7 = K_f > 2.11494 p_{\min} \qquad \text{or} \qquad p_{\min} < 0.472827 K_f, \tag{24}$$

$$\mathrm{QFF}_8 = K_f > 2.11375 p_{\min} \qquad \text{or} \qquad p_{\min} < 0.473093 K_f. \tag{25}$$

Each of these QFFs concerns the necessary and sufficient condition permitting the validity of the equations and the inequality constraints of the present crack problem, of course, with the use of the Lobatto-Chebyshev method with n nodes. We also observe the rapid convergence of the numerical coefficients in the right-hand sides of these QFFs for increasing values of the number of nodes n.

2.2. A periodic array of parallel cracks

Completely analogously, we can work with a periodic array of parallel cracks, which are assumed to be parallel to the Ox-axis (Fig. 2). Again each crack has length $2a$ and the period of the array

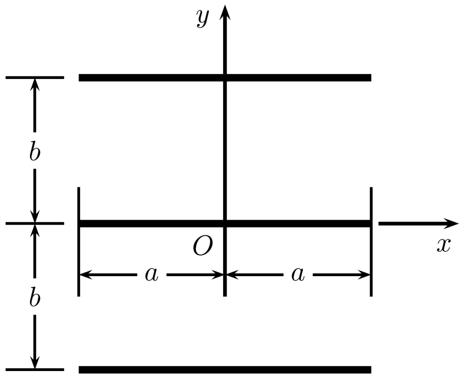

Fig. 2. A periodic array of parallel cracks.

(but now along the Oy-axis) is b (Fig. 2). We also use again the ratio $c = a/b$. In this case, the Cauchy-type singular integral equation (1) holds still true, but the singular kernel $K_s(t,x)$ is now given by the somewhat more complicated formula [29, 41]

$$K_s(t,x) = 2\coth[\pi c(t-x)] - \pi c(t-x)\,\mathrm{csch}^2[\pi c(t-x)]. \tag{26}$$

For $c = a/b = 0.50$ (i.e., with the parallel cracks moderately close to each other) and for $n = 2, 3, \ldots, 8$, exactly as previously by using the same approach based on the Lobatto-Chebyshev method [29, 41, 58] we obtain the approximate QFFs

$$\mathrm{QFF}_2 = K_f > 0.337898 p_{\min} \quad \text{or} \quad p_{\min} < 2.95948 K_f, \tag{27}$$

$$\mathrm{QFF}_3 = K_f > 0.554644 p_{\min} \quad \text{or} \quad p_{\min} < 1.80296 K_f, \tag{28}$$

$$\mathrm{QFF}_4 = K_f > 0.571915 p_{\min} \quad \text{or} \quad p_{\min} < 1.74851 K_f, \tag{29}$$

$$\mathrm{QFF}_5 = K_f > 0.570153 p_{\min} \quad \text{or} \quad p_{\min} < 1.75391 K_f, \tag{30}$$

$$\mathrm{QFF}_6 = K_f > 0.570188 p_{\min} \quad \text{or} \quad p_{\min} < 1.75381 K_f, \tag{31}$$

$$\mathrm{QFF}_7 = K_f > 0.570193 p_{\min} \quad \text{or} \quad p_{\min} < 1.75379 K_f, \tag{32}$$

$$\mathrm{QFF}_8 = K_f > 0.570193 p_{\min} \quad \text{or} \quad p_{\min} < 1.75379 K_f. \tag{33}$$

Again, we observe the rapid convergence of the numerical coefficients on the right-hand sides of these QFFs for increasing values of the number of nodes n.

It is understood that when we perform quantifier elimination to the quantified formulae and the related constraints without parameters (free variables), i.e., here with the present parameters $p_{\min}$ and K_f taking concrete, numerical values, then, obviously, the resulting QFF will be simply `True` or `False`. And naturally, exactly the same result, `True` or `False`, will be obtained simply by substituting the parameters, e.g., here $p_{\min}$ and K_f in the above QFFs QFF_n, with their numerical values provided that these values are available.

3. Application of quantifier elimination to singular integral inequalities under inequality constraints

A further and rather interesting use of the present computational approach based on quantifier elimination consists in its direct application to singular integral inequalities instead of equations. Two such applications to the same periodic arrays of cracks (collinear and parallel cracks) will be briefly presented in Subsec. 3.1. Next, the application in Subsec. 3.2 will concern the case of a parametric kernel.

3.1. Periodic arrays of collinear and parallel cracks

First, we consider again the problem of a periodic array of collinear cracks in an infinite plane isotropic elastic medium. This problem was already studied in the previous section on the basis of the singular integral equation (1) (with its singular kernel $K_s(t,x)$ given by Eq. (2)) and the accompanying condition of single-valuedness of displacements (3).

More explicitly, here instead of the singular integral equation (1) we consider the related singular integral inequality

$$c \fint_{-1}^{1} \frac{1}{\sqrt{1-t^2}} K_s(t,x) g(t) \,\mathrm{d}t \geq p_{\min}, \qquad -1 < x < 1 \tag{34}$$

with $p_{\min} > 0$. Here we have directly inserted the lower bound $p_{\min}$ into the constraint (14) concerning the compressive normal loading distribution $p(x)$ of the cracks in this singular integral inequality (34). Naturally, the condition (3) of single-valuedness of displacements should once more hold true.

Next, we performed again quantifier elimination by using the `Reduce` command of *Mathematica* analogously to what we did in the previous section and by using again the inequality constraints (13), i.e., $\pm g_n(\pm 1) < K_f$, but now in the singular integral inequality (34) instead of the related singular integral equation (1). Moreover, the Lobatto-Chebyshev quadrature rule was used in the singular integral inequality (34) and in the condition (3). As expected, the obtained QFFs were again those displayed in Eqs. (19)–(25).

Completely analogously, we studied the singular integral inequality (34) for an array of parallel cracks with the singular kernel $K_s(t,x)$ given by Eq. (26). As expected, the obtained QFFs were again those displayed in Eqs. (27)–(33).

3.2. A periodic array of collinear cracks with a parametric kernel

In this subsection, we consider again the problem of a periodic array of collinear cracks but now with a parametric kernel $K_s(t,x)$ with parameter the ratio $c = a/b$ under the geometric restriction that $0 \leq c < 1/2$. This is clear from Fig. 1 since $c = a/b$. Here for computational convenience we will also use the dimensionless unknown function $g^*(t) := g(t)/p_{\min}$ with $p_{\min} > 0$. As a result, the singular integral inequality (34) takes its simpler form

$$c \fint_{-1}^{1} \frac{1}{\sqrt{1-t^2}} K_s(t,x) g^*(t) \,\mathrm{d}t \geq 1, \qquad -1 < x < 1. \tag{35}$$

Moreover, here we assume that the actual compressive normal loading distribution $p(x)$ applied to the fundamental crack $x \in [-1,1]$ of the present periodic array of cracks is a symmetric loading with $p(-x) = p(x)$. Then, obviously, the unknown function $g^*(t)$, which is related to the slope $v'(t)$ of the crack opening displacement $v(t)$, will be an antisymmetric function with $g^*(-t) = -g^*(t)$. Therefore, the condition of single-valuedness of displacements (3) is automatically satisfied and does not need to be taken into account here.

Exactly as in the previous subsections, we will apply again the Lobatto-Chebyshev method [29, 41, 58] now to the singular integral inequality (35) here with $n = 6$ nodes $t_{i,n}$. Because of the presence of the parameter $c = a/b$ in the trigonometric singular kernel $K_s(t,x) = \cot[\pi c(t-x)]$ in Eq. (2), we use an approximation to this kernel. More explicitly, focusing our attention on not very high values of c, we use a generalized Taylor-Maclaurin series expansion of the singular kernel $K_s(t,x)$ in Eq. (2). For example, with an order $r = 10$ in this series expansion by using *Mathematica* we easily obtain the approximation $\widetilde{K}_s(t,x)$ to the kernel $K_s(t,x) = \cot s$:

$$\widetilde{K}_s(t,x) = \frac{1}{s} - \frac{1}{3}s - \frac{1}{45}s^3 - \frac{2}{945}s^5 - \frac{1}{4725}s^7 - \frac{2}{93555}s^9 \tag{36}$$

with $s := \pi c(t-x)$ and here with powers up to s^9.

By applying the already described Lobatto-Chebyshev method with $n = 6$ nodes to the singular integral inequality (35) we get the approximate system of linear inequalities

$$c \sum_{i=1}^{n/2} A_{i,n}[\widetilde{K}_s(t_{i,n}, x_{k,n}) - \widetilde{K}_s(t_{n-i+1,n}, x_{k,n})] g_n^*(t_{i,n}) \geq 1, \qquad k = 1, 2, \ldots, n/2 \tag{37}$$

with n being an even positive integer. The reason for using here only $n/2$ (here 3 since $n = 6$) inequalities is that we have already observed the antisymmetry $g^*(-t) = -g^*(t)$ of the unknown function $g^*(t)$ due to the assumed symmetry $p(-x) = p(x)$ of the loading $p(x)$. Therefore, we have

$$g_n^*(t_{n-i+1,n}) = -g_n^*(t_{i,n}), \qquad i = 1, 2, \ldots, n/2, \tag{38}$$

here with $n = 6$ nodes whence $n/2 = 3$ nodes. The above equations were already taken into account in the linear inequalities (37). In this way, with $n = 6$ we will use only $n/2 = 3$ linear inequalities of the form (37) and this constitutes a great simplification in the symbolic computations during quantifier elimination with *Mathematica*. In fact, the assumed symmetry of $p(x)$ and the resulting antisymmetry of $g(t)$ and further $g^*(t)$ permit us to restrict our attention on the right half $x \in [0, 1]$ of the fundamental crack $x \in [-1, 1]$ employing only the non-negative collocation points $x_{k,n}$ in Eqs. (8) with $k = 1, 2, \ldots, n/2$.

In addition to the singular integral inequality (35) we also assume again that the stress intensity factors $K(\pm 1)$ at the crack tips $x = \pm 1$ do not reach the fracture toughness K_f and therefore the fracture of the cracked specimen is avoided. Next, because of the assumed symmetry $p(-x) = p(x)$ of the present loading $p(x)$ only the first of the related two inequality constraints (13) should be taken into account. Therefore, here we must simply have (with $t_{1,n} = 1$ in the Lobatto-Chebyshev quadrature rule)

$$g_n(1) < K_f \qquad \text{or here equivalently} \qquad g_n^*(1) < K_f^* \tag{39}$$

with $K_f^* > 0$ and

$$K_f^* := \frac{K_f}{p_{\min}} \qquad \text{since} \qquad g_n^*(t) := \frac{g_n(t)}{p_{\min}} \qquad \text{with} \qquad p_{\min} > 0. \tag{40}$$

Summing up, under all of the previous assumptions here with $n = 6$ and therefore $n/2 = 3$ we have to perform quantifier elimination to the existentially quantified formula

$$\exists \{g_6^*(t_{1,6}), g_6^*(t_{2,6}), g_6^*(t_{3,6})\} \qquad \text{such that} \qquad \mathcal{I}_3 \wedge g_6^*(t_{1,6}) < K_f^* \tag{41}$$

under the geometric assumption $0 \leq c < 1/2$, where $t_{1,6} = 1$ and $\mathcal{I}_3$ denotes the system of the three linear inequalities (37) (here with $n/2 = 3$).

The related quantifier elimination was again performed with the help of the `Reduce` command of *Mathematica* with respect to the free variable $K_f^* := K_f/p_{\min}$. The auxiliary `Refine` command was also used again so that the geometric assumption $0 \leq c < 1/2$ can automatically be taken into account in the resulting QFF (quantifier-free formula). Moreover, the `Chop` command was also used so that extremely small quantities do not appear in the output.

After an appropriate simplification of the result with the help of the `Simplify` command the following final form of the QFF was obtained:

$$\frac{n_c(c)}{d_c(c)} < K_f^* \qquad \text{or equivalently} \qquad \frac{n_c(a/b)}{d_c(a/b)} p_{\min} < K_f, \tag{42}$$

including the free (not quantified) variables $K_f^* := K_f/p_{\min}$ and $c = a/b$. In this QFF the numerator $n_c(c)$ and the denominator $d_c(c)$ in the fraction denote the two polynomials

$$n_c(c) = -3.906960c^{20} - 3.341522c^{18} - 0.983512c^{16} + 1.847320c^{14} + 0.240696c^{12} + 26.277337c^{10} + 8.138855c^8 + 2.513184c^6 + 0.802112c^4 + 0.988136 \qquad \text{with} \qquad c = a/b, \tag{43}$$

$$d_c(c) = +0.756270c^{30} + 1.042278c^{28} + 0.337607c^{26} - 0.925124c^{24} + 9.818766c^{22} - 50.420814c^{20} - 27.735330c^{18} - 5.772990c^{16} + 4.487693c^{14} + 23.557404c^{12} - 30.222222c^{10} - 10.035338c^{8} - 3.769776c^{6} - 1.604224c^{4} - 1.625419c^{2} + 0.988136 \qquad \text{with} \qquad c = a/b. \quad (44)$$

The above approximate QFF (42) seems to be somewhat complicated. This happens because of the appearance of the fraction $n_c(c)/d_c(c) = n_c(a/b)/d_c(a/b)$ on its left-hand side. A simpler but even more approximate QFF can be derived by approximating this fraction. For example, by proceeding to a Taylor-Maclaurin series approximation to this fraction with terms of order up to $2r = 20$ by using *Mathematica* we easily obtain the approximate QFF

$$(60019.96486c^{20} + 19070.11945c^{18} + 6012.43216c^{16} + 1900.75837c^{14} + 613.57547c^{12} + 223.50332c^{10} + 61.77691c^{8} + 17.48558c^{6} + 5.14104c^{4} + 1.64493c^{2} + 1.00000)\, p_{\min} < K_f \qquad \text{with} \qquad c = a/b. \quad (45)$$

The above approximate QFF (45) constitutes a sufficiently good additional approximation to the original approximate QFF (42), especially for not very large values of the parameter $c = a/b$ and always under the constraint $0 \le c < 1/2$.

The above QFFs correspond to an order $r = 10$ in the generalized Taylor-Maclaurin series expansion (36) of the kernel $K_s(t,x)$. We have also derived the approximate QFFs corresponding to orders $r = 5$, $r = 20$ and $r = 30$, which will not be displayed here either in their original forms or after their further approximations, by using again Taylor-Maclaurin series expansions to the fraction $n_c(c)/d_c(c)$ on their left-hand sides. Naturally, it is clear that for increasing values of the order r of the approximation the related approximate QFFs become more complicated and more difficult for *Mathematica* to derive, but also simultaneously more accurate.

Now we denote by $f_{r,c}$ the fraction

$$f_{r,c} := \frac{n_c(c)}{d_c(c)} = \frac{n_c(a/b)}{d_c(a/b)} \quad (46)$$

in the approximate QFF (42) corresponding to a value r of the order of approximation in the generalized Taylor-Maclaurin series expansion (36) of the kernel $K_s(t,x)$. Below we just present the values of $f_{r,c}$ for $r = 5, 10, 20$ and 30 and $c = a/b = 0.10, 0.20, 0.30, 0.40$ and 0.45 having taken into account for geometric reasons that $0 \le c < 1/2$ because $2a < b$ or $c = a/b < 1/2$ as it is clear from Fig. 1, so that we can have separate cracks:

$$f_{5,\,0.10} = 1.01698, \quad f_{5,\,0.20} = 1.07526, \quad f_{5,\,0.30} = 1.20615, \quad f_{5,\,0.40} = 1.51624, \quad f_{5,\,0.45} = 1.85042 \quad \text{for} \quad r = 5, \quad (47)$$

$$f_{10,\,0.10} = 1.01698, \quad f_{10,\,0.20} = 1.07533, \quad f_{10,\,0.30} = 1.20826, \quad f_{10,\,0.40} = 1.55095, \quad f_{10,\,0.45} = 1.98470 \quad \text{for} \quad r = 10, \quad (48)$$

$$f_{20,\,0.10} = 1.01698, \quad f_{20,\,0.20} = 1.07533, \quad f_{20,\,0.30} = 1.20846, \quad f_{20,\,0.40} = 1.56427, \quad f_{20,\,0.45} = 2.08888 \quad \text{for} \quad r = 20, \quad (49)$$

$$f_{30,\,0.10} = 1.01698, \quad f_{30,\,0.20} = 1.07533, \quad f_{30,\,0.30} = 1.20847, \quad f_{30,\,0.40} = 1.56511, \quad f_{30,\,0.45} = 2.11153 \quad \text{for} \quad r = 30. \quad (50)$$

For $r = 10$ we display only the values $\widetilde{f}_{10,c} \approx f_{10,c}$ obtained on the basis of the second Taylor-Maclaurin series approximation (of order $2r = 20$) on the left-hand side of the approximate QFF (45) ($\widetilde{f}_{10,c}$ is defined as the multiplier of $p_{\min}$ there) for the same values of c. These values are

$$\begin{aligned} &\widetilde{f}_{10,\,0.10} = 1.01698, \qquad \widetilde{f}_{10,\,0.20} = 1.07533, \qquad \widetilde{f}_{10,\,0.30} = 1.20826, \\ &\widetilde{f}_{10,\,0.40} = 1.55029, \qquad \widetilde{f}_{10,\,0.45} = 1.97282 \qquad \text{for} \quad r = 10. \end{aligned} \tag{51}$$

By comparing these values to those displayed in Eqs. (48) we observe that the differences (with the present accuracy of five decimal digits) are restricted to the case of somewhat large values of the parameter c, here to $c = 0.40$ and 0.45 (with $0 \le c < 0.50$), and even in this case they are rather small. Therefore, in general the approximate QFF (45) can be used instead of the original but also approximate QFF (42).

4. Application of quantifier elimination to hypersingular integral equations under inequality constraints

In this section, we consider the hypersingular integral equation of the problem of a single straight crack in fracture mechanics again inside an infinite plane isotropic elastic medium. This equation has the form [23–26, 31, 32]

$$\frac{1}{\pi} \int_{-1}^{1} \sqrt{1-t^2}\, \frac{g(t)}{(t-x)^2}\, \mathrm{d}t = -p(x), \qquad -1 < x < 1. \tag{52}$$

In this equation, the product $\sqrt{1-t^2}\, g(t)$ of the weight function $\sqrt{1-t^2}$ and the unknown function $g(t)$ is proportional to the crack opening displacement $v(t)$. Moreover, the known function $p(x)$ on the right-hand side is again the normal loading distribution applied to the two crack edges. Because of the minus sign in $-p(x)$, this loading distribution is assumed to be positive when it is compressive and negative when it is tensile. We can add that the dimensionless stress intensity factors $K(\pm 1)$ at the crack tips $x = \pm 1$ are now determined by the very simple formulae [32]

$$K(\pm 1) = g(\pm 1). \tag{53}$$

Naturally, for the validity of the above hypersingular integral equation (52) the crack opening displacement $v(t)$, which is proportional to the product $\sqrt{1-t^2}\, g(t)$, should be a non-negative function along the whole crack $t \in [-1, 1]$. Therefore, the same should be also the case with the unknown function $g(t)$ itself and we must have

$$\forall t \in [-1, 1] \;\; g(t) \ge 0. \tag{54}$$

This is the inequality constraint (here a universally quantified formula because of the presence of the universal quantifier $\forall$, for all, in it) that accompanies the hypersingular integral equation (52). This inequality constraint should not be ignored for the validity of the approximate solution $g_n(t) \approx g(t)$ of this equation.

Two well-known methods for the approximate solution of the hypersingular integral equation (52) are (i) the collocation method and (ii) the Galerkin method [20, 21, 32, 63]. A third related method is the quadrature method, which is based on the simultaneous application of the Gauss-Chebyshev and the Lobatto-Chebyshev quadrature rules to Eq. (52) after appropriate selections of the collocation points during the application of both these quadrature rules. The quadrature method for the numerical solution of hypersingular integral equations constitutes a generalization of the related method for the numerical solution of Cauchy-type singular integrodifferential equations [42].

Here we will not use the quadrature method, which is somewhat complicated as far as hypersingular integral equations are concerned, and we will restrict our attention to the collocation and Galerkin method [20, 21, 32, 63].

4.1. Non-negativity of the crack opening displacement by the collocation method

First, we will use the collocation method. In this method the unknown function $g(t)$ in the hypersingular integral equation (52) is approximated by the sum [32]

$$g_n(t) = \sum_{i=0}^{n} a_i U_i(t), \tag{55}$$

where a_i are unknown coefficients to be determined and $U_i(t)$ denote the Chebyshev polynomials of the second kind of degree i. The $n+1$ unknown coefficients a_i are determined from the solution of the approximate system of $n+1$ linear algebraic equations [32]

$$\sum_{i=0}^{n}(i+1)U_i(x_{k,n})a_i = p(x_{k,n}), \qquad k = 0, 1, \dots, n. \tag{56}$$

In these equations, $x_{k,n}$ are the $n+1$ collocation points. Here these points are selected as the $n+1$ roots of the Chebyshev polynomial of the first kind $T_{n+1}(x)$ of degree $n+1$. Therefore [32]

$$x_{k,n} = \cos\frac{(2k+1)\pi}{2(n+1)}, \qquad k = 0, 1, \dots, n. \tag{57}$$

After the numerical solution of the system of Eqs. (56) and the determination of the approximation $g_n(t)$ in Eq. (55) to the unknown function $g(t)$ in Eq. (52) we are ready to perform quantifier elimination to the quantified formula (54) again by using the `Reduce` command of *Mathematica*. Here we selected the following exponential form of the normal loading distribution $p(x)$ of the crack:

$$p(x) = pe^{x} + qe^{-x} \tag{58}$$

with two parameters, p and q.

Here having worked with $n+1$ collocation points $x_{k,n}$ we display the values of the approximate stress intensity factors $K_n(\pm 1) \approx K(\pm 1)$ at the crack tips $x = \pm 1$ respectively directly obtained from Eqs. (53) as well as the related QFFs (quantifier-free formulae) QFF_n obtained by the application of quantifier elimination to the quantified formula (54), now for $g_n(t) \approx g(t)$. The related results for $K_n(\pm 1)$ and QFF_n with an accuracy of 12 decimal digits are displayed below:

$$K_0(+1) = p + q, \qquad K_0(-1) = p + q, \tag{59}$$

$$\mathrm{QFF}_0 = p + q \geq 0, \tag{60}$$

$$\begin{aligned} K_1(+1) &= 1.80331\,26571\,58p + 0.71787\,10158\,85q, \\ K_1(-1) &= 0.71787\,10158\,85p + 1.80331\,26571\,58q, \end{aligned} \tag{61}$$

$$\mathrm{QFF}_1 = (q \leq 0 \wedge p \geq -2.51202\,87868\,63q) \vee (q > 0 \wedge p \geq -0.39808\,46100\,29q), \tag{62}$$

$$\begin{aligned} K_2(+1) &= 1.83090\,69420\,61p + 0.70113\,48587\,99q, \\ K_2(-1) &= 0.70113\,48587\,99p + 1.83090\,69420\,61q, \end{aligned} \tag{63}$$

$$\mathrm{QFF}_2 = (q \leq 0 \wedge p \geq -2.61134\,77586\,84q) \vee (q > 0 \wedge p \geq -0.38294\,40168\,11q), \tag{64}$$

$$\begin{aligned} K_3(+1) &= 1.83122\,31777\,95p + 0.70090\,81792\,84q, \\ K_3(-1) &= 0.70090\,81792\,84p + 1.83122\,31777\,95q, \end{aligned} \tag{65}$$

$$\mathrm{QFF}_3 = (q \leq 0 \wedge p \geq -2.61264\,34701\,72q) \vee (q > 0 \wedge p \geq -0.38275\,40999\,82q), \tag{66}$$

$$\begin{aligned} K_4(+1) &= 1.83122\,49756\,63p + 0.70090\,67787\,40q, \\ K_4(-1) &= 0.70090\,67787\,40p + 1.83122\,49756\,63q, \end{aligned} \tag{67}$$

$$\mathrm{QFF}_4 = (q \le 0 \wedge p \ge -2.61265\ 12557\ 85q) \vee (q > 0 \wedge p \ge -0.38275\ 29593\ 88q), \tag{68}$$

$$\begin{aligned} K_5(+1) &= 1.83122\ 49817\ 31p + 0.70090\ 67737\ 71q, \\ K_5(-1) &= 0.70090\ 67737\ 71p + 1.83122\ 49817\ 31q, \end{aligned} \tag{69}$$

$$\mathrm{QFF}_5 = (q \le 0 \wedge p \ge -2.61265\ 12829\ 64q) \vee (q > 0 \wedge p \ge -0.38275\ 29554\ 06q), \tag{70}$$

$$\begin{aligned} K_n(+1) &= 1.83122\ 49817\ 44p + 0.70090\ 67737\ 60q, \\ K_n(-1) &= 0.70090\ 67737\ 60p + 1.83122\ 49817\ 44q, \end{aligned} \tag{71}$$

$$\mathrm{QFF}_n = (q \le 0 \wedge p \ge -2.61265\ 12830\ 26q) \vee (q > 0 \wedge p \ge -0.38275\ 29553\ 97q) \tag{72}$$

with $n = 6, 7, \dots, 12$ in Eqs. (71) and (72).

At this point, we can also mention that the theoretical values of the dimensionless stress intensity factors $K(\pm 1)$ in the present crack problem are determined as follows:

$$\begin{aligned} K(+1) &= \frac{1}{\pi} \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\,(pe^{x} + qe^{-x})\,\mathrm{d}x = [I_0(1) + I_1(1)]p + [I_0(1) - I_1(1)]q \\ &\approx 1.83122\ 49817\ 44p + 0.70090\ 67737\ 60q, \end{aligned} \tag{73}$$

$$\begin{aligned} K(-1) &= \frac{1}{\pi} \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}}\,(pe^{x} + qe^{-x})\,\mathrm{d}x = [I_0(1) - I_1(1)]p + [I_0(1) + I_1(1)]q \\ &\approx 0.70090\ 67737\ 60p + 1.83122\ 49817\ 44q. \end{aligned} \tag{74}$$

In these equations, $I_0(x)$ and $I_1(x)$ denote the modified Bessel functions of the first kind and of orders 0 and 1 respectively. Moreover, the computation of the stress intensity factors $K(\pm 1)$ was made analogously to the computation of the same factors in [32], where the same Bessel functions $I_0(x)$ and $I_1(x)$ appear in the formulae for $K(\pm 1)$.

Naturally, the previous numerical results for the approximate stress intensity factors $K_n(\pm 1)$ (here in a dimensionless form) obtained by the collocation method converge to the above theoretical values (73) and (74) respectively. Moreover, the same results in the special, non-parametric case $p = 1$ and $q = 0$ in Eq. (58) for the normal loading distribution $p(x)$, i.e., simply with $p(x) = e^{x}$, are in complete agreement with the corresponding results obtained by Ioakimidis [32].

4.2. Non-negativity of the crack opening displacement by the Galerkin method

Instead of the collocation method we can use the Galerkin method for the numerical solution of hypersingular integral equations [32]. In this method the unknown function $g(t)$ in Eq. (52) is again approximated by the sum $g_n(t)$ in Eq. (55). In the present simple crack problem, the $n+1$ coefficients a_i in this sum are determined from the very simple system of $n+1$ linear algebraic equations [32, p. 174, Eqs. (38)]

$$(i+1)a_i = \frac{2}{\pi}\gamma_i, \qquad i = 0, 1, \dots, n \tag{75}$$

with the quantities γ_i determined by [32, p. 172, Eqs. (20)]

$$\gamma_i = \int_{-1}^{1} \sqrt{1-x^2}\, p(x) U_i(x)\,\mathrm{d}x, \qquad i = 0, 1, \dots, n. \tag{76}$$

Evidently, in many and more difficult crack problems the resulting system of linear algebraic equations becomes more complicated [32, p. 172, Eqs. (18)] compared with its present extremely simple form (75). Here the normal loading distribution $p(x)$ is again assumed to be of the form (58) with two parameters, p and q.

Exactly as previously in the collocation method, here using the Galerkin method after the solution of the system of Eqs. (75) and the determination of the approximation $g_n(t)$ in Eq. (55) to the unknown function $g(t)$ we can perform quantifier elimination to the quantified formula (54) again by using the `Reduce` command of *Mathematica*. The obtained results are similar to those derived by the collocation method in the previous subsection and are displayed below:

$$K_0(+1) = 1.13031\,82079\,85(p+q), \qquad K_0(-1) = 1.13031\,82079\,85(p+q), \tag{77}$$

$$\mathrm{QFF}_0 = p \geq -1.00000\,00000\,00q, \tag{78}$$

$$\begin{aligned} K_1(+1) &= 1.67330\,88870\,53p + 0.58732\,75289\,17q,\\ K_1(-1) &= 0.58732\,75289\,17p + 1.67330\,88870\,53q, \end{aligned} \tag{79}$$

$$\mathrm{QFF}_1 = (q \leq 0 \wedge p \geq -2.84902\,17207\,07q) \vee (q > 0 \wedge p \geq -0.35099\,76750\,03q), \tag{80}$$

$$\begin{aligned} K_2(+1) &= 1.80631\,94365\,99p + 0.72033\,80784\,63q,\\ K_2(-1) &= 0.72033\,80784\,63p + 1.80631\,94365\,99q, \end{aligned} \tag{81}$$

$$\mathrm{QFF}_2 = (q \leq 0 \wedge p \geq -2.50759\,95433\,34q) \vee (q > 0 \wedge p \geq -0.39878\,77580\,61q), \tag{82}$$

$$\begin{aligned} K_3(+1) &= 1.82821\,63983\,67p + 0.69844\,11166\,94q,\\ K_3(-1) &= 0.69844\,11166\,94p + 1.82821\,63983\,67q, \end{aligned} \tag{83}$$

$$\mathrm{QFF}_3 = (q \leq 0 \wedge p \geq -2.61756\,69711\,72q) \vee (q > 0 \wedge p \geq -0.38203\,41603\,53q), \tag{84}$$

$$\begin{aligned} K_4(+1) &= 1.83093\,10299\,27p + 0.70115\,57482\,54q,\\ K_4(-1) &= 0.70115\,57482\,54p + 1.83093\,10299\,27q, \end{aligned} \tag{85}$$

$$\mathrm{QFF}_4 = (q \leq 0 \wedge p \geq -2.61130\,43136\,08q) \vee (q > 0 \wedge p \geq -0.38295\,03879\,68q), \tag{86}$$

$$\begin{aligned} K_5(+1) &= 1.83120\,08938\,65p + 0.70088\,58843\,16q,\\ K_5(-1) &= 0.70088\,58843\,16p + 1.83120\,08938\,65q, \end{aligned} \tag{87}$$

$$\mathrm{QFF}_5 = (q \leq 0 \wedge p \geq -2.61269\,47836\,18q) \vee (q > 0 \wedge p \geq -0.38274\,65826\,74q), \tag{88}$$

$$\begin{aligned} K_6(+1) &= 1.83122\,32829\,20p + 0.70090\,82733\,72q,\\ K_6(-1) &= 0.70090\,82733\,72p + 1.83122\,32829\,20q, \end{aligned} \tag{89}$$

$$\mathrm{QFF}_6 = (q \leq 0 \wedge p \geq -2.61264\,32694\,42q) \vee (q > 0 \wedge p \geq -0.38275\,41293\,89q), \tag{90}$$

$$\begin{aligned} K_7(+1) &= 1.83122\,48766\,20p + 0.70090\,66796\,72q,\\ K_7(-1) &= 0.70090\,66796\,72p + 1.83122\,48766\,20q, \end{aligned} \tag{91}$$

$$\mathrm{QFF}_7 = (q \leq 0 \wedge p \geq -2.61265\,14837\,58q) \vee (q > 0 \wedge p \geq -0.38275\,29259\,90q), \tag{92}$$

$$\begin{aligned} K_8(+1) &= 1.83122\,49759\,51p + 0.70090\,67790\,03q,\\ K_8(-1) &= 0.70090\,67790\,03p + 1.83122\,49759\,51q, \end{aligned} \tag{93}$$

$$\mathrm{QFF}_8 = (q \leq 0 \wedge p \geq -2.61265\,12552\,16q) \vee (q > 0 \wedge p \geq -0.38275\,29594\,71q), \tag{94}$$

$$\begin{aligned} K_9(+1) &= 1.83122\,49814\,57p + 0.70090\,67734\,97q,\\ K_9(-1) &= 0.70090\,67734\,97p + 1.83122\,49814\,57q, \end{aligned} \tag{95}$$

$$\mathrm{QFF}_9 = (q \leq 0 \wedge p \geq -2.61265\,12835\,95q) \vee (q > 0 \wedge p \geq -0.38275\,29553\,14q), \tag{96}$$

$$\begin{aligned} K_{10}(+1) &= 1.83122\,49817\,31p + 0.70090\,67737\,71q,\\ K_{10}(-1) &= 0.70090\,67737\,71p + 1.83122\,49817\,31q, \end{aligned} \tag{97}$$

$$\mathrm{QFF}_{10} = (q \leq 0 \wedge p \geq -2.61265\,12829\,63q) \vee (q > 0 \wedge p \geq -0.38275\,29554\,06q). \tag{98}$$

Naturally, the previous numerical results $K_n(\pm 1)$ for the approximate dimensionless stress intensity factors $K(\pm 1)$, now obtained by using the Galerkin method, also converge to the corresponding theoretical values (73) and (74) respectively although a little more slowly than the related results obtained by the collocation method in the previous subsection. Moreover, the same results in the special, non-parametric case $p = 1$ and $q = 0$ in Eq. (58) for the loading distribution $p(x)$, i.e., simply with $p(x) = e^x$, are again in complete agreement with the corresponding results obtained by Ioakimidis [32].

4.3. Non-negativity of the crack opening displacement and fracture-related inequality constraints

As an extension of the previous results, we consider the interesting and physically reasonable case where we wish in the present crack problem: (i) the non-negativity of the crack opening displacement $v(t)$, in our case with $g(t) \geq 0$ for $t \in [-1,1]$, and, simultaneously, (ii) the values of the stress intensity factors $K(\pm 1)$ at the crack tips $x = \pm 1$ not to reach the positive critical value K_f (the fracture toughness of the elastic material) so that fracture can be avoided. Therefore, because of Eqs. (53) now we have to satisfy the following three inequality constraints:

$$\forall t \in [-1,1] \;\; g(t) \geq 0 \;\wedge\; g(1) < K_f \;\wedge\; g(-1) < K_f, \tag{99}$$

instead of the single inequality constraint (54) previously.

In this problem, by using again the Galerkin method [32], the related approximation $g_n(t)$ in Eq. (55) to the unknown function $g(t)$ of the hypersingular integral equation (52) as well as using again the `Reduce` command of *Mathematica* for quantifier elimination, we obtain the following approximate QFFs (quantifier-free formulae) QFF_n for the related necessary and sufficient conditions:

$$\mathrm{QFF}_0 = p \geq -q \wedge K_f > 1.13031\,82079\,85p + 1.13031\,82079\,85q, \tag{100}$$

$$\begin{aligned}\mathrm{QFF}_1 = \big\{&K_f > 1.67330\,88870\,53p + 0.58732\,75289\,17q \\ &\wedge [(p + 2.84902\,17207\,07q \geq 0 \wedge q \leq 0) \vee p > q > 0]\big\} \\ &\vee (K_f > 0.58732\,75289\,17p + 1.67330\,88870\,53q \\ &\wedge p + 0.35099\,76750\,03q \geq 0 \wedge p \leq q \wedge q > 0),\end{aligned} \tag{101}$$

$$\begin{aligned}\mathrm{QFF}_2 = \big\{&K_f > 1.80631\,94365\,99p + 0.72033\,80784\,63q \\ &\wedge [(p + 2.50759\,95433\,34q \geq 0 \wedge q \leq 0) \vee p > q > 0]\big\} \\ &\vee (K_f > 0.72033\,80784\,63p + 1.80631\,94365\,99q \\ &\wedge p + 0.39878\,77580\,61q \geq 0 \wedge p \leq q \wedge q > 0),\end{aligned} \tag{102}$$

$$\begin{aligned}\mathrm{QFF}_3 = \big\{&K_f > 1.82821\,63983\,67p + 0.69844\,11166\,94q \\ &\wedge [(p + 2.61756\,69711\,72q \geq 0 \wedge q \leq 0) \vee p > q > 0]\big\} \\ &\vee (K_f > 0.69844\,11166\,94p + 1.82821\,63983\,67q \\ &\wedge p + 0.38203\,41603\,53q \geq 0 \wedge p \leq q \wedge q > 0),\end{aligned} \tag{103}$$

$$\begin{aligned}\mathrm{QFF}_4 = \big\{&K_f > 1.83093\,10299\,27p + 0.70115\,57482\,54q \\ &\wedge [(p + 2.61130\,43136\,08q \geq 0 \wedge q \leq 0) \vee p > q > 0]\big\} \\ &\vee (K_f > 0.70115\,57482\,54p + 1.83093\,10299\,27q \\ &\wedge p + 0.38295\,03879\,68q \geq 0 \wedge p \leq q \wedge q > 0),\end{aligned} \tag{104}$$

$$\mathrm{QFF}_5 = \{K_f > 1.83120\,08938\,65p + 0.70088\,58843\,16q \\ \wedge [(p + 2.61269\,47836\,18q \geq 0 \wedge q \leq 0) \vee p > q > 0]\} \\ \vee (K_f > 0.70088\,58843\,16p + 1.83120\,08938\,65q \\ \wedge p + 0.38274\,65826\,74q \geq 0 \wedge p \leq q \wedge q > 0), \quad (105)$$

$$\mathrm{QFF}_6 = \{K_f > 1.83122\,32829\,20p + 0.70090\,82733\,72q \\ \wedge [(p + 2.61264\,32694\,42q \geq 0 \wedge q \leq 0) \vee p > q > 0]\} \\ \vee (K_f > 0.70090\,82733\,72p + 1.83122\,32829\,20q \\ \wedge p + 0.38275\,41293\,89q \geq 0 \wedge p \leq q \wedge q > 0), \quad (106)$$

$$\mathrm{QFF}_7 = \{K_f > 1.83122\,48766\,20p + 0.70090\,66796\,72q \\ \wedge [(p + 2.61265\,14837\,58q \geq 0 \wedge q \leq 0) \vee p > q > 0]\} \\ \vee (K_f > 0.70090\,66796\,72p + 1.83122\,48766\,20q \\ \wedge p + 0.38275\,29259\,90q \geq 0 \wedge p \leq q \wedge q > 0), \quad (107)$$

$$\mathrm{QFF}_8 = \{K_f > 1.83122\,49759\,51p + 0.70090\,67790\,03q \\ \wedge [(p + 2.61265\,12552\,16q \geq 0 \wedge q \leq 0) \vee p > q > 0]\} \\ \vee (K_f > 0.70090\,67790\,03p + 1.83122\,49759\,51q \\ \wedge p + 0.38275\,29594\,71q \geq 0 \wedge p \leq q \wedge q > 0), \quad (108)$$

$$\mathrm{QFF}_9 = \{K_f > 1.83122\,49814\,57p + 0.70090\,67734\,97q \\ \wedge [(p + 2.61265\,12835\,95q \geq 0 \wedge q \leq 0) \vee p > q > 0]\} \\ \vee (K_f > 0.70090\,67734\,97p + 1.83122\,49814\,57q \\ \wedge p + 0.38275\,29553\,14q \geq 0 \wedge p \leq q \wedge q > 0), \quad (109)$$

$$\mathrm{QFF}_{10} = \{K_f > 1.83122\,49817\,31p + 0.70090\,67737\,71q \\ \wedge [(p + 2.61265\,12829\,63q \geq 0 \wedge q \leq 0) \vee p > q > 0]\} \\ \vee (K_f > 0.70090\,67737\,71p + 1.83122\,49817\,31q \\ \wedge p + 0.38275\,29554\,06q \geq 0 \wedge p \leq q \wedge q > 0). \quad (110)$$

First, from the above approximate QFFs we observe their sufficiently rapid convergence for increasing values of n. Moreover, we observe that apart from the obvious non-negativity of the crack opening displacement $v(t)$, which is proportional to the product $\sqrt{1-t^2}\,g(t)$ with $g(t)$ being the present unknown function in Eq. (52), the above QFFs take automatically into account which of the two approximate stress intensity factors $K_n(\pm 1)$ is greater than the other, this of course depending on the values of the two parameters p and q.

Therefore only one of these approximate stress intensity factors $K_n(\pm 1)$ appears in each disjunctive term of the QFF (with two disjunctive terms for $n \geq 1$). More explicitly, in the first disjunctive term, i.e., the term that includes the constraints $p > q > 0$ (although it is not restricted to this case), we observe the appearance of $K_n(+1)$. By contrast, in the second disjunctive term, i.e., the term including the constraints $p \leq q$ and simultaneously $q > 0$, we observe the appearance of $K_n(-1)$. These remarks are clear from the formulae for the approximate stress intensity factors $K_n(\pm 1)$ already displayed in the previous subsection and obtained again by using the Galerkin method [32].

4.4. Quadratic loading of the crack

As a further application of the Galerkin method, we consider the problem of the same single straight crack, but now under the parametric quadratic normal loading distribution

$$p(x) = px^2 + qx + r \tag{111}$$

with three parameters: p, q and r. In this application, we first use the already briefly described Galerkin method [32] and next quantifier elimination to the quantified formula

$$\forall t \in [-1,1]\ \ g(t) > 0\ \wedge\ g(1) < K_f\ \wedge\ g(-1) < K_f, \tag{112}$$

where we preferred to use the symbol $>$ instead of the symbol $\geq$ that has been used previously in the quantified formula (99). The related basic *Mathematica* command is displayed in Appendix, paragraph A2, where again the approximation $g_n(t) \approx g(t)$ has been used.

The obtained results for the unknown function $g_n(t) \approx g(t)$ and the related QFF (quantifier-free formula) QFF_n are

$$g_0(t) = \frac{1}{4}(p + 4r), \tag{113}$$

$$\mathrm{QFF}_0 = 4K_f > p + 4r, \tag{114}$$

$$g_1(t) = \frac{1}{4}(2qt + p + 4r), \tag{115}$$

$$\begin{aligned}\mathrm{QFF}_1 = p > -4r \wedge [(q \leq 0 \wedge p + 2q + 4r > 0 \wedge 4K_f + 2q > p + 4r)\\ \vee (q > 0 \wedge p + 4r > 2q \wedge 4K_f > p + 2q + 4r)],\end{aligned} \tag{116}$$

$$g_n(t) = \frac{1}{6}(2pt^2 + 3qt + p + 6r), \qquad n = 2,3,4,5,6, \tag{117}$$

$$\begin{aligned}\mathrm{QFF}_n = \{q \leq 0 \wedge 2K_f + q > p + 2r \wedge [(p + q + 2r > 0 \wedge 4p + 3q \leq 0)\\ \vee (4p + 3q > 0 \wedge p + 6r > 9q^2/(8p))]\} \vee \{q > 0 \wedge 2K_f > p + q + 2r\\ \wedge [(p + 2r > q \wedge 4p < 3q) \vee (p + 6r > 9q^2/(8p) \wedge 4p \geq 3q)]\}, \qquad n = 2,3,4,5,6.\end{aligned} \tag{118}$$

We observe that *Mathematica* has been completely successful in quantifier elimination even in this case with the presence of four parameters (free variables in the quantified formula). These parameters are the three loading parameters p, q and r in Eq. (111) for the loading distribution $p(x)$ on the crack and the upper bound K_f (the fracture toughness of the cracked isotropic elastic material) for the stress intensity factors $K(\pm 1)$. Moreover, since the present loading distribution $p(x)$ in Eq. (111) is a quadratic polynomial, $g_n(t)$ and QFF_n for $n \geq 2$ are the exact unknown function $g(t)$ in Eq. (117) and the related exact QFF (118) respectively for the present crack problem.

4.5. Cubic loading of the crack

In quite a similar manner, we proceed (again by using the Galerkin method) to study the slightly more difficult cubic polynomial

$$p(x) = x^3 + px^2 + qx + r \tag{119}$$

as the normal loading distribution of the crack $[-1,1]$. We also use the three inequality constraints (112) related both to the positivity of the crack opening displacement $v(t)$ along the whole crack $[-1,1]$ and to the fracture toughness K_f of the isotropic elastic material.

With the above cubic normal loading distribution $p(x)$ the Galerkin method derives the exact results both for the unknown function $g_n(t)$ and for the QFF (quantifier-free formula) QFF_n for $n \ge 3$. For the present unknown function $g(t)$, which is naturally a cubic polynomial, we found that

$$g(t) = g_n(t) = \frac{1}{24}\,[6t^3 + 8pt^2 + (12q+3)t + 4p + 24r], \qquad n = 3,4,5,6. \tag{120}$$

Additionally, for the related QFF we found that

$$\mathrm{QFF} = [A_1 \wedge (A_{21} \vee A_{22})] \vee [B_1 \wedge (B_{21} \vee B_{22})] \vee C \vee D. \tag{121}$$

This QFF is displayed here in a more convenient yet completely equivalent and verified with *Mathematica* form with the help of the eight algebraic-logical quantities A_1, A_{21}, A_{22}, B_1, B_{21}, B_{22}, C and D. These quantities are given by

$$A_1 = 2K_f > p + q + 2r + \frac{3}{4}, \tag{122}$$

$$A_{21} = p + 2r > q + \frac{3}{4} \wedge \Big[(4p+3<0 \wedge 4q+3>0) \vee (4p>9 \wedge 16p \le 12q+21) \vee \Big(4p \le 9 \wedge 4p+3>0 \wedge p(2p+3) \le 9q + \frac{45}{8}\Big)\Big], \tag{123}$$

$$A_{22} = 4q+3>0 \wedge \sqrt{2}(32p^2 - 108q - 27)^{3/2} + 1296pq < 8(32p^3 + 81p + 729r) \wedge \Big[(4p>9 \wedge 16p > 12q+21) \vee \Big(4p \le 9 \wedge 4p+3>0 \wedge p(2p+3) > 9q + \frac{45}{8}\Big)\Big], \tag{124}$$

$$B_1 = 2K_f + q + \frac{3}{4} > p + 2r, \tag{125}$$

$$B_{21} = p + q + 2r + \frac{3}{4} > 0 \wedge \Big[(4p+3>0 \wedge 16p+12q+21 \le 0) \vee (4p+27<0 \wedge 4q+3 \le 0) \vee \Big(-\frac{27}{4} \le p < -\frac{3}{4} \wedge 4q+3<0\Big)\Big], \tag{126}$$

$$B_{22} = 4p+3>0 \wedge 4q+3 \le 0 \wedge 16p+12q+21>0 \wedge \sqrt{2}(32p^2 - 108q - 27)^{3/2} + 1296pq < 8(32p^3 + 81p + 729r), \tag{127}$$

$$C = 4p+3=0 \wedge \big[(4q+3>0 \wedge 2K_f > q+2r \wedge 2q+3<4r) \vee (4q+3 \le 0 \wedge 4K_f + 2q + 3 > 4r \wedge q+2r>0)\big], \tag{128}$$

$$D = -\frac{27}{4} \le p < -\frac{3}{4} \wedge 4q+3=0 \wedge 2K_f > p+2r \wedge p+2r>0. \tag{129}$$

We observe that *Mathematica* has been again completely successful in quantifier elimination also in the present rather difficult crack problem again with four parameters (free variables in the quantified formula): p, q, r and K_f. But clearly, this will generally not be the case if we proceed to even more difficult problems with more than four parameters.

5. Application of quantifier elimination to hypersingular integral inequalities under inequality constraints

As a final possible use of the present approach, we consider the case of a hypersingular integral inequality again under parametric inequality constraints. More explicitly, we consider the hypersingular integral inequality

$$-\frac{1}{\pi}\int_{-1}^{1} \sqrt{1-t^2}\,\frac{g(t)}{(t-x)^2}\,\mathrm{d}t \geq p_b(x), \qquad -1 < x < 1. \tag{130}$$

This inequality is related to the hypersingular integral equation (52) of the previous section written here in an inequality form with the minus sign for convenience on its left-hand side and with $p_b(x)$ instead of $p(x)$ on its right-hand side. The above inequality (130) simply denotes that the normal loading $p(x)$ of the crack, which is determined by the left-hand side of this inequality, is greater than or equal to the loading $p_b(x)$ on its right-hand side along the whole crack $[-1, 1]$. Moreover, evidently the dimensionless stress intensity factors $K(\pm 1)$ at the crack tips $x = \pm 1$ are computed again by using Eqs. (53), i.e., $K(\pm 1) = g(\pm 1)$.

Here we will apply again the collocation method with the lower bound $p_b(x)$ of the normal loading $p(x)$ in the hypersingular integral inequality (130) given by the exponential function in Eq. (58), i.e.,

$$p_b(x) = pe^{x} + qe^{-x}, \tag{131}$$

but now under the assumption that $p,\ q > 0$. Of course, $p_b(x)$ does not denote the actual normal loading $p(x)$ of the crack, which because of Eq. (52) is given by

$$p(x) = -\frac{1}{\pi}\int_{-1}^{1} \sqrt{1-t^2}\,\frac{g(t)}{(t-x)^2}\,\mathrm{d}t, \qquad -1 < x < 1, \tag{132}$$

i.e., by the left-hand side of the singular integral inequality (130). Here $p_b(x)$ simply denotes the lower bound of $p(x)$ in the singular integral inequality (130), i.e.,

$$p(x) \geq p_b(x), \qquad -1 < x < 1. \tag{133}$$

In some sense, this inequality is an abbreviated form of the singular integral inequality (130). We can add that since we assumed that $p,\ q > 0$ in Eq. (131), the lower bound $p_b(x)$ of $p(x)$ in the above inequality (133) is continuously positive on the crack $[-1, 1]$. Therefore, because of the same inequality (133) the actual loading $p(x)$ of the crack will also be continuously positive, i.e., a continuously compressive normal loading.

Here we assume again that the dimensionless stress intensity factors $K(\pm 1)$ should not reach the fracture-related positive upper bound K_f ($K_f > 0$) so that fracture can be avoided. Therefore, because of Eq. (53) we must have

$$g(1) < K_f \qquad \text{and} \qquad g(-1) < K_f. \tag{134}$$

Under all the above conditions now we proceed to the application of the collocation method [32] already briefly described in Subsec. 4.1, which will permit us to derive the related QFF (quantifier-free formula). This QFF is again a logical-algebraic formula providing the necessary and sufficient conditions validating both the hypersingular integral inequality (130) and the two fracture-related inequality constraints (134) under the assumption (131) for the lower bound $p_b(x)$ of the actual loading distribution $p(x)$ of the crack in the inequality (133). Of course, it is understood that since we apply the collocation method, which is an approximate method, the resulting QFF will also be approximate.

Here the basic command in *Mathematica* for obtaining this QFF is again based on its `Reduce` command and is displayed in Appendix, paragraph A3, where we assume again the approximation $g_n(t)$ given by Eq. (55) to the unknown function $g(t)$ in the hypersingular integral inequality (130). Moreover, we use again the same collocation points $x_{k,n}$ in Eqs. (57). Therefore, the hypersingular integral inequality (130) is now approximated by the system of $n+1$ linear inequalities

$$\sum_{i=0}^{n}(i+1)U_i(x_{k,n})a_i \geq p_b(x_{k,n}), \qquad k = 0,1,\ldots,n. \tag{135}$$

Obviously, the system of these inequalities is completely analogous to the system of linear equations (56), but here with the inequality sign $\geq$ and with $p_b(x_{k,n})$ on the right-hand side. Here we also use the positivity assumptions

$$\mathcal{A} = p > 0 \wedge q > 0 \wedge K_f > 0. \tag{136}$$

After the above description of the present crack problem and the collocation method used for the derivation of the related approximate QFF, we display below the related results obtained by *Mathematica* for QFF_n by using $n+1$ collocation points $x_{k,n}$. For $n = 1,2,\ldots,5$ these results are

$$\begin{aligned}\mathrm{QFF}_1 = {} & (p \leq q \wedge K_f > 0.71787\,10158\,85p + 1.80331\,26571\,58q)\\ & \vee (p > q \wedge K_f > 1.80331\,26571\,58p + 0.71787\,10158\,85q),\end{aligned} \tag{137}$$

$$\begin{aligned}\mathrm{QFF}_2 = {} & (p \leq q \wedge K_f > 0.70113\,48587\,99p + 1.83090\,69420\,61q)\\ & \vee (p > q \wedge K_f > 1.83090\,69420\,61p + 0.70113\,48587\,99q),\end{aligned} \tag{138}$$

$$\begin{aligned}\mathrm{QFF}_3 = {} & (p \leq q \wedge K_f > 0.70090\,81792\,84p + 1.83122\,31777\,95q)\\ & \vee (p > q \wedge K_f > 1.83122\,31777\,95p + 0.70090\,81792\,84q),\end{aligned} \tag{139}$$

$$\begin{aligned}\mathrm{QFF}_4 = {} & (p \leq q \wedge K_f > 0.70090\,67787\,40p + 1.83122\,49756\,63q)\\ & \vee (p > q \wedge K_f > 1.83122\,49756\,63p + 0.70090\,67787\,40q),\end{aligned} \tag{140}$$

$$\begin{aligned}\mathrm{QFF}_5 = {} & (p \leq q \wedge K_f > 0.70090\,67737\,71p + 1.83122\,49817\,31q)\\ & \vee (p > q \wedge K_f > 1.83122\,49817\,31p + 0.70090\,67737\,71q).\end{aligned} \tag{141}$$

First from the above results we observe their rapid convergence for increasing values of n. Next, we observe that these results are in agreement with the results in Eqs. (61), (63), (65), (67) and (69) as far as the dimensionless stress intensity factors $K(\pm 1)$ are concerned. Finally, what is more important, in the above results a distinction of cases is made: (i) if $p < q$, then the lower bound $p_b(x) = pe^x + qe^{-x}$ in Eq. (131) is "dominated" by the parameter q and therefore it is the stress intensity factor $K(-1)$ that should not reach the fracture-related upper bound K_f, (ii) on the contrary, if $p > q$, then the lower bound $p_b(x)$ is "dominated" by the parameter p and therefore it is the stress intensity factor $K(+1)$ that should not reach the same upper bound K_f. And naturally, if $p = q$, then simply $K(-1) = K(+1) < K_f$.

6. Conclusions – discussion

From the above results it is concluded that the use of the computational method of quantifier elimination constitutes an interesting possibility when singular/hypersingular integral equations accompanied by inequality constraints (usually parametric inequality constraints) have to be solved by some numerical method such as the quadrature method, the collocation method and the Galerkin method. This approach permits to derive the necessary and sufficient conditions involving the parameters of the problem so that both the integral equation and the inequality constraint(s) are satisfied. Naturally, if no parameters are present, the result of quantifier elimination, the QFF (quantifier-free formula), is simply `True` or `False`.

The modern and powerful implementation of quantifier elimination in *Mathematica* [64] seems to offer a very good approach for quantifier elimination. Nonetheless, naturally improvements to this implementation in the future are very welcome and are expected to be made on the basis of new related results, see, e.g., the recent results by Strzeboński [56, 57]. Of course, although quantifier elimination is generally performed without approximations, in the presented results several approximations were made and this is completely natural since we based our results on numerical methods for the approximate solution of singular/hypersingular integral equations and the related appearance of floating-point numbers. Nevertheless, the obtained approximate results were seen to converge sufficiently rapidly.

Although the presented results concern crack problems in fracture mechanics, naturally the approach is completely general and can be applied to any problem in solid and fluid mechanics formulated through the use of singular/hypersingular integral equations and including inequality constraints. Here the constraints (i) of the non-negativity of the crack opening displacement $v(t)$ and (ii) of an upper bound K_f (the fracture toughness of the elastic material) for the stress intensity factors $K(\pm 1)$ at the crack tips were used since these constraints are extremely interesting in fracture mechanics. But evidently, many other inequality constraints also appear in a natural way in several solid and fluid mechanics problems.

Moreover, the present approach is not restricted only to singular/hypersingular integral equations, but, evidently, it is also applicable to any kind of integral equations such as weakly singular integral equations, ordinary (with a regular kernel) integral equations, etc. A typical example concerns contact problems between two elastic media (such as a beam and an elastic foundation), which should result in a compressive normal traction so that the contact can be assured.

As it was already seen, the application of quantifier elimination can be made either simultaneously with the numerical solution of the integral equation (here inside the `Reduce` command of *Mathematica*) or just after this numerical solution. On the other hand, it is clear that in many cases *Mathematica* is unable to perform quantifier elimination in a reasonable computer time. This is particularly the case when many variables are present (the quantified variables and the parameters) and generally the presence of few variables strongly facilitates the computations. On the other hand, the degrees of the variables are also extremely important, and linear equations and inequalities also strongly facilitate *Mathematica* in its computations. For example, in Sec. 2 it has been easily possible for $n = 8$ to use fifteen quantified variables $\mathcal{V}_n$ in Eq. (17) and additionally two free variables (the parameters $p_{\min}$ and K_f), and derive the related QFF_8 in Eqs. (25) and (33) for collinear and parallel cracks, respectively. These quantifier eliminations were easily possible since both the equations $\mathcal{E}_n$ and the inequality constraints $\mathcal{A}_n$ were linear (of the first degree) in all their variables.

Incidentally, it can be added that in some cases we can instruct *Mathematica* to use some appropriate method when performing quantifier elimination in a particular problem, which is appropriate for this problem. This can be very helpful from the computational point of view and it can easily be achieved through the `SetSystemOptions` command, which defines the related option. For example, the command

```
SetSystemOptions["InequalitySolvingOptions" -> "LinearQE" -> True]
```

instructs *Mathematica* to use an algorithm appropriate for linear quantifier elimination and this is the well-known Loos-Weispfenning linear quantifier elimination algorithm [48] instead of cylindrical algebraic decomposition, which is a general-purpose algorithm. Additionally, the related command

```
SystemOptions["InequalitySolvingOptions"]
```

displays all the 28 inequality solving options together with their default values. All the related details concerning the options for quantifier elimination can be found in the tutorial of *Mathematica* for real polynomial systems [65].

Moreover, naturally, it is completely evident that the appearance of several parameters (free variables) will generally lead to complicated or even extremely complicated QFFs independently of the efficiency of the quantifier elimination algorithms implemented in *Mathematica* or in another computer algebra system, and this unfortunate situation cannot be avoided. Therefore, the present computational method is generally useful when only one to four parameters are present. For example, the three QFFs (116), (118) and (121) in Sec. 4 include four parameters: the three coefficients p, q, r of the related polynomial, i.e., the quadratic polynomial (111) or the cubic polynomial (119), and the fracture toughness K_f of the elastic material.

Finally, perhaps an interesting possible use of the present computational approach based on quantifier elimination consists in its applicability to singular/hypersingular integral inequalities instead of singular/hypersingular integral equations. It seems that so far the numerical methods for the approximate solution of singular/hypersingular integral equations (such as the quadrature, collocation and Galerkin methods) were not extended to include the case of the related inequalities. Here, this has been achieved and it is illustrated in the applications presented in Sec. 3 and Sec. 5. What seems to be important is that this happened practically automatically, i.e., essentially without the necessity to modify the original numerical method (e.g., in Sec. 3 the Lobatto-Chebyshev method and in Sec. 5 the collocation method) so that it could become applicable to singular/hypersingular integral inequalities. Evidently, analogous is the case for any type of integral inequalities considered instead of integral equations.

APPENDIX: *MATHEMATICA* NOTEBOOK AND COMMANDS

A1. The complete *Mathematica* notebook (commands and approximate QFFs) for the crack problem of a periodic array of collinear cracks with $c = a/b = 0.45$ of Subsec. 2.1

```
t[i_,n_] = N[Cos[(i-1)Pi/(n-1)]];
A[i_,n_] = Pi/(n-1); A[1,n_] = A[n_,n_] = Pi/(2(n-1));
x[k_,n_] = N[Cos[(2 k-1)Pi/(2(n-1))]];
Ks[t_,x_] = Cot[Pi c(t-x)];
eqs[n_] := {Table[c Sum[A[i,n] Ks[t[i,n],x[k,n]] g[i,n], {i,1,n}] == p[k,n],
    {k,1,n-1}], Sum[A[i,n] g[i,n], {i,1,n}] == 0}/.List->And
var[n_] := {Table[p[k,n], {k,1,n-1}], Table[g[i,n], {i,1,n}]}//Flatten
ass0 = Kf > 0 && pmin > 0; ass[n_] := ass0 && Table[p[k,n] >= pmin,
    {k,1,n-1}]/.List->And//Flatten
QFF[n_,basicvar_] := Refine[Reduce[Exists[Evaluate[var[n]],
    ass[n], Evaluate[eqs[n] && g[1,n] < Kf && -g[n,n] < Kf]],
    basicvar, Reals], ass0]//Timing; Off[Reduce::ratnz];
c = 0.45; Table[{"n =", n, QFF[n,Kf], "or", QFF[n,pmin]}, {n,2,8}]

{{"n =", 2, {0.015, Kf > 4.46607 pmin}, "or", {0.,    pmin < 0.223911 Kf}},
 {"n =", 3, {0.016, Kf > 2.4518  pmin}, "or", {0.015, pmin < 0.407864 Kf}},
 {"n =", 4, {0.032, Kf > 2.19817 pmin}, "or", {0.031, pmin < 0.454923 Kf}},
 {"n =", 5, {0.062, Kf > 2.13585 pmin}, "or", {0.079, pmin < 0.468198 Kf}},
 {"n =", 6, {0.203, Kf > 2.11937 pmin}, "or", {0.203, pmin < 0.471839 Kf}},
 {"n =", 7, {0.703, Kf > 2.11494 pmin}, "or", {0.703, pmin < 0.472827 Kf}},
 {"n =", 8, {5.688, Kf > 2.11375 pmin}, "or", {5.734, pmin < 0.473093 Kf}}}
```

In the above *Mathematica* notebook, the actual quantifier elimination command is obviously the command defining the `QFF` function. In this command, besides the main command – the `Reduce` command, we also used some auxiliary commands: the `Evaluate` command for the evaluation of the list of our variables `var[n]` and our equations `eqs[n]`, the `Refine` command for the appearance of this QFF in its final form taking into account our assumptions `ass0` and, finally, the completely optional `Timing` command permitting us to be informed on the required CPU time for the execution of the above complete command. Moreover, naturally the symbol `Exists` denotes the existential quantifier ∃ (exists) and the option `Reals` instructs *Mathematica* to perform quantifier elimination exclusively inside the set of real numbers. Finally, the symbol `basicvar` instructs *Mathematica* to present the resulting QFF solved with respect to this variable, i.e., here either the fracture toughness K_f of the elastic material or alternatively the lower bound $p_{\min}$ of the normal compressive loading $p(x)$ of the cracks. Then, naturally we obtain completely equivalent QFFs but solved either with respect to K_f or with respect to $p_{\min}$ as it is also clear from the above derived QFFs.

A2. The basic *Mathematica* command that performed the quantifier elimination on the quantified formula (112) of Subsec. 4.4 with $g_n(t) \approx g(t)$ for the parametric quadratic loading (111) of a crack

```
Reduce[ForAll[t, -1<=t<=1, g[n,t] > 0 && g[n,1] < Kf && g[n,-1] < Kf,
    Reals]//Simplify
```

Here, in addition to the basic `Reduce` command the symbol `ForAll` denotes the universal quantifier ∀ (for all) and the `Simplify` command at the end is used for the simplification of the resulting QFF.

A3. The basic *Mathematica* command that performed the quantifier elimination in Sec. 5 for the hypersingular integral inequality (130) and the parametric exponential loading (131) of a crack under the inequality constraints (133) and (134) by using the collocation method

```
Refine[Reduce[Exists[Evaluate[var[n]], ass && Evaluate[ineqs[n]],
    Evaluate[g[n,1] < Kf && g[n,-1] < Kf]], Reals], ass]//Simplify//Timing
```

In this command, besides the aforementioned commands the symbol `ass` denotes the positivity assumptions $\mathcal{A}$ in Eq. (136) and the symbol `ineqs[n]` denotes the $n+1$ inequalities (135) resulting from the hypersingular integral inequality (130) through the application of the collocation method.

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