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Normal Contact Stiffness Based on Greenwood Unified Theory of Rough Surfaces and Accounting for Asperity Interaction

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We present the calculations of the normal contact stiffness based on the Greenwood unified theory of rough surfaces. In the model, the effect of asperity interaction is taken into account. The present results based on the Greenwood model were lower than those of both the experimental and fractal models presented in our earlier papers.

Keywords: unified theory of rough surfaces, normal contact stiffness.

1. Introduction

1.1. State of the art issues

The measurement and prediction of both normal and tangential stiffness have been the subject of the last few years of research by many authors [1–9]. Akarapu *et al.* [3] indicated that the contact area and normal stiffness increased linearly with the applied load. Medina *et al.* [4] proposed a simple analytical model based on the classical work of Greenwood and Williamson and predict that the tangential stiffness is proportional to the normal load and independent of the roughness radius and Young's modulus. Pohrt and Popov [5, 6] suggested a sublinear relationship between normal stiffness and nominal pressure. They found that the power relationship observed for slowly applied forces was valid for all applied forces, with an exponent ranging from 0.50 to 0.85, depending on the fractal dimension. The results presented by Pohrt and Popov [5, 6] were not confirmed by Pastewka *et al.* [7], who showed that contact stiffness cannot be described by the power law for all forces, and this would correspond to a straight line on a logarithmic scale only. Several authors note that the expression for the contribution of microasperity to the total elastic energy and elastic stiffness depends on the elastic coupling between the inequalities. Some of them state that any derivation that omits this interaction cannot describe the correct physics of realistic rough surfaces. Barber [10] has established an analogy between electrical conductivity and contact stiffness, in which the electrical conductivity at any load is proportional to the normal spring stiffness. This relationship has been extended to include finite body contact [11]. Measurements of tangential contact stiffness between rough surfaces made of titanium alloys using the digital image correlation method were recently investigated by Kartal *et al.* [12].

1.2. Greenwood unified theory of rough surfaces

The contact stiffness model presented here is based on a unified theory of rough surfaces proposed by Greenwood [13]. This description assumed that the surface height was a Gaussian random variable with a mean value of zero. Then any linear combination of height was also Gaussian. After Greenwood, we based it on four nearest neighbours, defining a summit as a point at which $z_0 > z_1$, z_2, z_3, and z_4. The distances between the five points are h, $2h$, and $h\sqrt{2}$. Next, we introduce new variables which are, as far as possible, uncorrelated. So we define the slopes (m_x, m_y) and curvatures (κ_x, κ_y) in two directions, x and y, approximated by finite difference as:

$$\begin{aligned} m_x &= (z_1 - z_2)/2h, & m_y &= (z_3 - z_4)/2h, \\ \kappa_x &= (z_1 - 2z_0 + z_2)/h^2, & \kappa_y &= (z_3 - 2z_0 + z_4)/h^2, \end{aligned} \tag{1}$$

and then the summit curvatures can be written as:

$$\kappa_s = -\frac{1}{2}(\kappa_x + \kappa_y), \qquad \kappa_d = -\frac{1}{2}(\kappa_x - \kappa_y), \tag{2}$$

where z_0, z_1, z_2, z_3, and z_4 defining a summit at a point at which $z_0 > z_1$, z_2, z_3, and z_4, and h is a sampling interval, κ_s and κ_d are the summit mean curvatures.

Assuming a Gaussian distribution of each of the variables, the joint probability distribution of a set of five random variables $y_1 = z_0$, $y_2 = m_x$, $y_3 = m_y$, $y_4 = \kappa_s$, $y_5 = \kappa_d$ is [13]

$$P(y_1, y_2, ..., y_5) = (1/(2\pi)^{\frac{1}{2}n}\Delta^{\frac{1}{2}})\exp\left(-\frac{1}{2}M_{ij}y_iy_j\right), \tag{3}$$

where M_{ij} is the inverse of the covariance matrix C_{ij}:

$$C_{ij} = \begin{bmatrix} E\{z_0^2\} & E\{z_0, m_x\} & E\{z_0, m_y\} & E\{z_0, \kappa_s\} & E\{z_0, \kappa_d\} \\ E\{m_x, z_0\} & E\{m_x^2\} & E\{m_x, m_y\} & E\{m_x, \kappa_x\} & E\{m_x, \kappa_y\} \\ E\{m_y, z_0\} & E\{m_y, m_x\} & E\{m_y^2\} & E\{m_y, \kappa_x\} & E\{m_y, \kappa_y\} \\ E\{\kappa_s, z_0\} & E\{\kappa_s, m_x\} & E\{\kappa_s, m_y\} & E\{\kappa_s^2\} & E\{\kappa_s, \kappa_d\} \\ E\{\kappa_d, z_0\} & E\{\kappa_d, m_x\} & E\{\kappa_d, m_y\} & E\{\kappa_d, \kappa_s\} & E\{\kappa_d^2\} \end{bmatrix}, \tag{4}$$

where the components of the matrix C_{ij} are the expectations of the y_i, y_j $(i, j = 1, 2, ..., 5)$, here $E(y_i, y_j)$ and Δ is the determinant of C_{ij}. After introducing standardized height and standard curvatures given by:

$$\zeta = z/\sigma, \qquad s = \kappa_s/\sigma_\kappa, \qquad t = -\kappa_d/\sigma_\kappa, \tag{5}$$

where σ defines the standard deviation of the rough surface (arithmetic roughness) and σ_κ is the standard deviation of the summits, so the probability that an ordinate is a peak of height ζ and curvature t is [13]

$$P(\zeta, s) = \frac{4}{(2\pi)^{\frac{3}{2}}\sqrt{(\cos^2\chi - r^2)}}\, F(\theta, \chi, s) \exp\left[-\frac{\zeta^2\cos^2\chi + s^2 - 2r\zeta s}{2(\cos^2\chi - r^2)}\right], \tag{6}$$

where

$$\begin{aligned} F(\theta, \chi, s) \;=\; & \frac{2}{\sin\chi}\int\limits_0^s \left[\varPhi(s-t)\tan\theta - \frac{1}{2}\right] \\ & \cdot \left[\varPhi(s+t)\tan\theta - \frac{1}{2}\right] \exp\left[-\frac{t^2}{2\sin^2\chi}\right] \mathrm{d}t, \end{aligned} \tag{7}$$

the parameter χ is defined by $\tau = \cos 2\chi$, where τ is the correlation coefficient between κ_x and κ_y, the parameter $\theta = \arcsin(h\sigma_\kappa/2\sigma_m)$ is a measure of the size of sampling interval and the variable r determines the character of the surface roughness. It is to be noted that the variable r^{-2} corresponds to Nayak's variable α.

Integrating (6) over the heights ζ and curvature s gives the density of summits D_s

$$D_s = \int\limits_{-\infty}^{+\infty}\int\limits_0^{+\infty} P(\zeta, s)\,\mathrm{d}\zeta\,\mathrm{d}s = \int\limits_0^{+\infty} \frac{2}{\pi\cos\chi} F(\theta, \chi, s) \exp\frac{-s^2}{2\cos^2\chi}\,\mathrm{d}s. \tag{8}$$

We remark, in the case of independent points ($\theta = \pi/3$ and $\tau = 2/3$), the probability that a point is a summit is then 0.2. The result was confirmed actually by numerical integration of (8). Dividing (6) by (8) we obtain the joint probability distribution of the heights and curvatures of the summits

$$P(\zeta, s) = \frac{\frac{4}{(2\pi)^{\frac{3}{2}}\sqrt{(\cos^2\chi - r^2)}}\, F(\theta, \chi, s)\, \exp\left[-\frac{\zeta^2\cos^2\chi + s^2 - 2r\zeta s}{2(\cos^2\chi - r^2)}\right]}{D_s}. \tag{9}$$

It is clear from (8) that the summit density D_s and the summit curvature distribution are independent of r. The summits of curvature s have a Gaussian height distribution with mean $2sr/(1+\tau)$ and variance $1 - 2r^2/(1+\tau)$.

2. Normal contanct

2.1. Elastic contact

The statistical model proposed by Greenwood [13] will be extended by relationship for the contact of two elastic single spheres. The behaviour of individual summit is described by Hertz theory, so the contact radius a_1, area A_1 and load F_1 related to one asperity can be expressed in terms of the compliance w (the distance which points outside the deforming zone move together during the deformation) as:

$$a_1 = \beta^{\frac{1}{2}} w^{\frac{1}{2}}, \qquad A_1 = \pi\beta w, \qquad F_1 = \frac{4}{3} E \beta^{\frac{1}{2}} w^{\frac{3}{2}}, \tag{10}$$

where

$$\frac{1}{E} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}, \tag{11}$$

where E and ν define the effective Young's modulus and Poisson's ratios, respectively. The indices 1 and 2 denote the contacting body 1 and 2.

If the surfaces come together until their reference planes are separated by the distance d, then all asperities are in contact if height z exceeds the separation d. Thus, the probability of making contact at any peak of height z with given curvature s is

$$\operatorname{prob}(z > d) = \int_d^\infty \int_0^\infty P(\zeta, s)\, \mathrm{d}s\, \mathrm{d}\zeta \tag{12}$$

and if the are D_s peaks in all taken from Eq. (8), the expected number of summits above a given height ζ can be calculated as

$$n(d) = D_s \int_d^\infty \int_0^\infty P(\zeta, s)\, \mathrm{d}s\, \mathrm{d}\zeta. \tag{13}$$

For $w = z - d$ and A_1 taken from Eq. (10) the mean contact area is

$$A = \pi D_s \sigma \int\limits_{\zeta}^{\infty} \int\limits_{0}^{\infty} \beta \left(\zeta - \frac{d}{\sigma} \right) P(\zeta, s) \, \mathrm{d}s \, \mathrm{d}\zeta. \tag{14}$$

Similarly, with the help of Eq. (10) we can find the expected load

$$F = \frac{4}{3} \pi E \, D_s \sigma^{\frac{3}{2}} \int\limits_{\zeta}^{\infty} \int\limits_{0}^{\infty} \beta^{\frac{1}{2}} \left(\zeta - \eta \right)^{\frac{3}{2}} P(\zeta, s) \, \mathrm{d}s \, \mathrm{d}\zeta, \tag{15}$$

where the normalized separation η is defined as follows

$$\eta = d/\sigma. \tag{16}$$

2.2. Normal stiffness

The coefficient of the normal stiffness for two contacting asperities (index i) can be obtained by differentiation of $(10)_3$ with respect to w, so

$$k_{ni} = \frac{\mathrm{d}F_i}{\mathrm{d}w} = 2E\beta^{\frac{1}{2}} w^{\frac{1}{2}}. \tag{17}$$

The normal stiffness for the joint is obtained by integration of Eq. (17) for all summits in contact, thus

$$K_n = 2\pi E \, D_s \sigma^{\frac{3}{2}} \int\limits_{\zeta}^{\infty} \int\limits_{0}^{\infty} \beta^{\frac{1}{2}} \left(\zeta - \eta \right)^{\frac{3}{2}} P(\zeta, s) \, \mathrm{d}s \, \mathrm{d}\zeta. \tag{18}$$

What is new in this formulation with respect to Greenwood's proposition? In the model, the effect of asperity interaction is taken into account. In Eq. (18), the correction is equivalent to an increase of the effective separation η by a quantity proportional to the nominal pressure, as has been suggested by Ciavarella *et al.* [14]

$$\eta_{\mathrm{new}} = \eta_{\mathrm{old}} - \frac{F(\eta_{\mathrm{old}})}{\sqrt{A_0} E \sigma}, \tag{19}$$

where A_0 is the nominal contact area, which is equal here to $A_0 = 195 \, \mathrm{mm}^2$ (this value was adopted in our experimental studies), $F(\eta_{\mathrm{old}})$ is the normal force related to the old separation. For details, we refer to the paper [14]. We note that another method of asperity interaction based on finite or discrete element models has been recently proposed by Mulvihill *et al.* [15], Yastrebov *et al.* [16], and Jerier and Molinari [17].

The values of the profilometric parameters of the tested surfaces are given in Table 1. The surfaces of the samples were subjected to three types of mechanical treatment: fine sandblasting (FSB), rough (coarse) sandblasting (CSB), and electrical discharge machining (EDM). The contact stiffness results calculated by formula (17) are shown in Fig. 1 to Fig. 3. We also present experimental results [18] and numerical results obtained from the fractal model [19]. In all cases, the presented results were lower than those of both models mentioned above. The fractal model better approximates the experimental results. The results obtained without taking into account the asperity interactions are several percent lower than those obtained with the interactions.

Table 1. Surface roughness values for different machining processes.

Surface roughness parameters	FSB	CSB	EDM
Arithmetic mean deviation, R_a [μm]	0.832	5.13	8.94
RMS deviation, $\sigma = R_q$ [μm]	1.08	6.67	11.62
Density of summits, D_p [1/μm^2]	500	230	160
Arithmetic mean peak radius, R [μm]	40	30	19

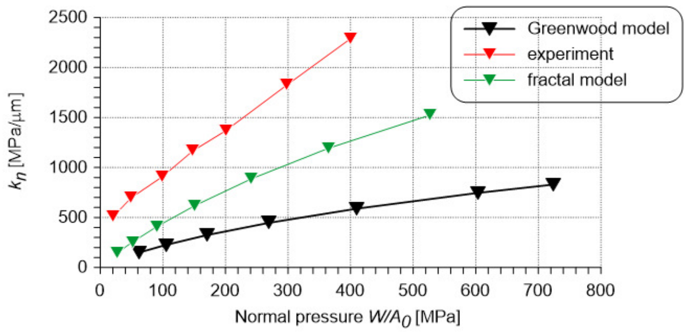

Fig. 1. Normal contact stiffness k_n versus normal pressure W/A_0 (sand blasted fine).

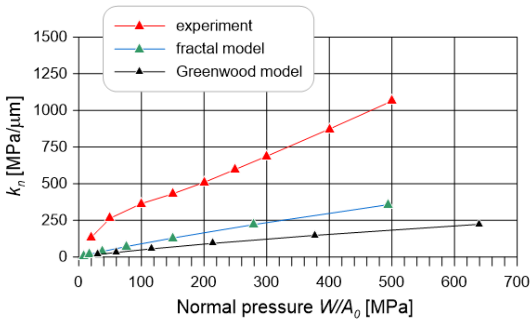

Fig. 2. Normal contact stiffness k_n versus normal pressure W/A_0 (coarse blasted fine).

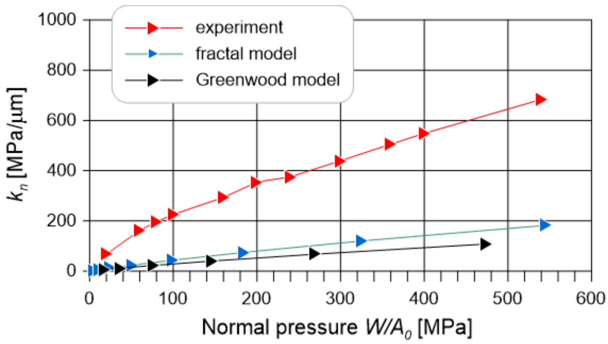

FIG. 3. Normal contact stiffness k_n versus normal pressure W/A_0 (EDM).

3. CONCLUSIONS

1. The calculations of the normal contact stiffness based on the unified theory of rough surfaces are given.
2. Normal stiffness depends directly on the value of the standard deviation of the surface σ.
3. The present results based on the Greenwood model were lower than those of both experimental and fractal models.

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