Computer Assisted Methods in Engineering and Science, **25**: 3–19, 2018, doi: 10.24423/cames.219

TWENTY-FIVE YEARS OF THE CAMES

Heuristics applying stochastic information as tools for thermoacoustic standing-wave engine optimization

Iwona Nowak
Institute of Mathematics
Silesian University of Technology
Kaszubska 23, 44-100 Gliwice, Poland
e-mail: iwona.nowak@polsl.pl

In this article, two numerical methods for solving engineering problems defined as multicriteria optimization and inverse problem are presented. In particular, this study deals with the optimization of the design of thermoacoustic engine in the frame in which both types of tasks are solved. The first proposed heuristic serves to find many p-optimal solutions simultaneously, which represents a compromise between usually mutually contradictory goals at work. Based on them, the full Pareto front is approximated. The inverse problem solution reproduces parameters for solutions located on a designated front but those that are not found in multicriteria optimization. In this article, the RACO heuristics are proposed for determining p-optimal solutions and the Bayesian approach is introduced as a method for solving ill-conditioned inverse problems. Optimization of the construction of the thermoacoustic engine is aimed at verifying proposed methodology and present the possibility of using both methods in engineering problems. The problem discussed in this article is formulated and the numerical methods used in the solution are presented in details.

Keywords: multicriteria optimization problem, ant colony optimization, inverse problem, Bayesian approach, thermoacoustic engine optimization, numerical modeling.

Nomenclature

p	–	pressure [N m^{-1}],
c_p	–	heat capacity [J kg^{-1}K^{-1}],
c	–	speed of sound [m s^{-1}],
u	–	velocity [m s^{-1}],
L	–	stack length [m],
Z	–	stack placement (along $0z$-axis), 0 at close end [m],
h	–	heat transfer coefficient [W m^{-2}K^{-1}],
H	–	cylindrical radius [m],
N	–	number of estimated parameters,
d_c	–	channel diameter [m],
T	–	temperature [K],
k_b	–	Boltzman constant,
k	–	thermal conductivity [W m^{-1}K^{-1}],
Q	–	heat flow [W],
W	–	acoustic work [W] (per channel).

Greek symbols:

ω	–	angular frequency [s^{-1}],
ρ	–	density [kg m^{-3}],
δ	–	penetration depth [m],
ε	–	stack heat capacity ratio,
ϵ	–	specific internal energy [J kg−1],
Γ	–	temperature gradient ratio,
γ	–	isentropic coefficient,
λ	–	wavelength [m],
μ	–	dynamic viscosity [kg $m^{-1}s^{-1}$],
∇T	–	temperature gradient [k m^{-1}].

Subscripts and superscripts:

∞	–	ambient,
c	–	cold side,
cond	–	conductive,
conv	–	conductive,
crit	–	critical,
h	–	hot side,
κ	–	thermal,
s	–	solid, surface,
rad	–	radiative,
rr, rz, zr, z	–	tensor directions,
v	–	viscous.

1. Introduction

In many engineering problems, the demand for optimization of parameters (decision variables) that in the best way simultaneously fulfills several objectives is increasing. In practice, this means finding a set of parameters by fixing a compromise between various objectives [5, 7, 25] for which the so-called multicriteria optimization should be used. The solution of such a problem, by its nature, is not unique. A solution of optimization means (infinitely) many, equivalent, nondominated solutions. Such a set of solutions is generally known as the Pareto-optimal or p-optimal set [15]. A pair of solutions is nondominated if one of them is better than the other with respect to at least one criterion and at the same time worse with respect to the other. These form a set whose individual solutions cannot be compared with each other. They are just possible variants of the solution of the optimization problem.

Having a set of p-optimal solutions and knowing that each of them should be treated as the solution of problem (optimal in Pareto sense), in fact, allows to have many variants available. For the final selection of solutions from the many proposed variants, an additional assessment is required by a human decision maker. This additional evaluation can be dictated by nonnumerical criterion (e.g., the ease of implementation or production and esthetics) and is particularly important at the stage of design of technology, devices, or any other type of problem. An ideal solution of a multicriteria problem is to designate the entire Pareto front, i.e., all the p-optimal variants of the solution from which the decision maker chooses the solution indicated by the additional assessment.

Multicriteria problems generally need to be solved using an appropriate numerical method or population heuristic. Depending on the approach adopted [5], a single solution on the Pareto front or, at best, a finite number of such solutions may be found. In order to simultaneously explore many

solutions lying on the Pareto front, it is appropriate to use population-based optimization methods [1, 3, 4, 10, 27, 28]. The appropriate modification of heuristics, such as evolutionary algorithm (EA) or other methods from the swarm intelligence group, allows to find many solutions simultaneously and collect them in a set S_{pOpt} (a finite set of p-optimal solutions).

Real ant colony optimization (RACO) as a multicriteria problem-solving tool was reviewed and modified in [13]. The evaluations were performed not only by solving benchmark tasks and comparing their results with known solutions, but also through the comparison of solutions obtained by using the scalar optimization objective function (aggregate function) and those obtained in the task of seeking full Pareto front. Regardless of whether the resulting optima are obtained as a single solution for aggregate function or as a result of work of population technique, many interesting Pareto points are missed. Similar observations were made by De Weck [23].

Since the p-optimal solutions found are usually "stacked" unequally at the front, the solutions from S_{pOpt} are those which may not be always interesting for a chosen assessment. Of course, in many cases, it is possible to approximate the course of the Pareto front based on the points from the set S_{pOpt}. Then, theoretically, there would be access to "intermediate" solutions on the front. It is important to remember that solutions, obtained in a numerical way, are pairs of p-optimal solutions (reconstructed parameters) together with corresponding values of criteria (points on the Pareto front) that are optimized. Any other point on the front represents the p-optimal solution, but without the information about the value of the parameters for which the solution would be achieved. In this case, for points outside of the set S_{pOpt}, it would be necessary to reconstruct the parameters for which the values of the criteria represented by the indicated point on the Pareto front are achieved. The problem of reproducing parameters based on the value of the criteria (determining the causes on the basis of the effects) is an inverse problem, which must be formulated and solved separately.

This two-step procedure allows to find any p-optimal solution (the values of the estimated parameters and the corresponding criteria values) even if it was found or missed in the multicriteria optimization method.

The optimization in problems related to thermoacoustics has rarely been discussed in the literature. One of the few examples is the problem of optimizing the engine structure [22]. The authors performed both single and multicriteria optimization. Consequently, they provided a tendency of structural variables when optimizing individual objective components, and their results fully coincide with an intuitive evaluation of the impact of individual variables on individual criteria (e.g., work grows along with the lengthening of the stack). The decision variables, which are the structural variables, have a contradictory effect on criteria, so it is more interesting to find the front of p-optimal solutions.

As far as the articles dedicated to inverse problems, although there are quite numerous works related to this subject, those dealing with thermoacoustics are missing too [6].

In [8], the inverse problem was formulated, but not related to the thermoacoustic engine. It dealt with the reconstruction of unsteady fluctuations of the heat release rate in a combustor. A typical approach to solve inverse problems is to use the regularization that is designed to restrict the impact of the input data error [24]. Another idea is to investigate the sensitivity of measured values to changes in the estimated parameters and use it for a proper selection of the searched values [16]. In some cases, the reformulation of the phenomenon model may be helpful [2]. In [12], the problem of reconstruction of unsteady fluctuations using the Bayesian stochastic approach was presented [9, 19]. This method allowed reconstruction of the parameters based on disturbed data, which is important in strong ill-conditioned problems.

Growing computational power and modern numerical methods are good tools used to solve engineering problems, even the ones whose mathematical model might be very complex.

This article deals with the thermoacoustic problem consisting of optimization of the position and length of the stack and the resonance tube diameter to maximize engine performance while minimizing heat loss. The task was to solve both the multicriteria and the inverse problem. The modified RACO algorithm was used as a multicriteria optimization tool (for finding S_{pOpt}) and

the Bayesian approach was used to solve the inverse problem, i.e., reconstructing the parameters determining the solution on the Pareto front but not belonging to the set S_{pOpt}.

In this study, results obtained for both problems, as well as the advantages and limitations of the proposed heuristics based on these results, are discussed.

2. Thermoacoustic engine – optimization of its construction

The basic goal of the thermoacoustic device design is to plan its construction for the most efficient operation properly.

In case of a thermoacoustic engine, it is usually a matter of maximizing work output while minimizing heat loss, which translates into its efficiency. Although the concept of the engine and thermoacoustic compressor is not new (the first engine was described by Stirling in 1816), the simplicity of their construction keeps the demand for such devices increasing. They can be used, e.g., in microcircuits as a pump heater or produce spot cooling of specific elements in the circuit [21], as a low-cost electricity generator for rural areas [26], or as a device for energy savings [18]. Thanks to optimization, the spectrum of applications for thermoacoustic devices can be very broad. Properly designed construction not only improves efficiency but also adapts the machine to the conditions in which it is designed to work.

The capabilities offered by numerical modeling, often designed to evaluate a project without the need for prototype construction are commonly used at the design stage. Modeling together with appropriate optimization methods provides a good chance of finding solutions that improve efficiency or performance of your design. Of course, the quality of the solution obtained is strongly correlated with the quality of the mathematical model. It should be borne in mind that optimization allows to find the best solution, but it is always a solution connected to the model in which the proposed variants are evaluated.

2.1. Mathematical model

In this study, the model of thermoacoustic engine proposed by Trapp *et al.* [22] is used. This is a two-dimensional nonlinear model of the real quarter-wavelength engine. This device consists of a glass resonance tube closed at one end. Inside the tube, there is the so-called stack, the key component in thermoacoustic devices. This unit is sandwiched between two heat exchangers, one of which is responsible for supplying heat to the system and the other for receiving it. The supplied heat forces the temperature gradient across the stuck in the axial direction of the engine and results in amplification of pressure disturbances in the working gas and a corresponding loud noise to be emitted once a steady state has been achieved. It is possible if the imposed temperature gradient is larger than the critical temperature gradient (this is achieved by appropriate heating and cooling of the ends of the stack). The expression for the critical temperature gradient was derived by Swift [20] and is given as follows:

$$\nabla T_{\text{crit}} = \frac{\omega p}{\rho_m c_p u}. \tag{1}$$

At the closed end of the resonance tube, there is the pressure arrow and the velocity node, whereas, at its open end, there is the pressure node and the velocity arrow. Placing the porous stack near the closed end causes the working gas to experience strong pressure oscillations (resulting from the heat supply) and relatively small shifts between the closed end and the stack.

"When a gas in the vicinity of the walls inside the regenerative unit is subject to a sound wave, it experiences compression, expansion, and displacement. Over the course of the cycle, heat is added to the gas at high pressure, and heat is withdrawn from it at low pressure. This energy imbalance causes an increase in the amplitude of pressure in subsequent cycles until the acoustic

energy dissipation equals the heat supplied to the system. In the steady state, the engine converts thermal energy into work, which is accompanied by an acoustic effect.

The construction and position of the stack have a significant impact on the operation of the thermoacoustic engine. With the increase in the length of the stack and the tube radius, the work done by the device increases, however, the heat loss also increases" [22]. The work of the device and the heat loss are thus mutually contradictory criteria.

Therefore, the purpose of this study was to find the placement for the values of construction parameters such as L – stack length, H – resonant tube radius, and Z – position of the hot end of the stack in order to maximize engine work while minimizing heat loss. This means that we are dealing with the optimization of the two-criteria function Φ, defined here as follows:

$$\begin{aligned} &\Phi\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2, \\ &\Phi(L, Z, H) = (W, -Q_{\text{all}}) \longrightarrow \max, \end{aligned} \tag{2}$$

where the decision variables are L, Z, and H, and the criteria for which optimization is made are W (maximization) and heat flow Q_{all} (minimization), understood as the sum $Q_{\text{all}} = Q_{\text{conv}} + Q_{\text{rad}} + Q_{\text{cond}}$.

The objective function Φ does not return one value that can serve as a measure of solution quality. However, it returns a vector of values whose constituents are called criteria. In other words, by controlling the parameters L, Z and H, we obtain the system response in the pair $(W, -Q_{\text{all}})$. Given the structural data of the engine, it is necessary to find the temperature distribution in the domain accepted for calculation, in order to calculate the work and the heat loss of the engine. Due to the axial symmetry of the engine, the calculations were performed in the two-dimensional domain (Fig. 1). According to the mathematical model adopted, the work of engine is calculated by the following formula:

$$W = \omega \left[\delta_k \frac{(\gamma - 1)p^2}{\rho c^2 (1 + \varepsilon)} (\Gamma - 1) - \delta_\nu \rho u^2 \right] L N d_c, \tag{3}$$

while the heat loss is

$$Q_{\text{all}} = Q_{\text{conv}} + Q_{\text{cond}} + Q_{\text{rad}}, \tag{4}$$

where

$$Q_{\text{conv}} = H \int_0^{2\pi} \int_0^{L} h(T_s)(T_s - T_\infty) dz d\phi, \tag{5}$$

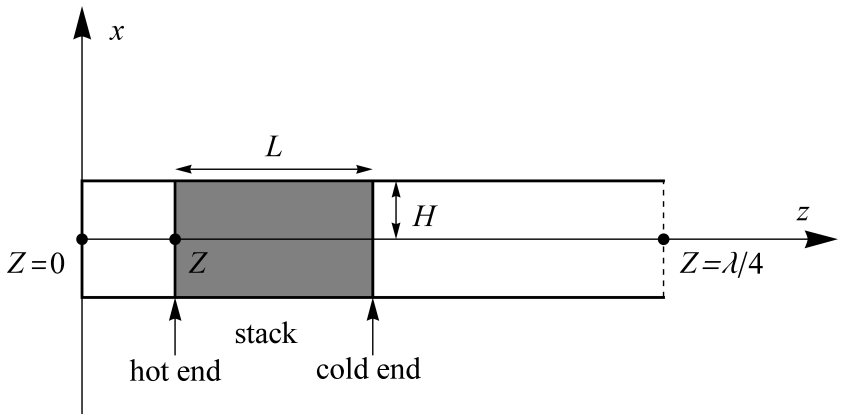

Fig. 1. Computational domain.

$$Q_{\text{cond}} = \int_0^{2\pi} \int_0^{H} \left(k_{rr} \frac{\partial T}{\partial r} + k_{zz} \frac{\partial T}{\partial r} \right) dz d\phi, \tag{6}$$

$$Q_{\text{rad}} = H k_b \int_0^{2\pi} \int_0^{L} \varepsilon \left(T_s^4 - T_\infty^4 \right) dz d\phi. \tag{7}$$

Optimized volumes depend on the parameters defined as follows:

$$\begin{aligned}
\varepsilon &= \frac{(\rho c_p \delta_k)_g}{(\rho c_p \delta_s)_s} \frac{\tanh\left((i+1) y_0/\delta_k\right)}{\tanh\left((i+1) l/\delta_s\right)}, \\
u_{\max} &= \frac{p_{\max}}{\rho c}, \\
p &= p_{\max} \cos\left(\frac{2\pi Z}{\lambda}\right), \\
u &= u_{\max} \sin\left(\frac{2\pi Z}{\lambda}\right), \\
k_{rr} &= \frac{k_s k_g (t_w + d_c)}{k_s t_w + k_g d_c}, \\
k_{zz} &= \frac{k_s t_w + k_g d_c}{t_w + d_c}.
\end{aligned} \tag{8}$$

In the model adopted, there is a constant temperature at both ends of the stack (T_h – hot side temperature and T_c – cold side temperature) and free convection and radiation to the surroundings (at T_∞). In addition, it is assumed that the stack structure is fixed (number of channels and its radius are constant), and the decision variables are constrained:

$$\begin{aligned}
Z_{\min} &\leqslant Z \leqslant Z_{\max}, \\
L_{\min} &\leqslant L \leqslant L_{TAE} - Z, \\
H_{\min} &\leqslant H \leqslant H_{\max},
\end{aligned} \tag{9}$$

where L_{TAE} is the length of the thermoacoustic engine.

Using the formulas (5)–(7) containing the integral is not convenient because this would require direct analytical solving, which is very difficult or even impossible to calculate. Because of this, following the work of Trapp *et al.* [22], the heat flow under consideration can be calculated by using its approximated forms:

$$Q_{\text{conv}} = 2\pi H L h \left[\frac{T_h}{\ln\left(\frac{T_c}{T_h}\right)} \left(\frac{T_c}{T_h} - 1 \right) - T_\infty \right], \tag{10}$$

$$Q_{\text{cond}} = \frac{k_{zz}}{L} \pi H^2 T_c \ln\left(\frac{T_h}{T_c}\right), \tag{11}$$

$$Q_{\text{rad}} = 2\pi HLk_b\epsilon\left[\frac{T_h^4\left(e^{4\ln\left(\frac{T_c}{T_h}\right)}-1\right)}{4\ln\left(\frac{T_c}{T_h}\right)} - T_\infty^4\right]. \tag{12}$$

It is worth noting that despite being approximated, the mathematical model is still nonlinear.

Since the decision variables, which are structural variables, affect the criteria contradictorily, the solution to the above problem is to find the front of p-optimal solutions. In this study, due to this reason, the modification of the ant algorithm has been proposed. As a population heuristic, this method is ideally suited to simultaneously determine the multi-element representation of the Pareto front, a set of equivalent solutions, each representing a compromise between the criteria.

3. Multicriteria optimization

3.1. Pareto-optimal solution

As mentioned, the problem formulation in the form (2) means that we are dealing with a multiobjective optimization.

In such tasks, a range of the objective function is usually a subset of the set $\mathbb{R}^m$ in which the order relation is not defined naturally. Consequently, it is not possible to compare each value of the objective function Φ to one another. However, the notion of optimization, by its very nature of searching the best solution, requires the ability to compare and unambiguously evaluate the variants. For this reason, in the multicriteria optimization, the relationship of dominance is introduced [25]. By definition, we say that the point $\mathbf{x}$ dominates $\mathbf{y}$ (or $\mathbf{y}$ is dominated by $\mathbf{x}$), which is noted by $\mathbf{x} \prec \mathbf{y}$ if

$$\forall\ k = 1,\ldots,m;\quad f_k(\mathbf{x}) \le f_k(\mathbf{y}) \ \wedge\ \ \exists\ k \in \{1,\ldots,m\}\ \ f_k(\mathbf{x}) < f_k(\mathbf{y}), \tag{13}$$

where f_i means i-th criteria (i-th component of the vector of objective function values Φ). The relation defined in this way introduces the order in the set $\mathbb{R}^m$. The solution which is dominated is treated as "worse" compared to the dominant solution. It is easy to see that there will be groups of solutions that cannot be compared to one another. These are *nondominated* solutions that lay in the so-called layers. Within one layer, solutions are better than others because of at least one criterion, and at the same time worse for another criterion.

As a result of well-performed optimization, nondominated solutions will "lie down" in one layer along, i.e., *Pareto front*. It is a collection of points representing all possible (but equivalent) variants

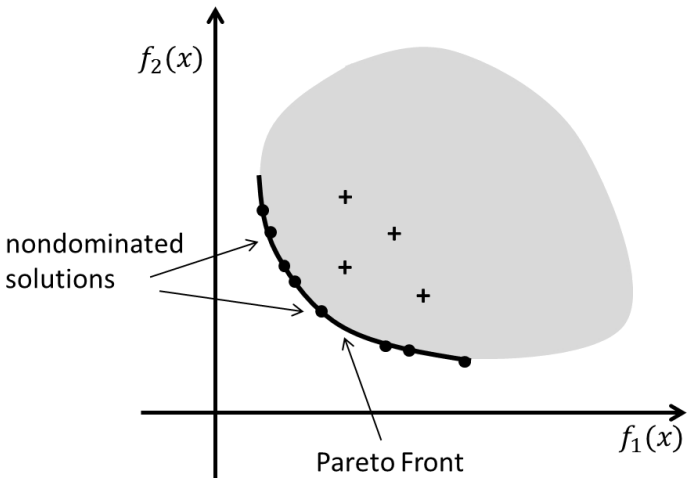

Fig. 2. Pareto front in two-objective function.

of the optimization problem solution. Because the Pareto front usually contains infinitely many solutions, finding all of them is not practically possible. The purpose of multicriteria optimization is to find, on the set of the feasible solutions, the most numerous (finite) group of nondominated points lying on this front. The set obtained can be called both "trade-off surface" or set of p-optimal solutions.

Because the multiobjective optimization does not produce a unique solution but a set of equivalent solutions on the Pareto front, it is best to use the heuristic population to find them. This allows for searching through space using multiple agents, allowing multiple points to be found on the Pareto front. Appropriate modifications such as genetic algorithms, particle swarm optimization, or other heuristics should be a good tool for multicriteria optimization.

Having many equivalents, i.e., optimal solutions in the Pareto sense, the decision maker has the opportunity to compare them and to choose the solution to be implemented (e.g., using additional nonnumerical criteria).

In this study, RACO (one of the heuristics inspired by the behavior of ant colonies) was used to search the space of possible solutions. The method was modified to be able to solve multicriteria optimization problems. The details of this approach are presented in the next section.

3.2. Real ant colony optimization

The original version of RACO was proposed by Socha and Dorigo in [17]. In general, the idea of this algorithm is based on the formation of the so-called pheromone stains (possible solutions) of increasing intensity, which are produced by artificial ants. The intensity of the stains is associated with the quality of the solution which it represents. In the iteration procedure, new and better solutions are the result of ants' work, where each of them imposes a trail pheromone (creates a new proposition of solution). Although the ants actions seem to be random, they are controlled by the use of an appropriate probabilistic model. The construction of such a model is an essential part of RACO.

The modification of RACO, for the purposes of multicriteria optimization, was to replace the standard selection, by the ranking one, in which the solution rank depends on belonging to a given layer of nondominated solutions. All solutions are divided into layers of mutually nondominated solutions. The solution position on the ranking list is determined by the number of layers in which it is located. In addition, the succession operation is used to keep the best stains (solutions) in the iteration and "evaporation" of the others. The process of creation of the new solutions was preserved in the form known from the original version of RACO [5, 17].

The effectiveness of the modified version of RACO for multicriteria problems was investigated and presented in [13].

In general, the RACO method consists of two parts:

1) **initialization** in which the parameters of the algorithm are defined and initial archive of k pheromone "stains" (solutions) is generated; the "stains" are arranged with respect to their quality as belonging to the appropriate layer of nondominated solutions;

2) **computational loop** is repeated until the stop criteria are not satisfied; in the single iteration new solutions are designed (by each of the ants), then the updated archive of solutions (pheromone "stains") is divided into layers containing mutually nondominated solutions.

The most important element of presented heuristics is, of course, a computational loop where new solutions are created as a result of work of each m ants. Each ant randomly selects a j-th solution ("stain" of pheromone) with probability

$$p_j = \frac{\omega_j}{\sum_{l=1}^{k} \omega_l}, \tag{14}$$

where ω_j is weight connected with j solution determined using Gaussian $g(\mu, \rho) = g(1, qk)$ (k – number of stains, q – parameter), i.e.,

$$\omega_j = \frac{1}{qk\sqrt{2\pi}} \cdot e^{\frac{-(j-1)^2}{2q^2k^2}}. \tag{15}$$

Weights reflect the pheromone "intensity" and the ants prefer to choose a seat/track more saturated with pheromones. Using the function (15) ensures that the weights ω_j have only positive values, which is important in determining the values p_j according to the function (14).

Then ants sample the subspace in the surroundings of selected stain (represented by vector $\mathbf{s}_j = (s_j^1, s_j^2, \cdots, s_j^l)$) applying a pheromone trail with random Gaussian distribution. The probability that r-th component of a new stain occurs at x is determined by the following formula:

$$p(x) = g(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \tag{16}$$

where the expected value $\mu = s_j^r$ is related to the position of the stain selected by ant, while the standard deviation $\sigma = \xi \sum_{p=1}^{k} \frac{|s_p^r - s_j^r|}{k-1}$ is the average distance between the r-th component of selected stain and the others within the population. Construction is repeated for each of m ants (m new pheromone "stains" are obtained). Using the Gaussian function makes new solutions that are not sampled with uniform distribution, but, they are produced in regions that are considered promising.

As it was mentioned, the modification of the method for the multiobjective optimization is the replacement of the standard selection by the ranking one, in which the solution rank depends on belonging to a given layer of nondominated solutions and assumption that the successive operation keeps the best stains (solutions) in the iteration and "evaporate" the others.

Such a change results in the best-evaluated solutions in the first layer. Heuristics "work" so that many solutions belong to this group. Pheromone stains move in successive iterations to avoid dominance. This ultimately places them on the Pareto front, which is part of the boundaries of the set of task solutions. The points obtained in this way are collected in S_{pOpt}.

It is also noteworthy that the above modification of the method may be an alternative to EA because of the more advanced model of space search. The probabilistic model for generating new solution proposals is more efficient than the one used by EA [9, 19].

It is assumed that the finite set S_{pOpt} obtained as a result of the optimization is a group of solutions lying on the Pareto front, which means that a set of points whose coordinates represent the values of the criteria for which optimization was made. It is important to note that only for solutions in the set S_{pOpt} (effects) the value of variables L, Z, and H are known, which are the cause of optimal solutions.

Unfortunately, the found p-optimal solutions are often located unevenly at the front. Although sufficiently numerous representation of the Pareto front usually allows determining its approximate shape, many interesting Pareto points are missed [13, 23]. If for some reason obtaining a solution on the front, but not belonging to the set S_{pOpt} would be of interest, it is impossible to determine easily for what parameters it is obtained.

Reconstruction of decision variables for solutions that do not belong to the set S_{pOpt} is an example of an inverse problem requiring a separate consideration.

4. Inverse problem

In inverse problems, we see wherever the cause of the result is to be reproduced on the basis of the effect.

For most engineering problems, solving inverse problems is a nontrivial task. Typically, such tasks are part of a group of ill-conditioned problems, in which even small inaccuracies in the readings

of the observed/measured values (effects) can make poor quality reconstruction of the parameters (causes) which depend on these values.

The same is also typical in the problems encountered in thermoacoustics. An example of such problem is to reproduce the structural data of the thermoacoustic engine, which affects the efficiency of its operation and heat loss. It is obvious that it is sensible and most interesting to reconstruct parameters for pairs $(W, -Q_{\text{all}})$, optimal in the Pareto-sense.

4.1. Formulation

In the problem under consideration, the inverse problems to determine for which values L, Z, and H, the heat loss Q_{all} and the work W of thermoacoustic motor will obtain the fixed values expected. The length and position of the stack, as well as the radius of the resonant tube, greatly influences the temperature distribution of the engine and hence its efficiency. Heat losses and work are mutually contradictory criteria, but due to the familiarity with the shape of the Pareto front, the compromises between them can be taken into account.

The formulation of the inverse problem, at first glance, is no different from the direct formulation that was presented in Subsec. 2.1. Nevertheless, from a mathematical point of view, this is a different kind of problem as it requires a solution because of other variables.

In this study, in order to reconstruct L, Z, and H on the basis of W and Q_{all}, again it was decided to employ heuristic population using stochastic information. This is a Bayesian approach that uses the probabilistic prior information about the data disturbance. This approach works well in inverse problems that are typically sensitive to data [12].

4.2. Bayesian approach

The characteristic feature of the inverse problems is their ill-conditioning, namely the high sensitivity of the resulting quality with respect to the measurement data (necessary to solve the task). One of the most popular approaches to this type of problems is the introduction of the regularization segment into the objective function, which allows overcoming the inaccuracies of the input data.

Another possibility is to use the Bayesian approach, which uses the probabilistic information about the measurement data disturbance. In this method, it is assumed that all the variables, both measurements $\mathbf{Y}$ and estimated parameters $\mathbf{P}$, are treated as random variables. This enables taking into account the uncertainty of measurements in the inverse problem solution. Measuring instruments, even of the highest quality, provide disturbed data. Usually, the class of the device allows specifying a distribution of error and the parameters of this distribution. The difficulty in determining the precise geometric coordinates of the point at which they are read may by another source of inaccuracy of measurement data. Although the attention of the researcher can really reduce the inaccuracy of the measurement, giving a feeling of complete control over them, there is no doubt that the deviation of measurements in comparison to the real values exist and is regarded as a factor hindering the correct solution. In contrast to other methods, in the Bayesian approach, information about the distribution of errors is actively used in problem-solving serving as additional information. The procedure of solving the inverse problem within the Bayesian framework is implemented according to the following scheme [14]:

- based on all the available information about the estimated parameters $\mathbf{P}$, a function $\pi(\mathbf{P})$ describing the density distribution of parameters is selected, so that it represents prior information;
- the next step is to select the likelihood function $\pi(\mathbf{Y}|\mathbf{P})$, which models the measurement errors and determines the response of the mathematical model of the considered phenomenon on the changes of parameters $\mathbf{P}$;

- finally, the method that determines the posterior density function $\pi(\mathbf{P}|\mathbf{Y})$ should be found/selected, which is the conditional probability distribution of the unknown parameters given by the measurements.

In the following steps, we discuss the aforementioned procedure in detail.

Any analyzed physical phenomenon or technology requires the construction of a mathematical model. In-depth analysis is need to build a robust model that corresponds to the real behavior of the system in question. Usually, it is not difficult to determine the limits and/or distributions of the parameters that should be reconstructed. Assuming, e.g., that the unknown parameters have a Gaussian distribution, the prior density model takes the following form:

$$\pi(\mathbf{P}) = \frac{1}{\sqrt{(2\pi)^N |\mathbf{V}|}} \exp\left[-\frac{1}{2}(\mathbf{P}-\mu)^T \mathbf{V}(\mathbf{P}-\mu)\right], \tag{17}$$

where μ, and $\mathbf{V}$ are known vector of averages and covariance matrix of parameters $\mathbf{P}$, respectively, and N is the number of estimated parameters.

In case, when only limitations α_l (lower) and α_u (upper) of the parameters are known, it can be assumed that the parameters have a uniform distribution at predetermined intervals ($\forall j : p_j \in [\alpha_{l,j}, \alpha_{u,j}]$). Then:

$$\pi(\mathbf{P}) = \begin{cases} \dfrac{1}{\prod_{i=1}^{N}(\alpha_{u,i}-\alpha_{l,i})}, & \mathbf{P} \in \langle \alpha_l, \alpha_u \rangle, \\ 0, & \text{in another case.} \end{cases} \tag{18}$$

Similarly, the density of conditional probability $\pi(\mathbf{Y}|\mathbf{P})$ may be determined, and it is a function defining the chances of getting different values of $\mathbf{Y}$ for the fixed values of $\mathbf{P}$. The density of such a likelihood can be determined on the basis of the following formula:

$$\pi(\mathbf{Y}|\mathbf{P}) = \frac{1}{\sqrt{(2\pi)^L |\mathbf{W}|}} \exp\left[-\frac{1}{2}\left(\mathbf{Y}-\mathbf{T}(\mathbf{P})\right)^T \mathbf{W}\left(\mathbf{Y}-\mathbf{T}(\mathbf{P})\right)\right], \tag{19}$$

where $\mathbf{W}$ is the covariance matrix of measurements, L is the number of measurements, and $\mathbf{T}(\mathbf{P})$ is a vector of observed values calculated in the direct model for fixed $\mathbf{P}$ at points where the data were collected. Formula (19) is most commonly used, because the measurement errors in the real conditions usually have a normal distribution. The possibility of using information about potential errors of measurement is one of the strengths of the Bayesian approach. A probabilistic model of the estimated values and measurements ensures that the technique discussed "remembers" and takes into account the fact that the available data are disturbed.

The final step of the proposed approach is to determine the posterior density function $\pi(\mathbf{P}|\mathbf{Y})$, defining the conditional probability distribution of the parameters searched, provided that the measurements are known. In consequence, the distribution of the expected values of estimated parameters can be determined, so that the inverse problem is solved. Determination of $\pi(\mathbf{P}|\mathbf{Y})$ is not difficult, because according to classical Bayes' theorem, if $\mathbf{P}$ and $\mathbf{Y}$ are continuous random variables, then,

$$\pi(\mathbf{P}|\mathbf{Y}) = \frac{\pi(\mathbf{Y}|\mathbf{P})\pi(\mathbf{P})}{\pi(\mathbf{Y})}. \tag{20}$$

In practical applications, the designation of $\pi(\mathbf{Y})$ is often difficult. However, it is not necessary, because in the formula (20) it plays only the role of a normalizing factor. Therefore, Bayes' theorem can be written as,

$$\pi(\mathbf{P}|\mathbf{Y}) \propto \pi(\mathbf{Y}|\mathbf{P})\pi(\mathbf{P}), \tag{21}$$

where symbol $\propto$ means the proportionality.

Unfortunately, the analytical description of the posterior probability density $\pi(\mathbf{P}|\mathbf{Y})$ is not always possible in practice. Therefore, usually, any algorithm from a group of Monte Carlo Markov chain (MCMC) methods is applied at this point. The main purpose of these methods is to generate the Markov chain, wherein the successive states $\mathbf{P}^{(t)}$ are sampled with the distribution $\pi(\mathbf{P}|\mathbf{Y})$. The expected value of the distribution $\pi(\mathbf{P}|\mathbf{Y})$ is defined as the arithmetic mean of the states of generated Markov chain. It is simultaneously regarded as $\mathbf{P}_{\text{opt}}$, ie, the solution of the inverse problem.

One of the simplest methods in the group of MCMC algorithms is the Metropolis-Hastings (MH) algorithm [11]. Generally speaking, this method produces a sequence of random samples from a probability distribution for which direct sampling is difficult. This sequence can be used to approximate the distribution. MH is used for sampling from a probability distribution by using the full joint density function and (independent) proposal distributions for each of the variables of interest. The operation of MH algorithm begins with drawing a vector of initial states $\mathbf{P}^{(0)}$, typically from the variables of the prior distribution. In the main loop (for $i = 1, 2, \ldots$), the algorithm consists of three components:

- generation of a proposal/candidate sample $\mathbf{P}^{\text{cand}}$ from the proposal distribution $\pi(\mathbf{P}^{(i)}|\mathbf{P}^{(i-1)})$;
- computation of the acceptance probability

$$\alpha\left(\mathbf{P}^{\text{cand}}|\mathbf{P}^{(i-1)}\right) = \min\left\{1, \frac{\pi\left(\mathbf{P}^{\text{cand}}\right)}{\pi\left(\mathbf{P}^{(i-1)}\right)}\right\};$$

- acceptance of the candidate sample:
  for pseudo random u with uniform distribution $U(0,1)$, if $u < \alpha(\mathbf{P}^{\text{cand}}|\mathbf{P}^{(i-1)})$ then the proposal is accepted, i.e., $\mathbf{P}^{(i)} = \mathbf{P}^{\text{cand}}$, else the proposal is rejected and $\mathbf{P}^{(i)} = \mathbf{P}^{(i-1)}$.

The Bayesian approach presented above is an alternative technique for solving inverse problems. As already mentioned, through the construction of a probabilistic model with Bayes' theorem, the method uses the calculations of irrelevant information (and often even disadvantageous) from the other methods' point of view. It is noteworthy that this method does not require the selection of multiple parameters that determine its performance. One of the major disadvantages of this method is the time required for optimization, as constructing the Markov chain of appropriate length is time-consuming. Unfortunately, this is the main disadvantage of the many methods based on probabilistic models.

In this study, the Markov chains were built using the MH algorithm, assuming that the measurements used in the calculation are burdened with normally distributed errors and the estimated parameters have uniform distributions at specified intervals.

5. Numerical results

For this work purposes, calculations were made to show that the numerical methods presented above are effective and that they can be an alternative to other methods used in solving engineering problems formulated as inverse problems and/or requiring multicriteria optimization.

For numerical tests, the thermoacoustic engine optimization problem described in Subsec. 2.1 was selected.

Numerical calculations were conducted in two stages:

- in the first stage, the multicriteria optimization problem (2) was solved and its solution was used to approximate the Pareto front;
- in the second stage, the inverse task formulated in Subsec. 4.1 was solved to reconstruct the engine's parameters for which it would receive the p-optimum values of the criteria lying on the Pareto front but not determined in the first calculation step.

5.1. Finding of Pareto front

In the mathematical model, it was assumed that the estimated values could take values from the intervals $Z \in \langle 0.01, 0.9\rangle$, $L \in \langle 0.01, L_{TAE} - Z\rangle$, and $H \in \langle 0.05, 0.5\rangle$, what was dictated by the engine's design capabilities. The other values of the physical parameters of the phenomenon assumed in the model are given in Table 1:

Table 1. Physical parameter values.

$p_{\max}$	10 000	Pa
λ	2π	m
ω	600	s^{-1}
γ	1.4	–
ρ	0.7	kg m^{-3}
c [1]	490.5	m s^{-1}
h	50.	W m^{-2} K^{-1}
T_h	700.	K
T_c, T_∞	300.	K
d_c	0.0005	m
c_p	1.02	kJ kg^{-1} K^{-1}
ϵ	0.05	J kg^{-1}
ε	$\sqrt{3}/2$	–
t_w	$2 \cdot d_c$	m

[1] for the average temperature.

Optimization was executed by using 20 pheromone stains (in single population) imposed by 20 ants. The iteration process was performed to obtain 100% stains in a set of nondominated solutions, but at least with 100 iterations. The obtained set S_{pOpt} was treated as a set of solutions on the Pareto front.

Figure 3 presents the solutions obtained in different start-ups. The charts show the points representing the criteria values for the start-up population (black dots) and the points from the set S_{pOpt}, ie, the obtained p-optimal solutions (green circles). The solid, green line represents an approximation of front (by third-order polynomial) determined on the basis of the set S_{pOpt}. The approximation used the standard tool available in Wolfram Mathematica.

Figure 4 presents the approximations of Pareto fronts determined on the basis of the obtained (in various start-ups) sets of non-dominated solutions. It is easy to see that the designated fronts are almost identical, proving that the found non-dominated solution sets ares located on the Pareto front.

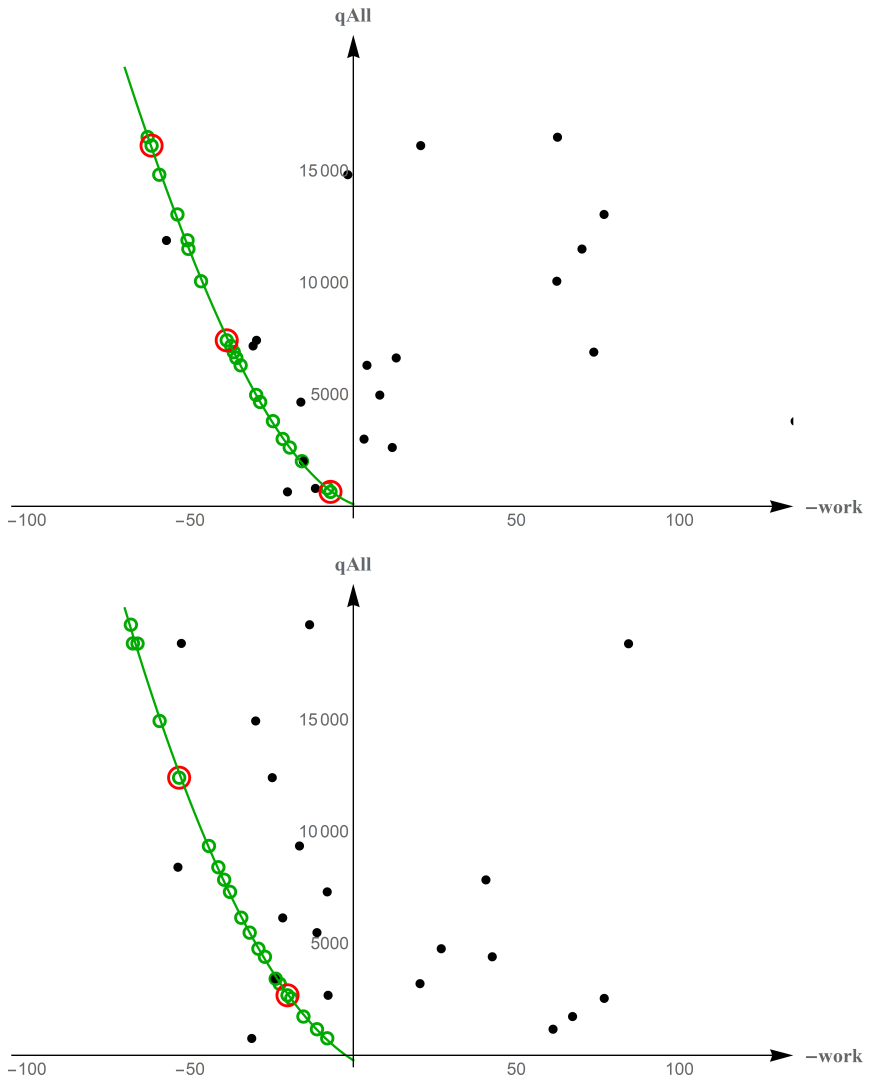

Fig. 3. Set of nondominated solutions on the background of starting population of pheromone stains (two different start-ups).

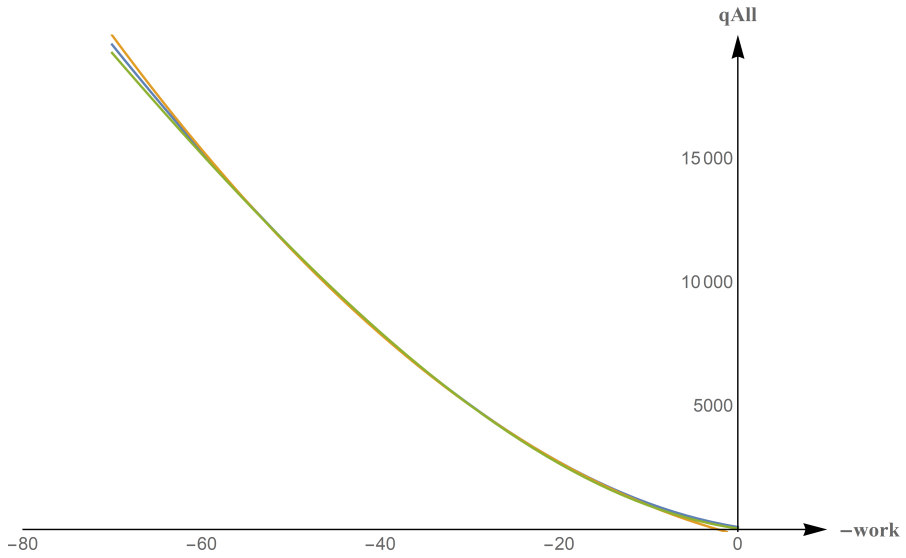

Fig. 4. Comparison of the approximated Pareto fronts.

In Table 2, the numerical values of the p-optimal solutions (marked in Fig. 3 by red circle) are presented together with the corresponding values of the estimated parameters. These are different variants of the geometry of the modeled thermoacoustic engine. It depends on the researcher's preference or other criteria chosen for the final assessment which solution will ultimately be selected.

Table 2. Chosen p-optimal solutions.

	Estimated parameters			p-optimal solution	
	Z	L	H	W	Q_{all}
1.	0.7544	0.1789	0.4644	61.7907	16123.94
2.	0.8393	0.1160	0.0545	6.9501	647.01
3.	0.7782	0.1608	0.2833	38.7399	7419.1
4.	0.7068	0.1029	0.1415	20.1194	2679.53
5.	0.8102	0.1808	0.3964	53.3386	12411.97

5.2. Solving inverse problem

In order to obtain the values L, Z, and H for solutions lying on the Pareto front but not belonging to the set S_{pOpt}, the inverse problem is formulated as given in Subsec. 4.1.

It was assumed that L, Z, and H are random variables with a uniform distribution respectively in the intervals $Z \in \langle 0.01, 0.9\rangle$, $L \in \langle 0.1, 0.9-Z\rangle$ and $H \in \langle 0.05, 0.5\rangle$. Since the Pareto front is determined by approximation, it is assumed that the data values (ie, the values read from the front) W and Q_{all} are treated as random variables with normal distribution respectively $\mathcal{N}(W^*, 0.1)$ and $\mathcal{N}(Q^*, 0.1)$, where W^* and Q^* are the coordinates of the point lying exactly on the approximated front.

In the numerical task, three configurations of L, Z, and H for chosen points lying on the approximate Pareto front are reconstructed. The first point belongs to the set S_{pOpt}, and the reconstruction of the engine parameters was performed to verify the Bayesian approach as an effective method for the inverse problem solution. The next two cases are points located on the approximated Pareto front, but not belonging to the set of solutions obtained in the multicriteria optimization.

The p-optimal solutions for which parameters L, Z, and H were reconstructed are marked with diamonds in Fig. 5.

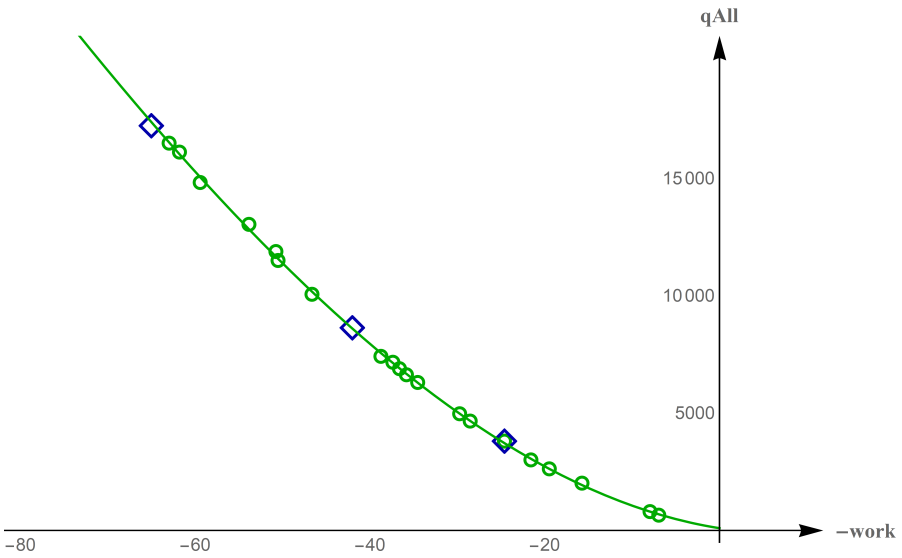

Fig. 5. Approximated Pareto front with solutions chosen for the inverse problem (diamonds) and points found in the multicriteria optimization (circles).

The results obtained are listed in Table 3.

Table 3. Results of the inverse problem.

	Retrieved parameters			Expected values	
	L	Z	H	W	Q_{all}
1.[2]	0.73888	0.14726	0.17846	24.5954	3 803.115
2.	0.74528	0.17649	0.31119	42.	8 634.886
3.	0.83921	0.17209	0.46944	−65.	17 244.6

[2] in comparison with (0.7647, 0.141, 0.1829) found in the optimization.

6. Summary

Contemporary engineering problems are often formulated as multicriteria tasks that by their nature have more than equivalent solutions. In case if we do not have a full mathematical model, or there is a need to reproduce the underlying causes of the problem, the problem is defined as the inverse. This work proposes numerical methods (heuristics) for solving both types of problems.

RACO belonging to a group of population methods was used to solve the multicriteria optimization problem. Previous tests [13] and calculations made for this work purposes have yielded satisfactory results and indicate the effectiveness of the proposed method in determining the Pareto-optimal front. The points on the Pareto front represent different but equivalent compromise versions between the criteria, which can be used, eg, in device design and technology. Using a more advanced space search model makes RACO a competitive method for EA. The probabilistic model for generating new solution proposals is more efficient than the EA's scheme.

Even good population heuristics cannot determine full information about the Pareto front. It can only find its finite representation and approximate shape. This means that there are many (often infinitely many) solutions that are known to be on the Pareto front, but there is no access to the parameters for which they are obtained.

In this situation, it is necessary to solve the inverse problem, for which the Bayesian approach was used in this study. The biggest advantage of the Bayesian approach is that it takes into account the existence and distribution of measurement errors. The main idea is to treat all variables, both measurements and estimated parameters, as random variables. The prior information about the error distribution was used in determining the posterior density function, which is the conditional probability distribution of the unknown parameters given by the measurements. This means that the data disorder is a kind of information which assists in determining the parameters. The solution is defined as the expected value of Markov chain generated according to the posterior information.

Both methods were used in the calculations related to the optimization of the geometry of the thermoacoustic engine. The results obtained and presented in this article demonstrate the usefulness of the discussed methods and the possibility of using them also in other engineering problems.

Calculations were mainly done in the in-home programs (Fortran) created for this work purposes. Other calculations and graphics were created with Wolfram Mathematica.

References

[1] J. Andersson. *Multiobjective Optimization in Engineering Design – Applications to Fluid Power Systems*. Dissertation, Linköping Studies in Science and Technology, Dissertation No. 675, Linköping University, Linköping, Sweden, 2001.

[2] P. Bala Subrahmanyam, R. Sujith, M. Ramakrishna. Determination of unsteady heat release distribution from acoustic pressure measurements: a reformulation of the inverse problem, *The Journal of the Acoustical Society of America*, **114**(2): 686–696, 2003.

[3] S. Bandyopadhyay, S. Saha, U. Maulik, K. Deb. A simulated annealing-based multiobjective optimization algorithm: AMOSA. *IEEE Transactions on Evolutionary Computation*, **12**(3): 269–283, 2008.

[4] B. Li, J. Li, K. Tang, X. Yao. Many-objective evolutionary algorithms: A survey. *ACM Computing Surveys*, **48**(1): 35 pages, 2015.
[5] U. Diwekar. *Introduction to Applied Optimization*. Springer, 2008.
[6] C. Heuberger. Inverse combinatorial optimization: a survey on problems, methods, and results. *Journal of Combinatorial Optimization*, **8**(3): 329–361, 2004.
[7] J. Horn. Multicriteria decision making. In: T. Back, D.B. Fogel, Z. Michalewicz [Eds.], *Handbook of Evolutionary Computation*. Institute of Physics Publishing, Bristol (UK), 1997.
[8] B. Kaltenbacher, W. Polifke. Some regularization methods for a thermoacoustic inverse problem. *Journal of Inverse and Ill-Posed Problem*. **18**: 997–1011, 2011.
[9] B.T. Knapik, A.W. van der Vaart, J.H. van Zanten. Bayesian inverse problems with Gaussian priors. *The Annals of Statistics*, **39**(5): 2626–2657, 2011.
[10] J. Knowles, D. Corne, D. Kalyanmoy. *Multiobjective Problem Solving from Nature: from Concepts to Applications*. Springer Science & Business Media, 2007.
[11] S.M. Lynch. *Introduction to Applied Bayesian Statistics and Estimation for Social Scientists*. Springer, New York, 2007.
[12] I. Nowak. Bayesian approach applied for thermoacoustic inverse problem. *Energy*, **141**: 2519–2527, 2017.
[13] I. Nowak, G. Nowak. Real ant colony optimization as a tool for multi-criteria problems. *Computer Assisted Methods in Engineering and Science*, **22**: 255-–265, 2015.
[14] H.R.B. Orlande, O. Fudym, D. Maillet, R.M. Cotta. *Thermal Measurements and Inverse Techniques*, CRC Press, 2011.
[15] V. Pareto. *Manuale di Economia Politica*. Societa Editrice Libraria, Milano, Italy, 1906. Translated into English by A.S. Schwier as Manual of Political Economy, Macmillan, New York, 1971.
[16] S.A. Saltelli, F. Tarantola. *Sensitivity Analysis in Practice: A Guide to Assessing Scientific Models*. John Wiley & Sons, 2004.
[17] K. Socha, M. Dorigo. Ant colony optimization for continuous domains. *European Journal of Operational Research*, **185**: 1155–1173, 2008.
[18] S. Spoelstra, M.E.H. Tijani. *Thermoacoustic Heat Pumps for Energy Savings*. Presented at the seminar "Boundary crossing acoustics" of the Acoustical Society of the Netherlands, 2005.
[19] A.M. Stuart. Inverse problems: a Bayesian perspective. *Acta Numerica*, **19**: 451–559, 2010.
[20] G.W. Swift. Thermoacoustics: a unifying perspective for some engines and refrigerators. *The Journal of the Acoustical Society of America*, **113**(5): 2379–2381, 2003.
[21] O.G. Symko, E. Abdel-Rahman, Y.S. Kwon, M. Emmi, R. Behunin. Design and development of high-frequency thermoacoustic engines for thermal management in microelectronics. *Microelectronics Journal*, **35**(2): 185–191, 2004.
[22] A.C. Trapp, F. Zink, O.A. Prokopyev. Thermoacoustic heat engine modelling and design optimization. *Applied Thermal Engineering*, **31**: 2518–2528, 2011.
[23] O.L. de Weck. Multiobjective optimization: history and promise. Keynote Paper. The 3rd China-Japan-Korea Joint Symposium on Optimization of Structural and Mechanical Systems, Kanazawa, Japan, 2004.
[24] H. Werner, M. Hanke, A. Neubauer. *Regularization of Inverse Problems*. Vol. 375. Springer Science & Business Media, 1996.
[25] P.L. Yu. Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives. *Journal of Optimization Theory and Applications*, **14**(3): 319–377, 1974.
[26] Z. Yu, A.J. Jaworski, S. Backhaus. A low-cost electricity generator for rural areas using a travelling-wave looped-tube thermoacoustic engine. *Part A: Journal of Power and Energy*, **224**(6): 787–795, 2010.
[27] E. Zitzler. *Evolutionary Algorithms for Multiobjective Optimisation: Methods and Applications*. PhD thesis, Eidgenössische Technische Hochschule Zurich, 1999.
[28] E. Zitzler, K. Deb, L. Thiele. Comparison of multiobjective evolutionary algorithms: empirical results. *Evolutionary Computation*, **8**(2): 173–195, 2000.