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Some continuous stress mixed formulations and inf–sup test

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The most frequent motivation for the use of mixed methods is their robustness in the presence of certain limiting and extreme situations. At variance, the main goal of the present paper is to reconsider the use of mixed formulation as a tool for wider application, i.e., to study the stability of the proposed procedure treating problems in elasticity otherwise well suited for the solution by the usual displacement method.Computational procedure for the inf–sup test is outlined, and the results are given.

1. Introduction

Mixed finite element methods, in the mechanics of solids, are based on formulations having also stresses and/or strains as fundamental variables, at variance with the classical (primal) finite element method where fundamental unknowns are only displacements. There are opinions [1] that mixed methods have some serious drawbacks. For instance, a fact that discrete mixed system involves more degrees of freedom than a primal one, and hence the unacceptable execution time for the *same mesh*, is considered as one of main disadvantages of mixed methods. However it has been already shown [3, 4] that the present mixed model is about two orders of magnitude faster than classical finite element analysis for the *same accuracy*.

A short overview of the underlying theory can be found in [1], and a detailed explanation in [10]. From the practical point of view, primal mixed formulation looks to be the most attractive one. This scheme can be easily combined with the classical finite elements. Primal mixed formulation, using continuous stress spaces, has been successfully applied in [13] and [3]. The next two steps in the development of a model were the introduction of stress constraints as essential boundary conditions [5], and stabilization of primal mixed elements by hierarchic interpolation of stresses [6]. The accuracy of the formulation has been studied in [7], and efficiency in [8].

In this paper we consider the stability of some mixed schemes more rigorously, by the use of numerical inf–sup test [2, 10, 11]. We limit our attention to the practically important problem with the continuous displacement and, where appropriate, also the continuous stress spaces.

2. Field equations

Let us consider a complete system of the field equations in the linear elasticity, where

$$\operatorname{div} \boldsymbol{T} + \boldsymbol{f} = 0 \qquad \text{in } \Omega, \tag{1}$$

$$2\boldsymbol{e}(\boldsymbol{u}) = \nabla \boldsymbol{u}^T + \nabla \boldsymbol{u} \qquad \text{in } \Omega, \tag{2}$$

$$\boldsymbol{T} - \mathbf{C}\boldsymbol{e}(\boldsymbol{u}) = 0 \qquad \text{in } \Omega, \tag{3}$$

$$\boldsymbol{T}\cdot\boldsymbol{n}-\boldsymbol{p}=0 \qquad \text{on } \partial\Omega_t\,, \tag{4}$$

$$\boldsymbol{u}-\boldsymbol{w}=0 \qquad \text{on } \partial\Omega_u\,, \tag{5}$$

are the equations of motion, strain–displacement and stress–strain relationships, boundary traction conditions and geometric boundary conditions respectively.In these expressions, $\boldsymbol{T}$ is the stress tensor, $\boldsymbol{f}$ the vector of the body forces, $\boldsymbol{e}$ is the strain tensor, $\boldsymbol{u}$ the displacement vector, C the elasticity tensor, $\boldsymbol{p}$ the vector of the boundary traction's, $\boldsymbol{w}$ the vector of the prescribed displacements. Finally, Ω is an open, bounded domain of the elastic body, $\boldsymbol{n}$ is the unit normal vector to the boundary $\partial\Omega$; $\partial\Omega_t$ and $\partial\Omega_u$ are the portions of $\partial\Omega$ where the stresses or the displacements are prescribed, respectively.

3. Weak form of the field equations

Let us suppose that both boundary conditions (4) and (5) are essential, and hence exactly satisfied by the trial functions of a problem. Then we need to consider only the weak forms of Eqs. (1)–(3).

3.1. The equations of equilibrium

By the use of the Galerkine procedure, from Eq. (1) follows

$$\int_\Omega \boldsymbol{v}\cdot\operatorname{div}\boldsymbol{T}\,\mathrm{d}\Omega=\int_\Omega \boldsymbol{v}\cdot\boldsymbol{f}\,\mathrm{d}\Omega \tag{6}$$

where $\boldsymbol{v}$ is taken from the space L_2 of all square integrable vector fields.

3.2. The strain displacement and the stress-strain relationships

We will consider the standard case, i.e. invertible constitutive equations (3), when one can write

$$\boldsymbol{e}(\boldsymbol{u})=\mathsf{A}:\boldsymbol{T} \tag{7}$$

where A is the *elastic compliance* tensor. From the comparison of (2) and (7) it follows that

$$\mathsf{A}:\boldsymbol{T}=\frac{1}{2}\left(\nabla\boldsymbol{u}^T+\nabla\boldsymbol{u}\right). \tag{8}$$

If the test functions $\boldsymbol{S}$ are taken from the space T^2 of all square integrable symmetric tensor fields, the weak solution of (8) can be determined by the simple expression

$$\int_\Omega \boldsymbol{S}:(\mathsf{A}:\boldsymbol{T}-\nabla\boldsymbol{u})\,\mathrm{d}\Omega=0. \tag{9}$$

3.3. Weak formulation of a mixed problem

By the simple summation of (6) and (9) one obtains a new and far-reaching expression (10), which allows *asymmetric* (primal-dual mixed) weak formulation of a mixed problem, associated with the mixed (Reissner's) variational principle:

Find $\boldsymbol{T}\in H(\mathrm{div})$ satisfying $\boldsymbol{T}\boldsymbol{n}|_{\partial\Omega_t}=\boldsymbol{p}$ and $\boldsymbol{u}\in(H^1)^n$ such that $\boldsymbol{u}|_{\partial\Omega_u}=\boldsymbol{w}$ and

$$\int_\Omega(\boldsymbol{S}:\mathsf{A}:\boldsymbol{T}-\boldsymbol{S}:\nabla\boldsymbol{u}+\boldsymbol{v}\cdot\operatorname{div}\boldsymbol{T})\,\mathrm{d}\Omega=-\int_\Omega\boldsymbol{v}\cdot\boldsymbol{f}\,\mathrm{d}\Omega \tag{10}$$

for all $\boldsymbol{S}\in(L_2)^{n\times n}$ and all $\boldsymbol{v}\in(L_2)^n$.

In this expression, $H(\mathrm{div})$ is the space of all symmetric tensorfields which are square integrable and have square integrable divergence [1], while $(H^1)^n$ is the space of all vector fields which are square integrable and have square integrable gradient.

By the use of the divergence theorem over the second integral in the above expression, we obtain the well known [1] symmetric (dual mixed) weak formulation for the mixed problem:

Find $\boldsymbol{T} \in H(\mathrm{div})$ satisfying $\boldsymbol{T}\boldsymbol{n}|_{\partial\Omega_t} = \boldsymbol{p}$ and $\boldsymbol{u} \in (L_2)^n$ such that

$$\int_\Omega (\boldsymbol{S} : \mathbf{A} : \boldsymbol{T} + \operatorname{div} \boldsymbol{S} \cdot \boldsymbol{u} + \boldsymbol{v} \cdot \operatorname{div} \boldsymbol{T})\, \mathrm{d}\Omega = -\int_\Omega \boldsymbol{v} \cdot \boldsymbol{f}\, \mathrm{d}\Omega + \int_{\partial\Omega_u} (\boldsymbol{S}\boldsymbol{n}) \cdot \boldsymbol{w}\, \mathrm{d}\partial\Omega \tag{11}$$

for all $\boldsymbol{S} \in H(\mathrm{div})$ satisfying $\boldsymbol{S}\boldsymbol{n}|_{\partial\Omega_t} = 0$ and all $\boldsymbol{v} \in (L_2)^n$.

Dual formulation is frequently elaborated by mathematicians, but not very popular amongst the finite element practitioners, at least for the two reasons. First, both the stresses $\boldsymbol{T}$ and their variations $\boldsymbol{S}$ are taken from $H(\mathrm{div})$, the space unfamiliar for an average finite element user, and second, the use of the discontinuous displacements from the space $(L_2)^n$ and natural displacement boundary conditions is awkward for the same people.

At variance with the preceding case, application of the divergence theorem over the third term of (10) gives a more popular (especially amongst the engineering-oriented scientists [15]) primal mixed form of a mixed problem:

Find $\boldsymbol{T} \in (L_2)^{n\times n}$ and $\boldsymbol{u} \in (H^1)^n$ such that $\boldsymbol{u}|_{\partial\Omega_u} = \boldsymbol{w}$ and

$$\int_\Omega (\boldsymbol{S} : \mathbf{A}\ \boldsymbol{T} - \boldsymbol{S} : \nabla\boldsymbol{u} - \nabla\boldsymbol{v} : \boldsymbol{T})\, \mathrm{d}\Omega = -\int_\Omega \boldsymbol{v} \cdot \boldsymbol{f}\, \mathrm{d}\Omega - \int_{\partial\Omega_t} \boldsymbol{v} \cdot \boldsymbol{p}\, \mathrm{d}\partial\Omega \tag{12}$$

for all $\boldsymbol{S} \in (L_2)^{n\times n}$ and $\boldsymbol{v} \in (H^1)^n$ such that $\boldsymbol{v}|_{\partial\Omega_u} = 0$.

If the constitutive equations are locally satisfied (see (3) and (7)) the above equation straightforwardly reduces to the primal (displacement) problem:

Find $\boldsymbol{u} \in (H^1)^n$ and such that $\boldsymbol{u}|_{\partial\Omega_u} = \boldsymbol{w}$ and

$$\int_\Omega \boldsymbol{e}(\boldsymbol{v}) : \mathbf{C} : \boldsymbol{e}(\boldsymbol{u})\, \mathrm{d}\Omega = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{f}\, \mathrm{d}\Omega + \int_{\partial\Omega_t} \boldsymbol{v} \cdot \boldsymbol{p}\, \mathrm{d}\partial\Omega \tag{13}$$

for all $\boldsymbol{v} \in (H^1)^n$ such that $\boldsymbol{v}|_{\partial\Omega_{\boldsymbol{u}}} = 0$.

4. Finite element approximations of the field equations

4.1. Classical mixed approach

We let $\mathcal{C}_h$ be the partitioning of Ω into elements Ω_i and define the finite element sets and subspaces for the displacement vector, the stress tensor and the appropriate weight functions respectively as

$$\begin{aligned}
U_h &= \{\boldsymbol{u} \in (H^1(\Omega))^n \mid \boldsymbol{u}|_{\partial\Omega_u} = \boldsymbol{w},\ \ \boldsymbol{u}|_{\Omega_i} = \Psi_U^K \boldsymbol{u}_K,\ \ \forall\Omega_i \in \mathcal{C}_h\}\\
V_h &= \{\boldsymbol{v} \in (H^1(\Omega))^n \mid \boldsymbol{v}|_{\partial\Omega_u} = 0,\ \ \boldsymbol{v}|_{\Omega_i} = \Psi_V^K \boldsymbol{v}_K,\ \ \forall\Omega_i \in \mathcal{C}_h\}\\
T_h &= \{\boldsymbol{T} \in (L_2(\Omega))^{n\times n} \mid \boldsymbol{T}\cdot\boldsymbol{n}|_{\partial\Omega_t} = \boldsymbol{p},\ \ \boldsymbol{T}|_{\Omega_i} = \Psi_{TL}\boldsymbol{T}^L,\ \ \forall\Omega_i \in \mathcal{C}_h\}\\
S_h &= \{\boldsymbol{S} \in (L_2(\Omega))^{n\times n} \mid \boldsymbol{S}\cdot\boldsymbol{n}|_{\partial\Omega_t} = 0,\ \ \boldsymbol{S}|_{\Omega_i} = \Psi_{SL}\boldsymbol{S}^L,\ \forall\Omega_i \in \mathcal{C}_h\}
\end{aligned} \tag{14}$$

In these expressions $\boldsymbol{u}_K$ and $\boldsymbol{T}^L$ are the nodal values of the displacement vector $\boldsymbol{u}$ and stress tensor $\boldsymbol{T}$ respectively. Accordingly, Ψ_U^K and Ψ_{TL} are the corresponding values of the interpolation functions, connecting the displacements and stresses at an arbitrary point in Ω_i (the body of an element), and the nodal values of these quantities. The complete analogy holds for the displacement and stress variations (weight functions) $\boldsymbol{v}$ and $\boldsymbol{S}$ respectively.

4.2. Continous stress and displacement mixed approach

The stresses and stress variations here are taken from a smaller space, the same as in (11) or even more restricted $\boldsymbol{T} \in (H)^{n \times n} \subset H(\mathrm{div}) \subset T^2$, i.e.

$$T_h = \{\boldsymbol{T} \in (H(\Omega))^{n \times n} \mid \boldsymbol{T}|_{\Omega_i} = \Psi_{TL}\boldsymbol{T}^L, \quad \forall \Omega_i \in \mathcal{C}_h\}, \tag{15}$$

$$S_h = \{\boldsymbol{S} \in (H(\Omega))^{n \times n} \mid \boldsymbol{S}|_{\Omega_i} = \Psi_{SN}\boldsymbol{S}^N, \quad \forall \Omega_i \in \mathcal{C}_h\}, \tag{16}$$

Technically, the boundary traction conditions are treated as the natural ones (identically as in the displacement method), while at the interfaces the stress components are disconnected (again as in the displacement method). From the point of view of known finite element techniques, the part of a mesh limited by the boundaries and interfaces of a body under consideration, can be understood as a large composite mixed hybrid element, having continuous stress and displacement continuity at the boundaries.

The advantage of the described approach is in a fact that for the displacements and stresses one can use interpolation functions of the same type, and that for an average finite element user the stress continuity, where appropriate, is an obvious and attractive property of a model.

4.3. Satisfaction of the boundary traction conditions

In this paper we consider the situation when stresses satisfy also the boundary traction conditions. Although this approach is not necessary from the point of view of the variational principle (12), where the stress boundary conditions are satisfied as the natural ones, we will see that, in the finite element equations (26), one can separate known and unknown nodal stress values, and hence also satisfy the stress boundary conditions as the essential ones, i.e. equilibrate the boundary traction's (components of the stress vector).

Hence (only) at the boundaries and interfaces we allow that our $\boldsymbol{T} \in H^1$ behaves as $\boldsymbol{T} \in H(\mathrm{div})$, instead of $\boldsymbol{T} \in T^2$ from the preceding case. Finally, the stress and its variation subspaces will be

$$T_h = \{\boldsymbol{T} \in (L_2)^{n \times n}(\Omega) \mid \boldsymbol{T} \cdot \boldsymbol{n}|_{\partial\Omega_t} = \boldsymbol{p}, \quad \boldsymbol{T}|_{\Omega_i} = \Psi_{TL}(\Omega_i)\boldsymbol{T}^L, \quad \forall \Omega_i \in \mathcal{C}_h\} \tag{17}$$

$$S_h = \{\boldsymbol{S} \in (L_2)^{n \times n}(\Omega) \mid \boldsymbol{S} \cdot \boldsymbol{n}|_{\partial\Omega_t} = 0, \quad \boldsymbol{S}|_{\Omega_i} = \Psi_{SL}(\Omega_i)\boldsymbol{S}^L, \quad \forall \Omega_i \in \mathcal{C}_h\} \tag{18}$$

As it has been already mentioned, where the stress boundary conditions were satisfied in an iterative manner, it has been shown that such an approach can significantly improve results.

4.4. Compact matrix form of the finite element equations

As it has been shown in [5], the finite element equations based on (12) can be written in a symbolic matrix form as

$$\begin{bmatrix} \mathbf{A} & -\mathbf{D} \\ -\mathbf{D}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{u} \end{bmatrix} = - \begin{bmatrix} \mathbf{0} \\ \mathbf{f} + \mathbf{p} \end{bmatrix} \tag{19}$$

In this expressions $\mathbf{t}$ and $\mathbf{u}$ are the column matrices of the stress and displacement components nodal values. The members of the matrices $\mathbf{A}$ and $\mathbf{D}$ and of the vectors (column matrices) $\mathbf{f}$ and $\mathbf{p}$ (discretized body and surface forces) are, respectively,

$$A_{\Lambda uv\Gamma st} = \sum_e \int_{\Omega_i} \Omega_\Lambda^N \Psi_{SN}\, g^a_{(\Lambda)u}\, g^b_{(\Lambda)v}\, \mathrm{A}_{abcd}\, g^c_{(\Gamma)s}\, g^d_{(\Gamma)t}\, \Psi_{TL} \Omega_\Gamma^L \,\mathrm{d}\Omega \tag{20}$$

$$D^{\Gamma q}_{\Lambda uv} = \sum_e \int_{\Omega_i} \Omega_\Lambda^N \Psi_{SN} \Psi^K_{Ua}\, g^a_{(\Lambda)u}\, g^{(\Gamma)q}_{(\Lambda)v} \,\mathrm{d}\Omega \tag{21}$$

$$f^{\Lambda q} \quad = \sum_{e} \int_{\Omega_i} g_a^{(\Lambda)q}\, \Omega_M^{\Lambda}\, \Psi_V^M\, f^a \,\mathrm{d}\Omega \tag{22}$$

$$p^{\Lambda q} \quad = \sum_{e} \int_{\Omega_{it}} g_a^{(\Lambda)q}\, \Omega_M^{\Lambda}\, \Psi_V^M\, p^a \,\mathrm{d}\partial\Omega \tag{23}$$

where

$$g_{(L)s}^{(K)m} = \delta_{kl}\, g^{(K)mn} \frac{\partial z^k}{\partial x^{(K)n}} \frac{\partial z^l}{\partial y^{(L)s}}\,, \tag{24}$$

$$g_{(L)s}^{a} = \delta_{kl}\, g^{ab} \frac{\partial z^k}{\partial \xi^b} \frac{\partial z^l}{\partial y^{(L)s}}\,, \tag{25}$$

$$g_b^{(K)q} = \delta_{kl}\, g^{(K)qp} \frac{\partial z^k}{\partial \xi^b} \frac{\partial z^l}{\partial x^{(K)p}}\,, \tag{26}$$

are the Euclidean shifters. In this paper we will consider isotropic elastic examples where the compliance tensor components are of the form

$$\mathsf{A}_{abcd} = \frac{1}{2E}\left[(1+\nu)(g_{ca}g_{db} + g_{cb}g_{da}) - 2\nu g_{ab}g_{cd}\right] \tag{27}$$

In the above expressions z^i $(i,j,k,l = 1,2,3)$ are the global Cartesian coordinates, while $x^{(K)n}$ $(m,n,p,q = 1,2,3)$ and $y^{(L)s}$ $(r,s,t,u,v = 1,2,3)$ are local (nodal) coordinates, used for determination of the nodal displacements and stresses respectively. Commonly used notions, ξ^a $(a,b,c,d = 1,2,3)$ are taken for the local (element) coordinates, usualy convected (parametric, isoparametric). Further, $g^{(K)mn}$ and g^{ab} are the components of the contravariant fundamental metric tensors, the first one with respect to $x^{(K)n}$ and the second to ξ^b. Computation of these quantities is described in detail per instance in [6]. Furthermore, $U_a^K = \partial U^K/\partial \xi^a$. Also, A_{abcd} are the components of the elastic compliance tensor A, while f^a and p^a are the body forces and boundary traction's, respectively. Integration is performed over the domain Ω_i of each element, or over the part of the boundary surface $\partial\Omega_{it}$ where the traction are given, while summation is over all the e elements of a system. Finally, Ω_M^Λ is a connectivity operator between global nodes and element nodes.

Because the tensorial character of (12) is fully respected, one can easily choose at each global node different coordinate systems for the stresses and/or displacements, for the most convenient application of boundary conditions and interpretation of output results.

It has been shown in [5] that (14) can be decomposed for unknown (variable) and known (prescribed) values of the stresses and displacements denoted by the indices v and p, respectively,

$$\begin{bmatrix} \mathbf{A}_{vv} & -\mathbf{D}_{vv} \\ -\mathbf{D}_{vv}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{t}_v \\ \mathbf{u}_v \end{bmatrix} = \begin{bmatrix} -\mathbf{A}_{vp} & \mathbf{D}_{vp} \\ \mathbf{D}_{pv}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{t}_p \\ \mathbf{u}_p \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_p + \mathbf{p}_p \end{bmatrix}. \tag{28}$$

In these expressions the nodal stresses T^{Lst} and displacements u_{Kq} components are consecutively ordered in the column matrices $\mathbf{t}$ and $\mathbf{u}$ respectively.

4.5. Triangular elements

In this subsection, and next all displacement degrees of freedom at node, i.e., u_α or $\{u_1, u_2\}$ are considered to be active and the *displacement node* is denoted by circle. The *stress nodes* with all three stress components $T^{\alpha\beta}$ or $\{T^{11}, T^{12}, T^{22}\}$ active are denoted by the rectangular triangle. If only one stress component is active, the stress node is denoted by the appropriate dash, i.e., — $\{T^{11},0,0\}$, \ $\{0,T^{12},0\}$, | $\{0,0,T^{22}\}$. At the midside nodes it has been taken that T^{11} component is parallel to the boundary, while T^{22} is orthogonal to it. The expressions for the interpolation functions, corresponding to the element nodes are given in Appendix A.

1. Triangular element TC 3/4 (MINI element [1])

$$T^{\alpha\beta} = \sum_{i=1}^{3} \Psi_i T_i^{\alpha\beta} + \Psi_7 T_7^{\alpha\beta} \tag{29}$$

2. TC 3/6

$$T^{\alpha\beta} = \sum_{i=1}^{6} \Psi_i T_i^{\alpha\beta} \tag{30}$$

3. TC 3/6r

$$T^{\alpha\beta} = \sum_{i=1}^{6} \Psi_i T_i^{\alpha\beta} \tag{31}$$

This expresion is finally the same as (30). However, from Fig. 3 it is evident that at midside nodes only the stress component parallel to the boundary is retained, and hence three components $T^{\alpha\beta}$ at each of nodes 4, 5, and 6 are mutually linearly dependent.

4.6. Quadrilateral elements

1. QC 4/5

$$T^{\alpha\beta} = \sum_{i=1}^{4} \Psi_i T_i^{\alpha\beta} + \Psi_9 T_9^{\alpha\beta} \tag{32}$$

2. QC 4/7

$$T^{\alpha\beta} = \sum_{i=1}^{4} \Psi_i T_i^{\alpha\beta} + \sum_{j=9}^{11} \Psi_j T_j^{\alpha\beta} \tag{33}$$

3. $QC4/7r$

$$\begin{aligned} T^{11} &= \sum_{i=1}^{4} \Psi_i T_i^{11} + \Psi_9 T_9^{11} + \Psi_{11} T_{11}^{11} \\ T^{12} &= \sum_{i=1}^{4} \Psi_i T_i^{12} + \Psi_9 T_9^{12} \\ T^{22} &= \sum_{i=1}^{4} \Psi_i T_i^{22} + \Psi_9 T_9^{22} + \Psi_{10} T_{10}^{22} \end{aligned} \tag{34}$$

4. QC 4/9

$$T^{\alpha\beta} = \sum_{i=1}^{9} \Psi_i T_i^{\alpha\beta} \tag{35}$$

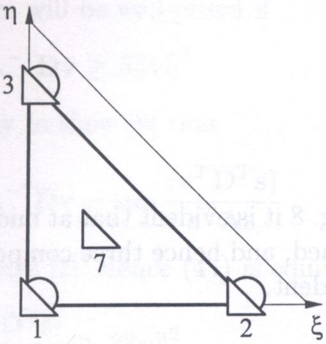

Fig. 1. Element TC 3/4

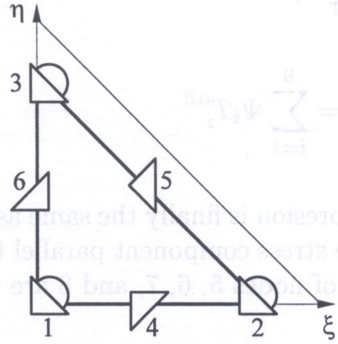

Fig. 2. Element TC 3/6

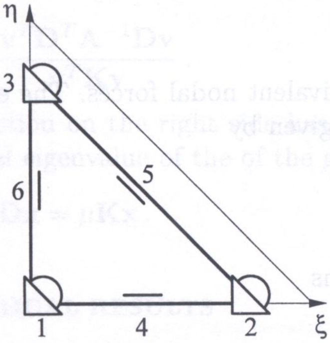

Fig. 3. Element TC 3/6r

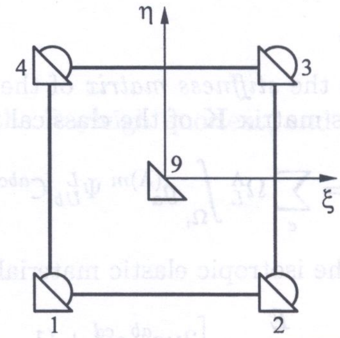

Fig. 4. Element QC 4/5

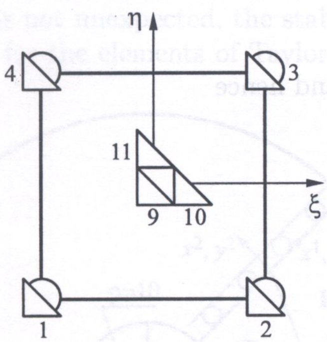

Fig. 5. Element QC 4/7

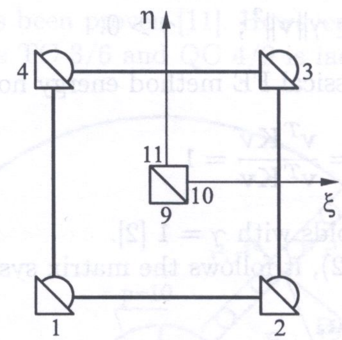

Fig. 6. Element QC 4/7r

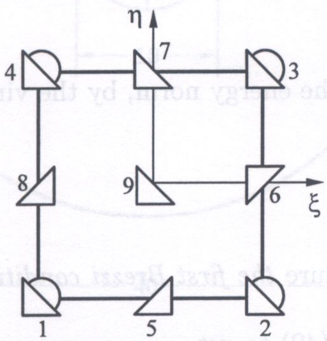

Fig. 7. Element QC 4/9

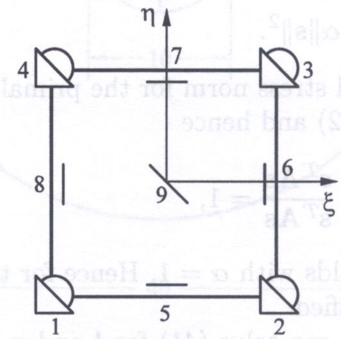

Fig. 8. Element QC 4/9r

5. QC 4/9r

$$T^{\alpha\beta} = \sum_{i=1}^{9} \Psi_i T_i^{\alpha\beta} \tag{36}$$

This expresion is finally the same as (35). However, from Fig. 8 it is evident that at midside nodes only the stress component parallel to the boundary is retained, and hence three components $T^{\alpha\beta}$ at each of nodes 5, 6, 7, and 8 are mutually linearly dependent.

5. Some algebraic implications of the Brezzi conditions (inf–sup test)

Let start from the classical FE formulation (13), leading to the matrix equation

$$\mathbf{K}\mathbf{u} = \mathbf{r}, \tag{37}$$

where $\mathbf{K}$ is the *stiffness matrix* of the problem and $\mathbf{r}$ the equivalent nodal forces. The elements of the stiffness matrix $\mathbf{K}$ of the classical primal formulation are given by

$$K^{\Lambda m \Gamma n} = \sum_e \Omega_L^\Lambda \int_{\Omega_i} g_a^{(\Lambda)m}\, \Psi_{Ub}^L\, \mathsf{C}^{abcd}\, \Psi_{Ud}^K\, g_c^{(\Gamma)n}\, d\Omega\, \Omega_K^\Gamma \tag{38}$$

where for the isotropic elastic material in plane stress problems

$$\mathsf{C}^{abcd} = \frac{E}{2\left(1-\nu^2\right)}\left[2\nu g^{ab} g^{cd} + (1-\nu)\left(g^{ac} g^{bd} + g^{ad} g^{bc}\right)\right]. \tag{39}$$

This system is well-posed provided the bilinear form $\mathbf{v}^T\mathbf{K}\mathbf{v}$ is *coercive*, i.e.

$$\mathbf{v}^T\mathbf{K}\mathbf{v} \geq \gamma \|\mathbf{v}\|^2; \qquad \gamma > 0. \tag{40}$$

For the classical FE method energy norm is a natural norm, and hence

$$\frac{\mathbf{v}^T\mathbf{K}\mathbf{v}}{\|\mathbf{v}\|^2} = \frac{\mathbf{v}^T\mathbf{K}\mathbf{v}}{\mathbf{v}^T\mathbf{K}\mathbf{v}} = 1$$

and (40) holds with $\gamma = 1$ [2].

From (12), it follows the matrix system

$$\mathbf{A}\mathbf{t} = \mathbf{D}\mathbf{u}, \tag{41}$$

$$\mathbf{D}^T\mathbf{t} = \mathbf{r}. \tag{42}$$

Equation (41) will be well posed if

$$\mathbf{s}^T\mathbf{A}\mathbf{s} \geq \alpha \|\mathbf{s}\|^2. \tag{43}$$

The natural stress norm for the primal mixed method is also the energy norm, by the virtue of the principle (12) and hence

$$\frac{\mathbf{s}^T\mathbf{A}\mathbf{s}}{\|\mathbf{s}\|^2} = \frac{\mathbf{s}^T\mathbf{A}\mathbf{s}}{\mathbf{s}^T\mathbf{A}\mathbf{s}} = 1,$$

and (43) holds with $\alpha = 1$. Hence for the primal-mixed procedure *the first Brezzi condition* (43) is always satisfied.

Now one can solve (41) for $\boldsymbol{t}$ and substitute the result into (42) to get

$$\mathbf{D}^T\mathbf{A}^{-1}\mathbf{D}\mathbf{u} = \mathbf{r}. \tag{44}$$

This system will be well-posed if

$$\mathbf{v}^T\mathbf{D}^T\mathbf{A}^{-1}\mathbf{D}\mathbf{v} \geq \beta\|\mathbf{v}\|^2. \tag{45}$$

It is easy to show [9] that

$$\mathbf{v}^T\mathbf{D}^T\mathbf{A}^{-1}\mathbf{D}\mathbf{v} = \sup \frac{(\mathbf{v}^T\mathbf{D}^T\mathbf{s})}{\mathbf{s}^T\mathbf{A}\mathbf{s}}$$

and vice versa [2]. Hence (44) is equivalent to

$$\sup \frac{(\mathbf{v}^T\mathbf{D}^T\mathbf{s})}{\mathbf{s}^T\mathbf{A}\mathbf{s}} \geq \beta\|\mathbf{v}\|^2, \tag{46}$$

the discrete *LBB (second Brezzi) condition.* For the primal mixed method (12) the natural displacement norm is also the energy norm, and hence from (44) it follows that

$$\beta = \inf \frac{\mathbf{v}^T\mathbf{D}^T\mathbf{A}^{-1}\mathbf{D}\mathbf{v}}{\mathbf{v}^T\mathbf{K}\mathbf{v}}. \tag{47}$$

The fraction on the right side has evidently the form of the Rayleigh quotient, and hence β is the minimal eigenvalue of the of the generalized problem

$$\mathbf{D}^T\mathbf{A}^{-1}\mathbf{D}\mathbf{x} = \mu\mathbf{K}\mathbf{x}. \tag{48}$$

6. Numerical results

For the purpose of stability analysis in this paper we consider the problem of a thin uniformly loaded ring (Lamé, problem – Figs. 9, 10). The results shown in the Figs. 11, 12 indicate that the elements analyzed can be stable, with the exception of QC 4/5. As concerns TC 3/4 MINI element this result is not unexpected, the stability of this element has been proven [11]. However the proof of stability for the elements of Taylor–Hood type, in our case TC 3/6 and QC 4/9 is lacking. The

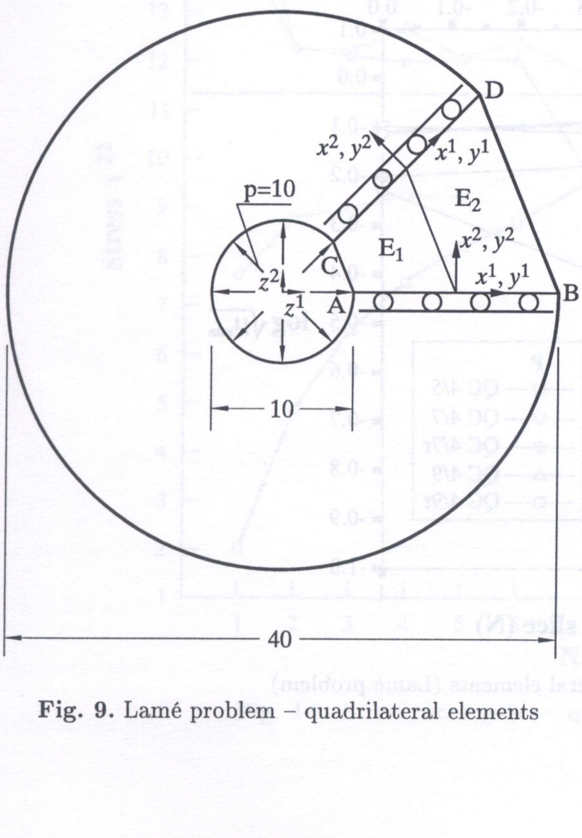

Fig. 9. Lamé problem – quadrilateral elements

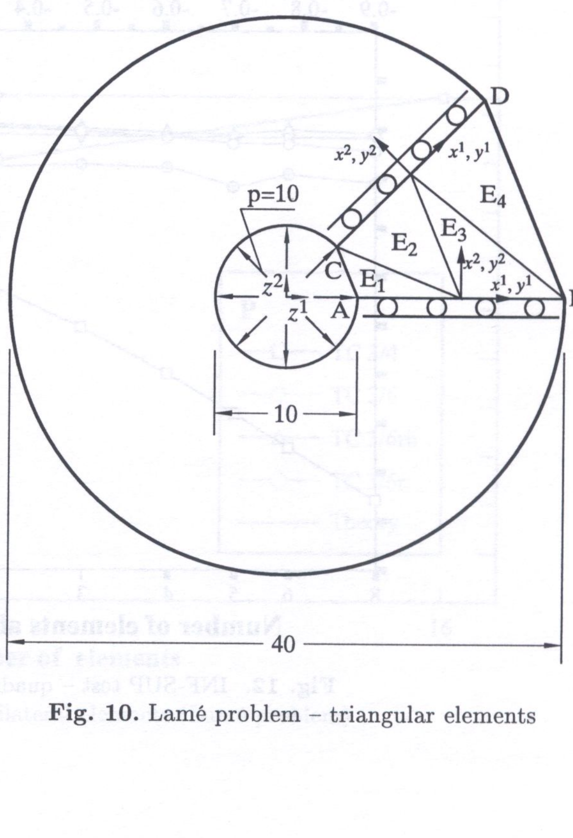

Fig. 10. Lamé problem – triangular elements

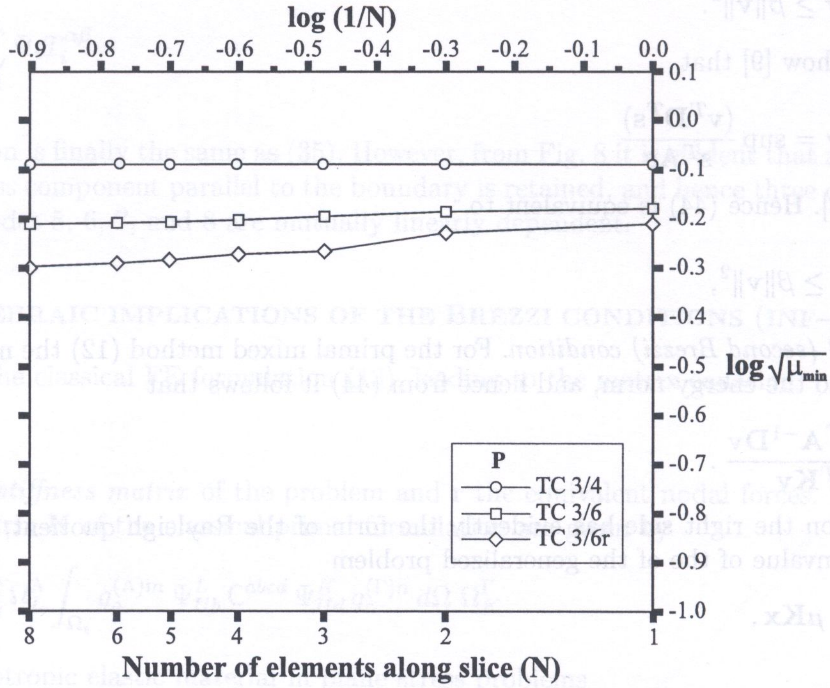

Fig. 11. INF-SUP test – triangular elements (Lamé problem)

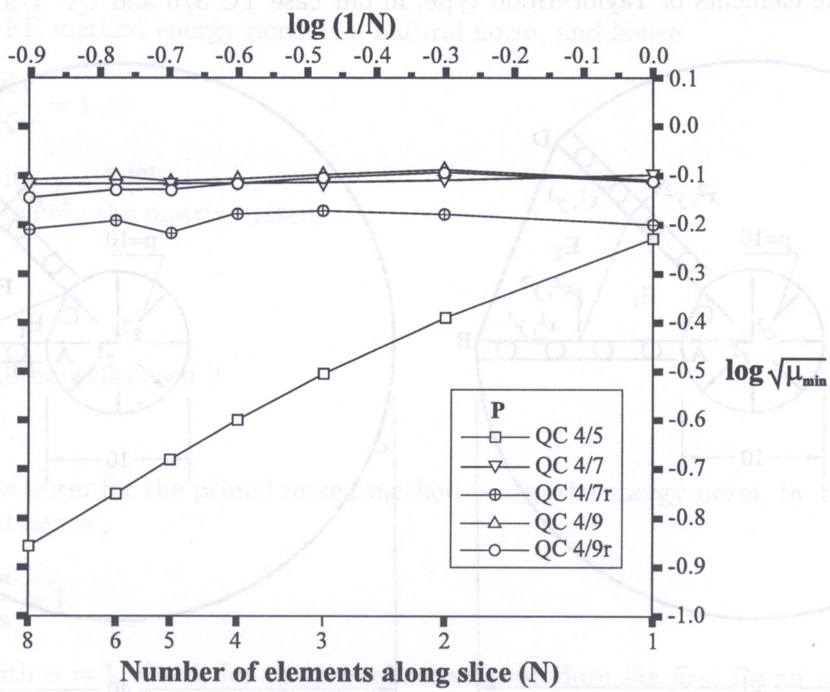

Fig. 12. INF-SUP test – quadrilateral elements (Lamé problem)

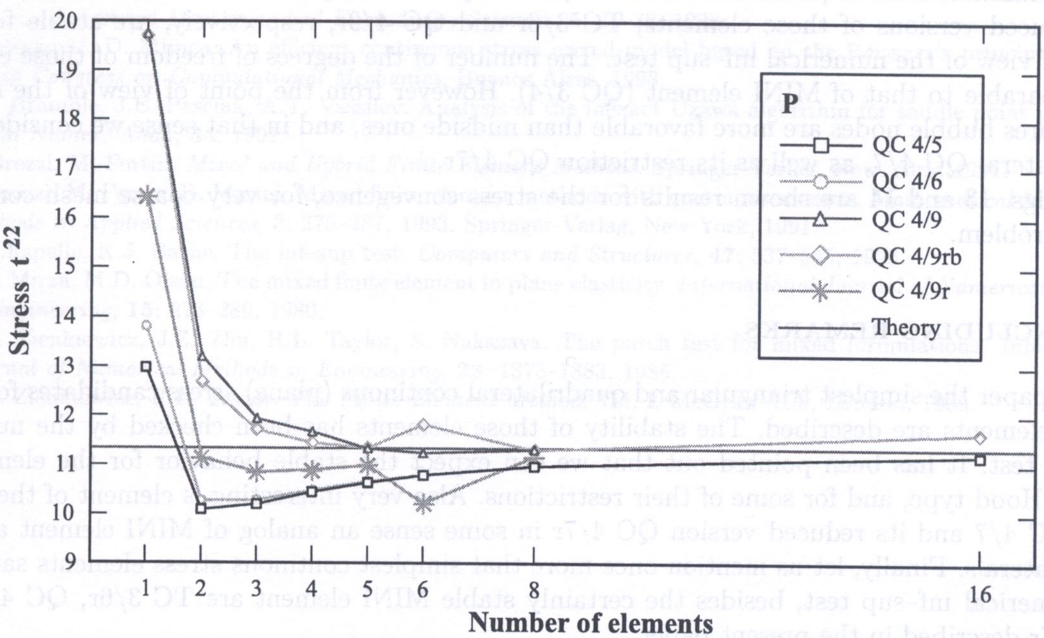

Fig. 13. Stress convergence – triangular elements (Lamé problem)

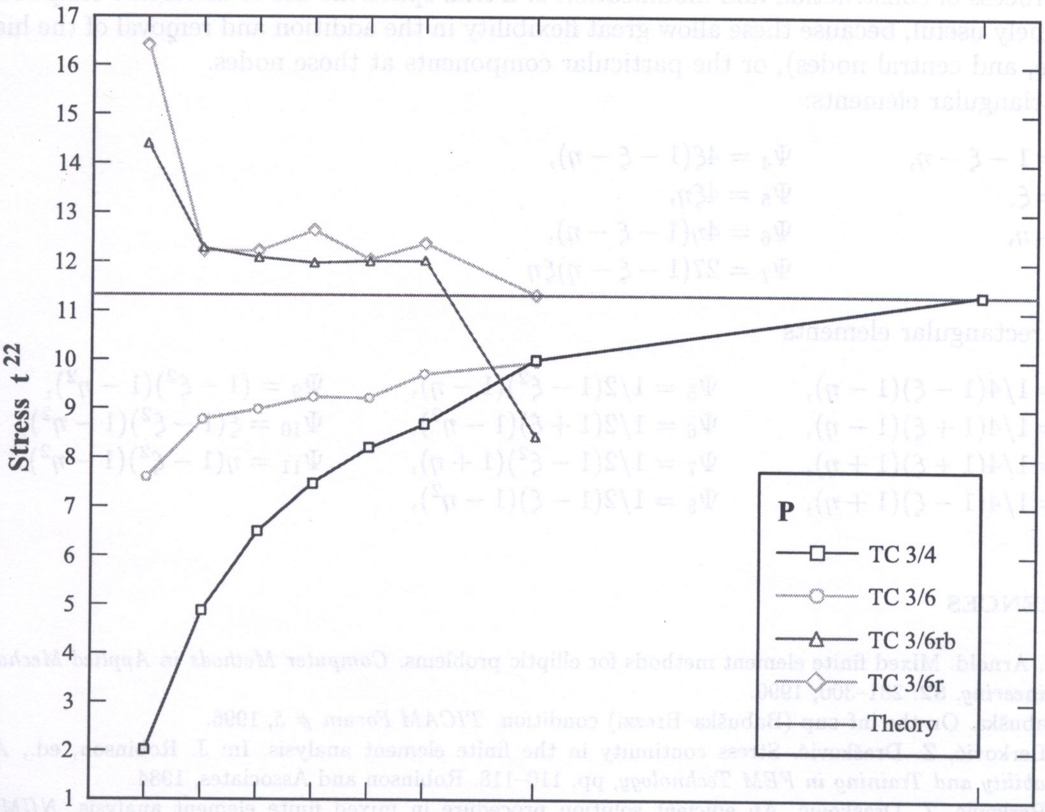

Fig. 14. Stress convergence – quadrilateral elements (Lamé problem)

present numerical inf–sup tests show that we can expect stable behavior of those elements. Even the reduced versions of those elements, TC 3/6r and QC 4/9r, respectively, are stable from the point of view of the numerical inf–sup test. The number of the degrees of freedom of those elements is comparable to that of MINI element (QC 3/4). However from the point of view of the solution procedures bubble nodes are more favorable than midside ones, and in that sense we considered the quadrilateral QC 4/7, as well as its restriction QC 4/7r.

In Figs. 13 and 14 are shown results for the stress convegence, for very coarse mesh considered Lamé problem.

7. Concluding remarks

In this paper the simplest triangular and quadrilateral continous (plane) stress candidates for stable mixed elements are described. The stability of those elements has been checked by the numerical inf–sup test. It has been pointed out that we can expect the stable behavior for the elements of Taylor–Hood type, and for some of their restrictions. Also very interesting is element of the bubble type QC 4/7 and its reduced version QC 4/7r in some sense an analog of MINI element amongst quadrilaterals. Finally, let us mention once more that simplest continous stress elements satysfying the numerical inf–sup test, besides the certainly stable MINI element are TC 3/6r, QC 4/9r and QC 4/7r described in the present paper.

Appendix A. Hierarchic shape functions

In the process of construction and modification of a trial space the use of hierarchic shape functions is extremely useful, because these allow great flexibility in the addition and removal of the hierarchic (midside, and central nodes), or the particular components at those nodes.

For triangular elements:

$$\begin{aligned}
&\Psi_1 = 1-\xi-\eta, && \Psi_4 = 4\xi(1-\xi-\eta),\\
&\Psi_2 = \xi, && \Psi_5 = 4\xi\eta,\\
&\Psi_3 = \eta, && \Psi_6 = 4\eta(1-\xi-\eta),\\
& && \Psi_7 = 27(1-\xi-\eta)\xi\eta
\end{aligned} \tag{49}$$

and for rectangular elements

$$\begin{aligned}
&\Psi_1 = 1/4(1-\xi)(1-\eta), && \Psi_5 = 1/2(1-\xi^2)(1-\eta), && \Psi_9 = (1-\xi^2)(1-\eta^2),\\
&\Psi_2 = 1/4(1+\xi)(1-\eta), && \Psi_6 = 1/2(1+\xi)(1-\eta^2), && \Psi_{10} = \xi(1-\xi^2)(1-\eta^2),\\
&\Psi_3 = 1/4(1+\xi)(1+\eta), && \Psi_7 = 1/2(1-\xi^2)(1+\eta), && \Psi_{11} = \eta(1-\xi^2)(1-\eta^2).\\
&\Psi_4 = 1/4(1-\xi)(1+\eta), && \Psi_8 = 1/2(1-\xi)(1-\eta^2), &&
\end{aligned} \tag{50}$$

References

[1] D.N. Arnold. Mixed finite element methods for elliptic problems. *Computer Methods in Applied Mechanics and Engineering*, **82**: 281–300, 1990.

[2] I. Babuška. On the inf–sup (Babuška–Brezzi) condition. *TICAM Forum # 5*, 1996.

[3] M. Berković, Z. Drašković. Stress continuity in the finite element analysis. In: J. Robinson, ed., *Accuracy, Reliability and Training in FEM Technology*, pp. 110–118. Robinson and Associates, 1984.

[4] M. Berković, Z. Drašković. An efficient solution procedure in mixed finite element analysis. *NUMETA 85*, pp. 625–633. Balkema, Rotterdam, 1985.

[5] M. Berković, Z. Drašković. On the essential mechanical boundary conditions in two-field finite element approximations. *Computer Methods in Applied Mechanics and Engineering*, **91**: 1339–1355, 1991.

[6] M. Berković, Z. Drašković. A two-field finite element model related to the Reissner's principle. *Theoretical and Applied Mechanics*, **20**: 17–36, 1994.

[7] M. Berković, Z. Drašković, D. Mijuca. A direct block sparse solution of the mixed finite element equations. *Computer Assisted Mechanics and Engineering Sciences*, **5**: 21–30, 1998.

[8] M. Berković, D. Mijuca. An efficient continuous stress mixed model based on the Reissner's principle. *Fourth World Congress on Computational Mechanics*, Buenos Aires, 1998.

[9] J.H. Bramble, J.E. Pasciak, A.T. Vassilev. Analysis of the inexact Uzawa algorithm for saddle point problems. *SIAM Numer. Anal.*, **34**, 1997.

[10] F. Brezzi, M. Fortin. *Mixed and Hybrid Finite Element Methods*. Springer-Verlag, New York, 1991.

[11] F. Brezzi, M. Fortin, D. Marini. Mixed finite element methods with continous stresses. *Mathematical Models and Methods in Applied Sciences*, **3**: 275–287, 1993. Springer-Verlag, New York, 1991.

[12] D. Chapelle, K.J. Bathe. The inf–sup test. *Computers and Structures*, **47**: 537–545, 1993.

[13] F.A. Mirza, M.D. Olson. The mixed finite element in plane elasticity. *International Journal of Numerical Methods in Engineering*, **15**: 273–289, 1980.

[14] O.C. Zienkiewicz, J.Z. Zhu, R.L. Taylor, S. Nakazava. The patch test for mixed formulations. *International Journal of Numerical Methods in Engineering*, **23**: 1873–1883, 1986.

[15] O.C. Zienkiewicz, R.L. Taylor. *The Finite Element Method*, Vol. I. McGraw–Hill, London, 1989.