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Milestones in the 150-Year History of Topology Optimization: A Review

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Structural optimization is one of the most intensively investigated research areas in engineering. Recently, topology optimization has become the most popular engineering subfield. The starting date of structural optimization cannot be precisely determined. Michell's optimization paper, published in 1904, is considered as the first publication in this subfield. However, his paper starts with a statement that his work is a generalization of Maxwell's idea presented in the paper published in 1870.

The authors of this review paper consider that this date can be accepted as the starting date of topology optimization. This paper is an overview of subjectively selected state-of-art achievements in topology optimization during its history of 150 years. The selection of the achievements is a rather difficult task because, in the early period of the history of topology optimization, a lot of meetings were classified and the results were not available for the public. The optimization community has almost no knowledge about the publications in topology optimization in the 1950s. Around that time, one can find some information on workshops and meetings connected to the Cambridge University or Oxford University with researchers such as Foulkes, Cox, Hemp, and Shield, who published significant results and these communications are generally not known for the reason mentioned above. After the 1970s, this situation has changed and there were more possibilities to find publications due to the changes and thanks to digitalization. As indicated earlier here subjectively selected works are overviewed from the 150-year history focusing on the first hundred twenty years.

Keywords: topology optimization, optimal layout, optimality criteria method, level set method, heuristic optimal design.

1. Introduction

Topology optimization is a very complex computational procedure because it includes the elements of the size and shape optimization. In the case of skeletal structures, it integrates the layout optimization and the optimal cross-section design simultaneously. This paper is an extended, revised version of the workshop presentation of the first author [1] and includes some overviews presented in his earlier papers [2–4]. In addition, this paper incorporates the overview of the early stage of topology design based on the work of Hemp [5].

Usually, the majority of the papers cite the work of Michell [6] from 1904 as the first in topology optimization, but in reality, the first important work was written by Maxwell [7]. This can be derived from the first sentences written by Michell in his paper, which indicate that Maxwell's result was "only" extended by Michell [6] 34 years later. They presented closed- form solutions for the minimum volume structures.

Before the history of topology optimization is discussed, one has to mention two important theories in mathematical programming [8]: the Farkas lemma for linear programming [9] and the Karush–Khun–Tucker conditions for nonlinear programming [10, 11]. In addition, the basic idea of multicriteria mathematical programming presented by Pareto [12] also plays a pivotal role. Without these theorems, the numerical procedures in topology optimization could not develop so effectively and rapidly.

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The authors of this review paper consider that this date can be accepted as the starting date of topology optimization. This paper is a review of subjectively selected state-of-art achievements during 150 years of topology optimization. The selection of the achievements in this field is a rather difficult task because in the early period of the topology optimization history many meetings were classified and the results were not available for the public. The optimization community has almost no knowledge about the publications in topology optimization in the 1950s. Around that time, one can find some information on workshops and meetings connected to the Cambridge University or Oxford University with researchers such as Foulkes, Cox, Hemp, and Shield, who published significant results and these communications are generally not known for the reason mentioned above. After the 1970s of the 20th century, this situation has changed and there were more possibilities to find publications due to the changes and thanks to the digitalization. As indicated earlier, here subjectively selected works are overviewed from the optimization topology history focusing on the first 120 years.

Maxwell [7] in 1870, Cilley [13] in 1900 and Michell [6] in 1904 used the principle of full stress design (FSD) criteria in the case of statically determinate structures and obtained the same results as the minimum weight design. It is necessary to mention that for the final result the procedure took one iteration only. They concluded that FSD was the same as minimum weight design for statically determinate structures.

According to the literature research on the theorem of the optimal design, the topology design remained unnoticed for some fifty years until the publication of the papers of Foulkes [14], Cox [15–17] and Hemp [5, 18, 19]. The years around 1955 can be called the "golden age" of the optimal layout design of trusses. Independently from the English school, one can find several papers that can be named as original ideas of different topology branches. Among others, the papers written by Sved [20] and Barta [21] are the most significant ones. These two works have a lot of similarities in contents and conclusions. Barta's paper discussed the minimum volume design of plane and space structures (trusses). He proved the following theorem: "by removing a given number of properly chosen redundant bars from a given network, it was possible to obtain such a statically determinate structure, which yields a structure with the least weight". It has to be noted that this conclusion was stated for single and deterministic loads as well. Barta also concluded that "the proof did not guarantee that only the statically determinate structure could be the least weight solution". Sved's paper can be considered as the origin of the stress limited and minimum volume design. It is also important to note that the minimum weight designs of different types of structures were studied by Drucker and Shield [22], Mróz [23], Prager and Shield [24] and their results are significant to understand the optimality in the case of complex problems.

The true surge in layout theory research was in the 1960s and 1970s. In the 1960s, the significant publications helped to derive optimality conditions for minimum volume designs. Shield [25] presented optimum design methods for multiple loading. He used variational principles to prove the optimality. In 1960 Schmit [26] applied FSD to statically indeterminate structures and found that FSD provides the exact optimum in a single sizing operation for statically determinate structures where the internal forces remain constant during resizing. However, for indeterminate structures, the number of resizing iterations can vary from a few to many as a function of the sensitivity of internal forces to changes in member sizes. The reason why these different results were obtained is known now: it mainly depends on the redundancy of structures or in other words, it depends on whether the internal forces remain almost constant during resizing, as it happens in most well designed practical structures. Lansing *et al.* [27] applied FSD to statically indeterminate structures. This procedure was used to design wing and empennage structures. In 1966 Gellatly and Gallagher [28] suggested that FSD should be used to create a "starting point" of nonlinear programming methods. Furthermore, some important remarks on FSD were reported by Gallagher [29], who pointed out that FSD was inadequate for minimum weight design. Berke and Khot [30] concluded that minimum weight design should be at the point including "fully stressed elements, lower bound elements, and neither of them". During these years, Cox [15–17], Hemp [5, 18, 19], Prager and Shield [24], Prager and Taylor [31], Prager [32–35], Prager and Rozvany [36] elaborated several theories that can be named as the origins of the exact structural topologies. Nagtegaal and Prager [37] investigated the optimal truss layout theory in the case of alternative loads. Prager [33] derived an optimality condition for beams and frames subjected to alternating loading using the Foulkes mechanism. His results were based on the extension of the optimality conditions presented by Chan [38]. Prager and Rozvany [36] extended the existing optimal layout theory originally used for low volume fraction to grid-like structures (trusses, grillages, shell-grids, etc.). The method was validated for the case of not restricting to low volume fraction structures. Rozvany *et al.* [39] provided a solution on exact optimal topologies of perforated plates. The field of truss topology design was investigated with significant findings by Achtziger [40, 41] and Achtziger and Stolpe [42, 43]. In layout and size optimization, the three-bar truss example was investigated by several authors (e.g., Save [44]) due to the complexity of the problem. This problem was also investigated by Sokół and Lewiński [45] several years later.

The origins of the numerical solution technique of the constrained optimality criteria (COC) methods were presented by Berke and Khot [30] in 1974. This provided the mathematical background of an effective solution technique in topology optimization. The first numerical procedure for finite element (FE) based topology optimization was elaborated by Rossow and Taylor [46] in 1973, but its biggest development started at the end of the 1980s represented by works of Bendsøe and Kikuchi [47, 48] and Rozvany [49, 50].

Almost two decades later, the probabilistic topology optimization was a new direction in this field. In the first two decades of the 21st century Kharmanda *et al.* [51] published a reliability-based topology paper. Later, Lógó *et al.* [52, 54], Lógó [53, 55], Dunning *et al.* [56, 57], Guest and Igusa [58], Csébfalvi [59, 60], Csébfalvi and Lógó [61] introduced some efficient and accurate approaches to probabilistic and/or robust structural topology optimization. Generally, the objective was to minimize the expected compliance or volume with uncertainty in loading magnitude and applied direction, where uncertainties are assumed normally distributed values and statistically independent. This approach is analogous to a multiple load case problem where load cases and weights are derived analytically to accurately and efficiently compute the expected compliance and sensitivities.

In the following sections, some selected achievements are presented in more detail. This paper is divided into three parts: first, some selected milestones are presented from 1870–1940. Secondly, the period of 1940–1988 is discussed and, then, the last 30 years are briefly reviewed. The more detailed overviews are concentrated on publications published in the first hundred twenty years.

2. Fundamental results in the first period (1870–1940)

The topology design started with the problem class of layout optimization of trusses and the first work was called a minimum volume design of frames. As it was indicated earlier, the first optimal solution was elaborated by Maxwell in 1870 and was later extended by Michell [6] in 1904. It was almost unknown for the optimization community that Michell [6] started his paper by referring to Maxwell's achievements [7]. He determined the optimal layout of a truss for a single load case when the absolute value of the axial stress in any bar was not to exceed a given limit. His solution and the design condition have received a lot of attention during the 20th century, but the optimal layout of a truss for alternative loads seems to get less attention. Somehow the publications in this topic have remained hidden. Here a brief overview is also presented. It has to be noted that a Michell truss is statically determinate and the problem class can be handled as one in either elastic design or limit design. However, the two methods no longer lead to the same result when alternative loading or multiple loading cases have to be considered.

Next, we have to mention the works of Kazinczy – the inventor of the plastic hinge theory [62] in 1914 – who produced significant results in this field but unfortunately his publications have remained unexplored. He investigated the volume minimization of trusses as an object of the economical design. Kazinczy was also among the first researchers who investigated the problem of the statically indeterminate trusses in the case of multiple load conditions [63]. Using the Cremona-type solution procedure, he investigated the case of the pre-stressing technique to reach the uniform collapse of the member forces in the case of statically indeterminate structures. With this technique, he used the shakedown theory without naming it, much earlier than Melan [64] published his work on it in 1936. This Cremona-type solution technique has recently been used again as a numerical procedure in topology design [65]. Kazinczy also discussed the questions of safety and reliability designs much earlier than anybody else in the world.

Returning to Michell structures and theories connected to them, the recent book of Lewiński *et al.* [66] is a very complex and extensive study with new findings on this topic.

The overview of the works mentioned above is based on Hemp's presentation [5] in 1958 and the first author's research papers [2–4].

2.1. Maxwell's theorem on minimum volume design

The first important work in truss optimization was presented by Maxwell [7]. He proved a theorem about the equilibrium of a series of attracting repelling centers of force and applied it to trusses (in his original words: to frame structure) in which the bars replaced the action at a distance except in the case of external forces. Maxwell commented on the scientific significance of his theorem by using of the following words: "The importance of the theorem to the engineer arises from the circumstance that the strength of a piece is in general proportional to its section, so that if the strength of each piece is proportional to the stress which it has to bear, its weight will be proportional to the product of stress multiplied by the length of the piece. Hence these sums of products give an estimate of the total quantity of material which must be used in sustaining tension and pressure respectively." We have to notice that Maxwell used the word "stress" for what we should term "load". His result or comment has drawn the practical conclusion about the required weight of the truss.

Maxwell's problem can be described as follows: consider a truss which maintains equilibrium with a set of forces $\overline{F}_i$ acting at the points $\overline{r}_i$, $(i = 1, 2, ..., n)$. Denote by T_t the load carried in a typical tension member with length L_t and the section area A_t while in the case of typical compression members these variables are T_c, L_c and A_c. The permissible stresses were denoted by f_t and f_c, respectively. By the use of the principle of virtual work, Maxwell derived the optimality condition of the lightest structure, which has the volume given by:

$$V = V_c\left(1 + \frac{f_c}{f_t}\right) + \frac{1}{f_t}\sum_i \overline{F}_i\overline{r}_i = V_t\left(1 + \frac{f_t}{f_c}\right) - \frac{1}{f_c}\sum_i \overline{F}_i\overline{r}_i. \quad (1)$$

Here V_t is the volume of all the tension members and V_c is the volume of all the compression members.

2.2. Michell's formulation on minimum volume design

In contradiction to the common knowledge, Michell "only" generalized Maxwell's theorem and did not invent the theory of topology optimization. He recognized the importance of Maxwell's result and applied it to calculate the optimum structural weight. This led him to determine sufficient conditions for a structure to be an optimum. He proved the geometric restriction that determines the classes of orthogonal sets of curves along which the members of an optimum structure must lie.

The Michell problem can be described as follows: in addition to Maxwell's problem above, let us consider a series of external forces $\overline{F}_i$ acting at the points $\overline{r}_i$, $(i = 1, 2, ..., n)$. Let D be a domain of space containing the points $\overline{r}_i$, (it should be noted that D can be the whole of feasible space). Consider then all possible frameworks (trusses) S, contained in D, which equilibrate the forces $\overline{F}_i$ and satisfy the limiting conditions on stresses. Let us assume that there is a framework S^* that satisfies the following condition of Michell: "There exists a virtual deformation of the domain D such that the strain along all members of S^* is equal to $\pm e$, where e is a small positive number, and where the sign agrees with the sign of the end load carried by the particular member, and further that no linear element of D has strain numerically greater than e." Michell's theorem states that the volume V^* of S^* is less than or equal to the volume V of any of the frameworks S

$$V = \frac{(f_t + f_c)}{2 f_t f_c} \left(\sum_t L_t T_t + \sum_c L_c T_c \right) - \frac{(f_t - f_c)}{2 f_t f_c} \sum_i \overline{F}_i \overline{r}_i. \tag{2}$$

The actual value of V^* follows from the principle of virtual work. If the virtual displacements corresponding to Michell's statement above are $e\overline{v}_i$ at $\overline{r}_i$ this volume V^* is:

$$V^* = \frac{(f_t + f_c)}{2 f_t f_c} \sum_i \overline{F}_i \overline{v}_i - \frac{(f_t - f_c)}{2 f_t f_c} \sum_i \overline{F}_i \overline{r}_i. \tag{3}$$

The character of the deformation e imposes certain restrictions upon the layout of members in S^*. At a node of this framework, the directions of the strains $\pm e$, which are along the lines of members of S^* and are principal directions of strain, must satisfy certain orthogonality conditions. In a three-dimensional truss, at a node with three members, there are no restrictions if the loads in the members have the same sign, since in that case the virtual deformation is a pure dilatation and therefore isotropic. If one load is of opposite sign to the others, it must be at right angles to them. For a node with four members, there is again no restriction if all the loads have the same sign. If one member has an opposite load to the other three, then it must be orthogonal to them all and so forces them to lie in a plane. Finally, if the members fall into pairs with opposite-signed loads, then one of these pairs must be in line and normal to the other two.

Michell also presented a very important property of the optimal structure: the optimum structure S^* has greater stiffness than any other structure of S. He also presented the value of the strain energy stored in the optimal structure.

It is noted that the original formulas of Michell are not valid for different allowable stresses in tension and compression. In 1960 A.S.L. Chan [67] and in 1963 H.S.Y. Chan [68], wrote down correctly Michell's theorem. The validity and the critical examination of Michell's theorem can be read in Rozvany's work [69] published almost 40 years later, too. One can also find a very specific overview of Michell's theorem and its extension in the book of Lewiński et al. [66].

2.3. The minimum-potential criterion of Wasiutynski

The presentation of this theorem is based on the book of Brandt [70]. In this book, one can find the different theorems and applications of the structural design. Until this time, generally, the optimization papers were based on plastic design while Wasiutynski used elastic material.

The mechanical criterion of optimizing elastic structures for given loads can be expressed as the criterion of maximum rigidity against deformations or minimum deformability. The minimum-potential criterion (which is often named as the least-deformability criterion) was presented first by Wasiutynski [71] in 1939. It involves:

- formulation of preliminary assumptions appropriate to the given problem and determination of the given set of admissible structures,
- analysis of the influence of different design variables on the deformability of the structure,
- derivation of the necessary and sufficient conditions for the least deformability of the structure from the relations provided by that analysis,
- determination of the unknown optimum variables and the stress and stress fields of the optimum structure from the least-deformability conditions and deformation equations.

The following theorems were presented:

THEOREM 1: *The addition of a new, active element reduces the elastic strain potential of the system. The greater are the displacements of the original structure on the surface contiguous to the new element and the stronger are the reactions of the structure to the new element, the greater is the reduction of the potential.*

THEOREM 2: *The elastic strain potential of a structure reinforced by an additional, arbitrary small element is reduced by a value equal to the potential contained in the new element; the potential of the original part of the structure is decreased by the double value of the potential of the new element.*

THEOREM 3: *The reinforcement of a structure by the addition of an arbitrarily small volume of material reduces the potential of the structure the more, the greater is the pre-reinforcement unit potential at the point of reinforcement.*

THEOREM 4: *Among all acceptable forms of a structure with a given volume that has least potential in which the unit potential takes equal values at all points between which it is possible to transfer material.*

THEOREM 5: *Equivalence of optimization for minimum potential with optimization for minimum volume. Namely, the design for minimum potential at a constant volume is equivalent to design for minimum volume at a constant potential.*

Theorem 6: *An elastic structure with a given volume V accumulates the least potential and has the greatest rigidity under a given useful load if statement, the greatest unit potential occurs at the points which suffer the largest displacements under the useful load, holds at all points of the structure.*

Theorem 7: *Reinforcement of a structure with a new element ΔV produces changes of order ΔV in deformations only in the immediate neighborhood of this element. The increase of deformations at finite distances from the reinforcing element is small of higher order.*

Similarly, the potential loss ΔU caused by the addition of a new element ΔV is concentrated in a differential neighbourhood of this element. The increments of the potential which occur at finite distances from the new element inside the original structure, are small of higher order.

By the use of the theorems above, the optimal solution of a new optimization procedure can be verified.

3. "Hidden" results from the middle of the 20th century

The period around the middle of the 20th century has produced impressive improvements in power and efficiency of optimization techniques, as applied to general structural design problems. However, these methods paid a price for their generality with a rapid increase in the number of computational requirements such as the increase of the number of design variables and number of constraints. In this subtitle, the "hidden" word means "not easily available".

The layout theory plays a primary role in structural optimization. The main difficulties are whether the obtained solution is unique or not, and whether the extremal point is a local or global one. This question becomes more difficult when several loading cases are taken into consideration. From the 50s to 70s of the 20th century, several papers were published to investigate the questions mentioned above. Generally, the variational calculus was the tool to prove the optimality and the uniqueness. Here a limited overview is presented by using the papers of Nagtegaal and Prager [37], Chan [68] and Shield [72].

3.1. Cox's optimal solutions

Applications of the theorems of Maxwell and Michell to simple design problems have been made by Cox [16]. He has considered, first of all, the problem of three coplanar forces. In the case where their point of intersection lies within the triangle formed by their points of application, the optimum framework can consist of tension or compression members only. Some of his layouts are given in Fig. 1. We have to note that all these structures have equal weights. One can recognize the non-uniqueness of the optimal layout and we can have an infinite

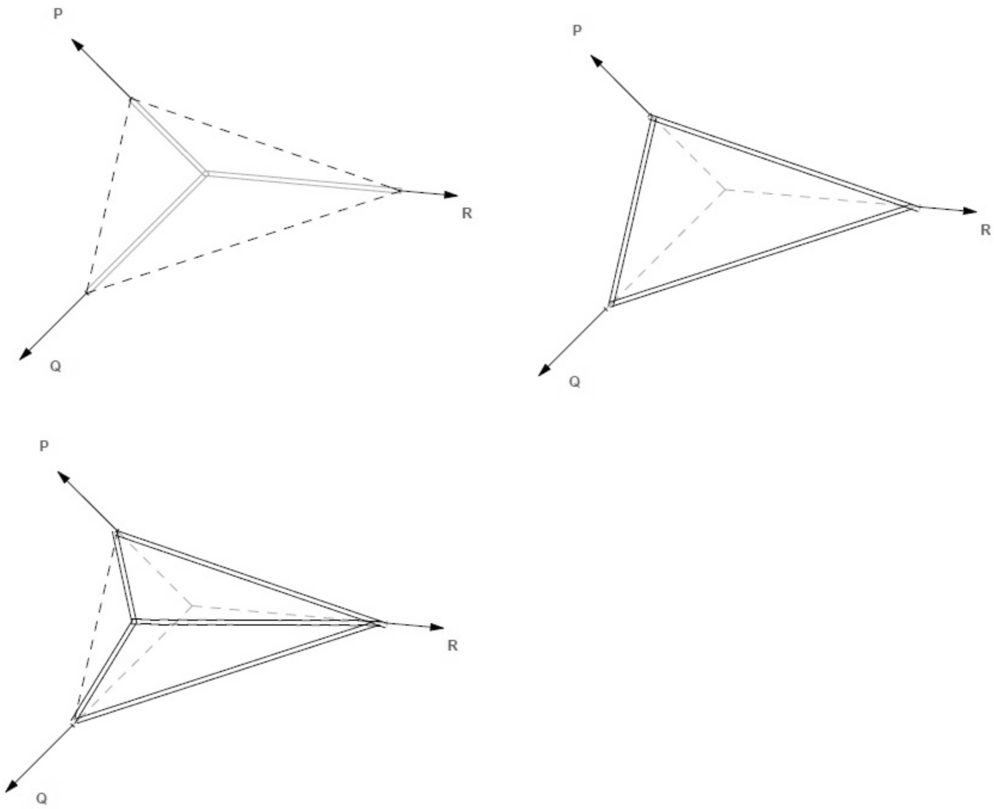

Fig. 1. Simple tension structures by Cox.

number of solutions ranging from mechanisms through simple stiff structures to structures of any degree of redundancy.

Cox [16] extended the theory presented above to build a structure for the transmission of a bending moment. He showed that if the proportion of length over height of the structure is greater than 4, this structure is considerably lighter than a "simple tie and strut", and that for larger values of length/height proportion, multiple constructions, along the lines of Fig. 2, can be even lighter. He produced a competitive 14-bar framework and a variation of Fig. 2a, in which the circles are replaced by spirals, which for length/height >4 is lighter than any other construction considered. These structures for the transmission of bending moments are not Michell's optimum structures, since they fail to satisfy the orthogonality conditions for members with opposite signed loads.

The derivation of the optimal layout of pure bending and the optimal solution of Cox's beam problem in Fig. 2a was presented by Chan [67] in 1960 (see Fig. 2b). The solution of Shield [72] for the same problem can be seen in Fig. 2c.

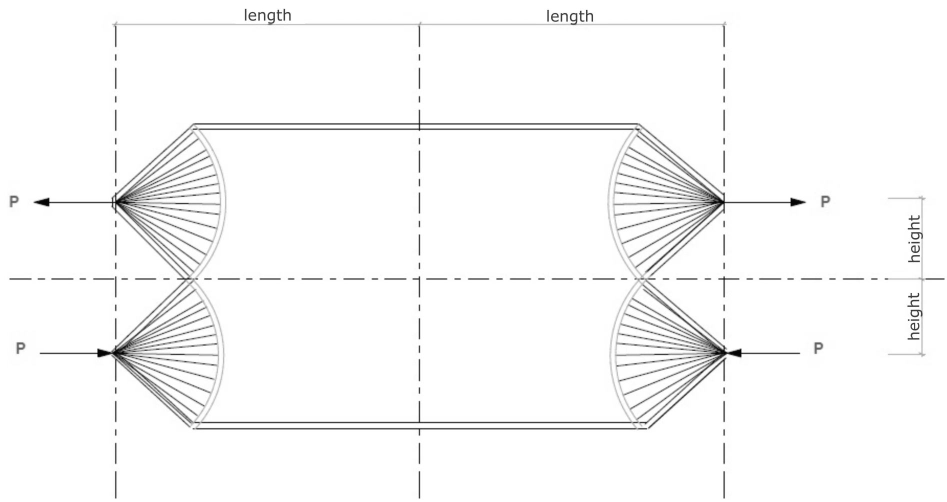

Fig. 2a. Cox's optimal beam for bending.

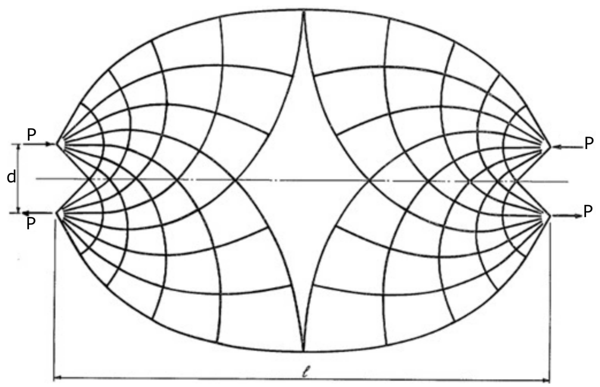

Fig. 2b. The optimal solution for pure bending presented by Chan [67].

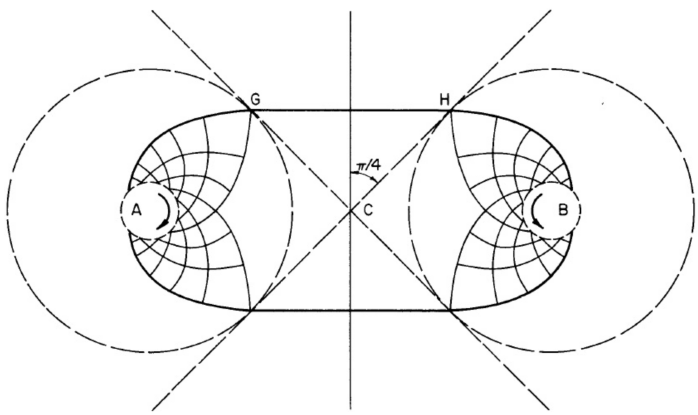

Fig. 2c. The optimal solution for pure bending presented by Shield [72].

3.2. Shield's optimal solutions

As it was indicated by Cox's work in the previous section, the optimal solutions are not unique in the case of Michell's structures. By using variational calculus, Shield [72] presented appropriate necessary conditions for the structural volume to be stationary, but he noted that there is no guarantee to get the global optimum. This section is based on Shield's original work [72], and below the outline of the original paper is presented.

Shield investigated the so-called Michell's structures and declared that the Michell-type design fails when kinematic constraints are taken into consideration. He presented an alternative approach, which does not have the limitation of the Michell method. The procedure is based on the idea to design a frame compatible with a reduced virtual small deformation in which the principal strains are of magnitude e/limit-tensional-stress (in case of tensioned members) and e/limit-compressive-stress, the directions of frame elements coinciding with the principal c directions of strain as before. Here e is the virtual deformation indicated by Michell. The virtual deformation must satisfy any imposed kinematic constraints.

Next, some examples are presented that he gave in [72] to show that minimum-volume frames are not necessarily unique, and Shield described some new additions to the list of Michell's structures. The diagram (Fig. 3a) shows the layout

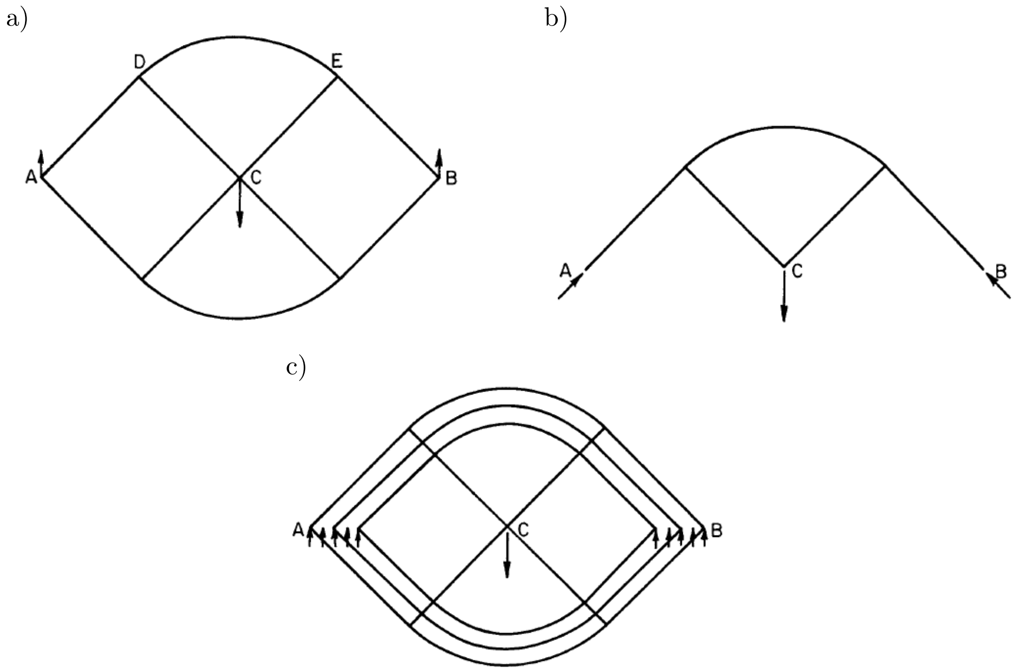

Fig. 3. Examples of non-uniqueness by Shield: a) sliding support direction, b) hinge support, c) distributed sliding support.

given by Michell for a single force applied at the midpoint C of the line AB and balanced by equal parallel forces at A and B.

The struts AD, EB and the curved bar DE carry a uniform compressive force and a quadrantal fan of tie-bars from C to DE maintains the equilibrium of the curved bar. The layout is symmetrical about AB with tie-bars replacing struts and vice-versa. The virtual deformation with principal strains $\pm e$ associated with the layout can be adjusted so that the displacement is zero at points A and B. If one assumes the equal tensional and compressive limit stress condition, one can use this virtual deformation for the case when one has the same force at C, but now A and B are fixed points of support. The optimum structure has the same volume as the structure with specified parallel forces at A, B, but the optimum design is not unique. For example, the load at C can be carried by a frame entirely above AB, as indicated in the diagram of Fig. 3b. An infinity of optimum designs results from arbitrarily assigning a fraction of the load at C to be carried by a structure above the line AB and the remainder by a structure below the line AB. It was noted that if one had specified that the load at C was carried by a beam with centerline AB and built-in at A and B, the optimum design would have bending moments at A and B. The Michell structure has no moments at the fixed points A, B. The minimum-volume design indicated at the bottom diagram (Fig. 3c) uses the same virtual deformation with principal strains $\pm e$, but now it is specified that distributed loads at A and B balance the load at C.

We have to note that Fig. 3 is presented to illustrate the non-uniqueness of design. However, this figure presents the design's dependence on support conditions: Fig. 3a: sliding support direction (vertical reaction), Fig. 3b: hinge support (reaction following member direction), and Fig. 3c: distributed sliding support. One can also read about the uniqueness theorem related to Michell's structure design in the paper of Kozlowski and Mróz [73].

3.3. Optimal solutions of Kozlowski and Mróz

The uniqueness theorem in topology design was investigated by Kozlowski and Mróz [73], where the authors presented the use of Michell's structures to design disks with thickness constraints.

More specifically, they presented a general formulation of the problem of optimal design with geometric constraints. The optimal design of perfectly plastic disks of Tresca material was considered, and two problems were discussed in detail: a disk simply supported at two points and a cantilever disk loaded by a concentrated force. Their optimal design lied in the semi-plane below the line joining supports or within a strip of prescribed width (Figs 4a and 4b).

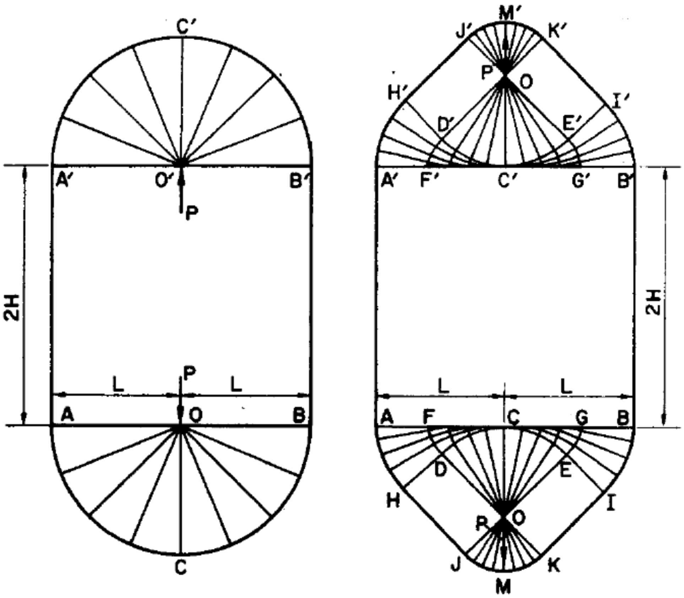

Fig. 4a. Optimal disks carrying two concentrated and opposite forces; the design is to lie beyond the region ABA′B′.

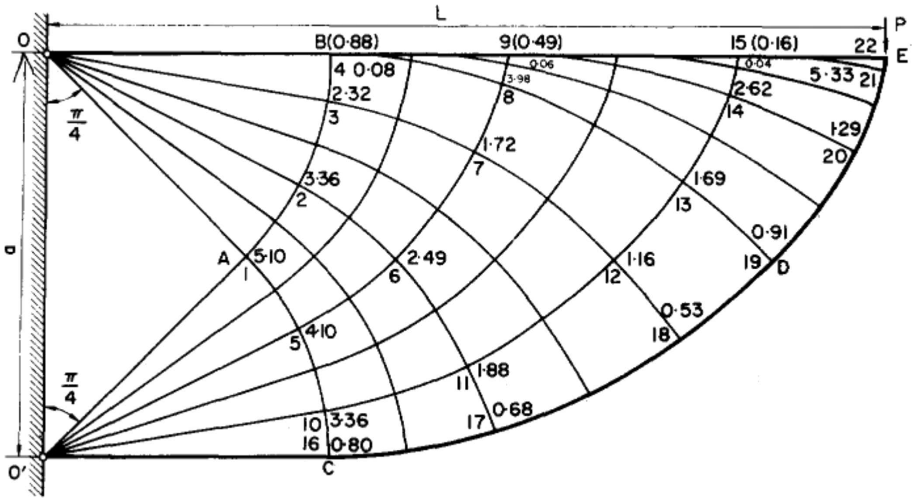

Fig. 4b. The optimal solution of a cantilever disk loaded by a concentrated force.

3.4. Design dependence of the optimal solutions in the case of alternating loads

In optimization, the investigation of the convergence of the applied procedure and the uniqueness of the solution are of primary importance. This question becomes more difficult whenever there are multiple loadings and/or the loading uncertainty is considered. In this latter case, the load can be considered as a quantity given in an interval with a certain possibility of location or/and direction, and/or magnitude. The design loads are usually selected based on worst loading cases.

To understand the topic here, the main achievements of the paper by Nagtegaal and Prager [37] are discussed briefly at first. The starting point is the minimum volume design of a truss. The results of this layout optimization, coming from this uniaxial case, can be generalized and a continuum type topology optimization problem with bi-axial stress state is investigated later. The paper of Nagtegaal and Prager is concerned with the following problem: two alternative loads with the same point of application are to be transmitted to a rigid foundation by a plane truss of minimum weight whose load factors for plastic collapse under one or the other load are not to exceed a given value. A necessary and sufficient condition for global optimality is established and used to determine the optimal layout of the truss. According to their optimality criteria (global optimality condition), in an active truss member (being non zero cross-section), the sum of the normalized strain rates is a unit, while in vanishing members, the summation of the strain rates results in a smaller value than a unit. In addition to the optimality conditions, Nagtegaal and Prager gave a short overview on how the optimal layout looks like in a special case of alternative loads (see Fig. 5). They considered a fixed force (say P′) and discussed the optimal types of trusses for all possible other forces (P″). In Fig. 5, "A" is the common application point of these two forces. Here vector AB′ represents the fixed force P′. The lines B′C and B′D form angles of 45° with the horizontal direction, and lines EF and EG are obtained by mirroring the lines B′C and B′D with respect to point "A".

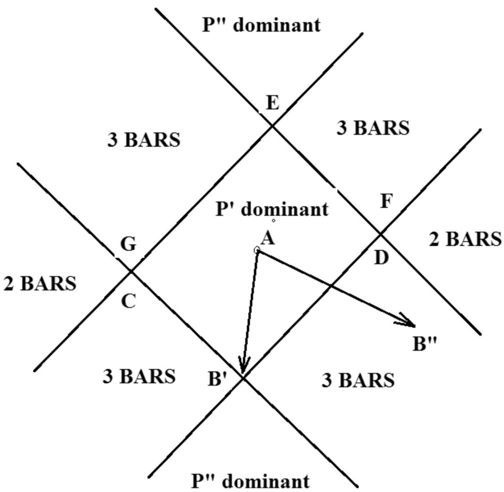

Fig. 5. Truss type as a function of the alternative loading.

These lines divide the plane of the figure into nine regions. If the load P″ is represented by the vector AB″, the label of the region that contains B″ indicates the type of the optimal truss. Where one of the loads is dominant, which exclusively determines the optimal design, the optimal truss consists of two bars that form angles of 45° with the wall.

The optimal layout problem of a minimum weight truss design problem with a single vertical force load was presented by Save [44] in the case of stress constraints. Singular situations in the minimum-volume elastic design are analyzed and illustrated as they occur in the optimization of a three-bar truss. Relationships between the minimum-volume design with bounds on stress intensity, assigned load factor at collapse, and assigned elastic compliance are analyzed. He concluded that the optimal layout could be one, two and three bar trusses depending on the design conditions.

These general layout theories can act as reference studies for topology optimization of truss-like structures in the case of multiple and/or stochastic loading. It is worth to note that in the case of trusses, the uniaxial stress state is considered, but the truss-like structures belong to the biaxial stress state problems.

3.5. Solution techniques in this period

The special approaches used to solve successfully the majority of the problems in the second half of the 20th century are known in the literature as optimality criteria methods. Optimality criteria methods are based on radically different thinking from those applied in the development of the mathematical programming (MP) methods. Most MP methods concentrate on obtaining information from conditions around the current design point in design space in order to answer two questions: in what direction and how far to go to best reduce the value of the objective function directly. This is repeated until no more reduction is produced in the iterations within some selected tolerance. On the other hand, optimality criteria methods, exact or heuristic, derive or state conditions characterizing the optimum design, then find or change the design to satisfy those conditions while indirectly optimizing the structure.

The origin of the classical optimality criteria method (COC) dates back to the 1970s. By turning attention to certain stiffness-related constraints, theoretically valid optimality criteria were derived for discretized structures with displacement constraints employing classical Lagrangian multiplier methods of mathematical optimization. To satisfy the optimality criteria, an algorithm was proposed by Berke [74] and later Berke and Khot [30], based on the approach that if it provides an exact direct formula for statically determinate structures, same as FSD, then it would converge in a few iterations for most practical structures, again same as FSD. The difference, however, is an important one; this criterion, unlike FSD, was theoretically correct also for indeterminate structures. Furthermore it resulted in the separability of variables leading to member-by-member resizing, similar in this respect to FSD, thus inheriting its often benign convergence behavior. Berke also suggested that the design variables and constraints should be separated into a passive and active set. One can see the similarity in terminology and general steps of the COC method and the SIMP (solid isotropic material with penalization) method-one of the leading solution procedures in topology optimization recently.

At the “beginning”, the OC and MP methods were still treated as different ones. It was the Fleury and Geradin paper [75] that pointed out the relationship between these two methods. They proposed two new methods in that paper: the first one is a mixed method, which is between OC and MP methods by introducing a parameter to control the algorithm between OC and MP methods and to control the stability of COC methods. The second method is an OC method that uses a first-order approximation to stress constraints instead of the classical FSD method. Fleury and Sander [76] proposed a “mixed method” combining OC and MP methods. In 1979, Khot, Berke and Venkayya [77] also applied an OC model, but they solved it by using iteration methods. Almost in the same time, a very efficient MP method called “dual method” was proposed by Fleury [78]. This method was derived from the sequential linear approximation of constraints with reciprocal variables, but it clearly explained the iteration difficulties of the Lagrange multipliers in dual space. He reported the correct identification of passive and active sets for both design variables and constraints in advance in this paper. Fleury and Schmit [79] combined dual method and approximation concepts, which resulted in an efficient method. Schmit and Fleury [80, 81] applied this method for mixed variables (discrete and continuous) and showed that the dual method was suitable for mixed variable problems. Furthermore, Fleury and Braibant [82] generalized this method to a more efficient one and named as “dual method using mixed variables”, which worked well in both size design and shape design. By the end of this period, the Schittkowski algorithm gave an appropriate tool to solve MP problems [83].

3.6. Monographs and the Prager–Rozvany’s layout theorem

This middle period produced several monographs such as Gerard [83], Prager [84], Cox [17], Johnson [85], Cohn [86], Gallagher and Zienkiewicz [87], Hemp [88], Cyras *et al.* [89], and Sawczuk and Mróz [90], which help to understand the basic principles and solution techniques, and verify the numerically obtained optimum solutions by the use of the limited number of analytical solutions.

Prager and Taylor [31] presented a uniform method of treating a variety of problems of the optimal design of sandwich structures. Their design procedure consists of two steps: the integration of an optimality condition, which is a differential equation for the optimal displacement field that does not involve any design parameters, and the subsequent determination of the optimal distribution of elastic stiffness or plastic resistance from the usual differential equations of the structure. Optimal elastic design for maximum stiffness, maximum fundamental frequency or maximum buckling load, and optimal plastic design for maximum safety are treated as examples. In this framework, the paper of Hegemeir and Prager [91] was an important study, where one can find some help to understand the correctness of the theorem stating that in single load case, minimizing the volume of the structure with respect to displacement constraint or compliance constraint, one can obtain the same optimum.

At that time, the optimization topic started its "career" at CISM (International Centre for Mechanical Sciences). The first meeting was coordinated by Prager [35]. In addition, Save and Prager [92] gave an extensive overview of the optimality criteria methods. Interestingly, this book was published in 1985, 5 years after Prager's death.

Also, this is the period when Rozvany has started his long career in this field. No doubt, he is one of the most important persons in topology optimization. His first book [93] deals extensively with the layout design of grillage structures. The existing layout theorems were published in compact form by Prager and Rozvany [36] in 1977. The theorem is based on the Prager–Shield [94] optimality condition for plastic design. The existing works were extended by a new element, namely, optimality conditions for vanishing members in terms of adjoint strains along these members were identified. The adjoint strains were given by the subgradient of the specific cost function with respect to stresses or stress resultants. The subgradient of a function was the usual gradient, but at discontinuities of the gradient, any convex combination of the adjacent gradient could be taken. For the sign-independent, stress-based design of trusses and grillages of given depth, the specific cost functions were

$$A = k\,|F| \quad \text{and} \quad A = k\,|M|\,, \tag{4}$$

where A is the cross-sectional area, k is a constant, F is a member force and M is a bending moment.

Then, e.g., for trusses, the optimality conditions reduce to those of Michell [6]

$$\overline{\varepsilon} = k\,\mathrm{sgn}\,F \;\; (\text{for } F \neq 0), \qquad |\overline{\varepsilon}| \leq K \;\; (\text{for } F = 0), \tag{5}$$

where $\overline{\varepsilon}$ is the adjoint strain.

For grillages

$$\overline{\kappa} = k \operatorname{sgn} M \quad (\text{for } M \neq 0), \qquad |\overline{\kappa}| \leq K \quad (\text{for } F = 0), \tag{6}$$

where $\overline{\kappa}$ is the adjoint beam curvature.

The theorem above was extended by Rozvany *et al.* [39] in 1987 and the design was not restricted to low volume fracture structures anymore.

At the beginning of the 1980s, several applications in optimal design were published to extend the achievements of layout optimization into the direction of topology design. The results of Olhoff [95, 96] and co-workers (Cheng, Taylor, Bendsøe) [97–99] in the field vibration provided a strong foundation for the continuum-type topology design.

4. Selected milestones in the last three decades

The research papers in the last period of this review are easily available and there are several review papers in different journals besides the earlier mentioned works. These lastly mentioned reviews were written, among others, by Rozvany, Bendsøe and Kirsch [100] in 1995, Eschenauer and Olhoff [101] in 2001, Sigmund and Maute [102] in 2013 and Zargham *et al.* [103] in 2016. Therefore, this period is covered only briefly in this chapter.

In 1988, a new generation of problem formulation was created by Bendsøe and Kikuchi [47] in topology optimization by using homogenization. A detailed description of the continuum-type optimality criteria method was reported in Rozvany's books [49, 50, 104] and [105] co-edited with Lewiński, and several reports [106–110]. The detailed description of theories in connection with Michell structures can be found in the recent book of Lewiński, Sokół, and Graczykowski [66], as indicated earlier. However, one can read brief summary of the selected milestones in the following paragraphs.

After the original homogenization technique to topology optimization was published by Bensoe and Kikuchi [47], several new computational concepts and procedures were elaborated. The most frequently applied computational methods are:

a) density-based method (Bendsøe [48], Zhou and Rozvany [110], Mlejnek [111]), which is very often called SIMP;
b) evolutionary approaches (Xie and Steven [112]), where the density and evolutionary approaches use simple element or nodal-based design variables, and the iteration scene is very similar to that of Berke's COC method (e.g., Zhou and Rozvany [113]). The general formulation was published by Bendsøe and Sigmund [114];
c) the topological derivative (Sokołowski and Zochowski [115]) method is based on the bubble-method. The basic idea is to predict the influence

(derivative) of introducing an infinitesimal hole at any point in the design domain and use this as the driver for the generation of new holes;

d) the level set method (Allaire *et al.* [116], Wang *et al.* [117]) aims to combine some advantages of the shape sensitivity method and the topology approach. The method uses shape derivatives for the development of the optimal topology. To overcome the numerical difficulties, hybrid approaches have appeared, such as level set approaches that use shape derivatives for design updates but, somewhat against the original definition of the level set concept, do allow for introducing holes without the use of topological derivatives (Yamada *et al.* [118]);

e) the phase-field (Bourdin and Chambolle [119]) approach works directly on the density variables and considers the minimization of a functional.

The methods mentioned above created the first general directions in the field of continuum-based topology optimization. To overcome a serious numerical problem, that is, to avoid the checkerboard pattern, several filtering techniques and procedures were developed [2, 120–125].

The evolutionary optimization inspired many researchers, and a wide range of heuristic methods was developed during these three decades. One can read a rather complex review in the book of Kaveh [126].

4.1. Monographs and some selected results

From the beginning of the 1990s, several books were published in connection to topology optimization. In addition to the books mentioned earlier in this paper, one can find monographs written by Haftka *et al.* [127], Kirsh [128], Bendsøe [129], Bendsøe and Sigmund [130].

As quick as the numerical methods in topology design developed, the demand to verify analytically the numerically obtained optimal topologies increased. The first analytical solutions come from Rozvany and his research colleagues. In Fig. 6, one can see the analytical solution published by Lewiński, Zhou and Rozvany [131].

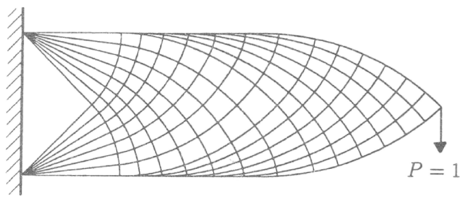

FIG. 6. The exact optimal solution by Lewiński, Zhou and Rozvany [131].

A very similar solution and detailed calculation were published by Melchers [132]. Another well-known problem, the L-shape structure, was also solved by

Lewiński and Rozvany [133]. In Fig. 7, one can see the analytically derived solution.

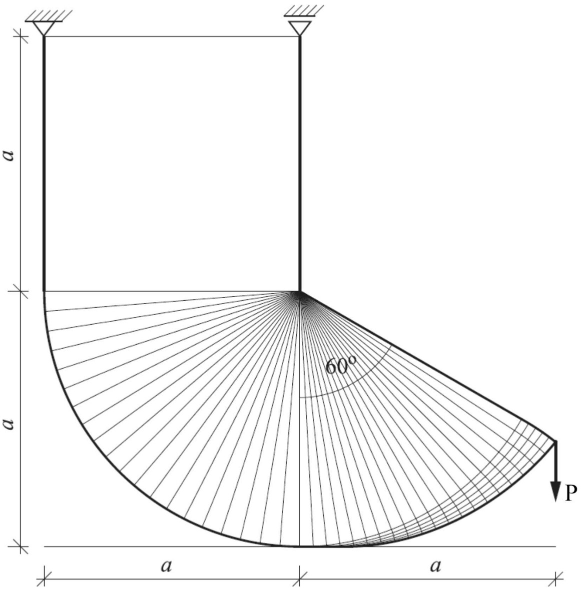

FIG. 7. The exact optimal solution of the L-shape beam by Lewiński and Rozvany [133].

Here one has to mention the Danish school achievements. Besides the work of Bendsøe, one has to mention N. Olhoff [134], who has numerous applications of SIMP (design dependent loads, vibration problems, 3D structures). Another important person in topology design is O. Sigmund, who contributed greatly to the general acceptance of SIMP.

His milestone works are: educational articles, checkerboard control, multi-physics applications, material model for SIMP. The pressure load solution calculated by Clausen and Sigmund [135] can be seen in Fig. 8.

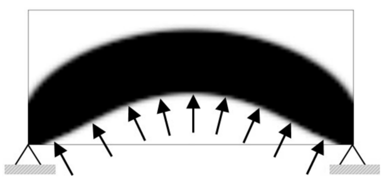

FIG. 8. The pressure load solution by Clausen and Sigmund [135].

The SIMP method was also applied extensively by Helder Rodrigues. Some of his major research topics have been: hierarchical topology optimization on macro- and micro-scales [136], applications to bio-mechanics (e.g., bone tissue adaptation) [137] and optimization of piezoelectric properties [138]. Topology optimization problems in the case of fluid-structure interaction were solved by Maute *et al.* [139, 140]. Their aim was to find the layout of the internal structure, actuation system, and the overall design with the optimal flow and structural characteristics. Their approaches were high-fidelity analysis of fluid-structure interaction problems based on Navier–Stokes and nonlinear solid mechanics models, SIMP-type interpolation of structural and flow parameters, and adjoint sensitivity analysis.

As one can see there are many different fields in topology optimization. It is no surprise that all these branches can be found under the name of N. Kikuchi. His research is connected to this last period and he is still active [141]. Kikuchi's 1988 milestone publication together with Bendsøe [47] was reported earlier. His Research Lab at the University of Michigan, Ann Arbor, under his guidance, has "produced" a number of important researchers, who are currently leading scientists (K. Suzuki, S. Nishiwaki) in topology optimization. Together with K. Suzuki, the general theory of the homogenization method was elaborated (Suzuki and Kikuchi [142]). He and Gueddes [143] used the topology optimization for material design. Kikuchi and Nishiwaki *et al.* [144] extended the topology design to compliant design, and with Lee [145], the structural topology was used in electrical machinery. Having the leading scientist position at the Research Department of Toyota Motor Company, he successfully introduced topology optimization into the car industry [146].

4.2. Other methods in structural topology optimization

Besides the numerical procedures mentioned above, Svanberg [147] in 1987 elaborated a mathematical programming subroutine – the method of moving asymptotes, which is one of the most frequently applied computational tools in topology optimization.

The method has been extensively used in multi-constrained problems of topology optimization, such as the minimization of the weight under local stress constraints (Duysinx and Bendsøe [148]), addressing symmetric and non-symmetric strength criteria also in conjunction with displacement constraints (Bruggi and Duysinx [149, 150], Bruggi [151, 152]).

As an example of the wide range of applications of topology optimization, the free material design was introduced into this field at the beginning of the 1990s. The applications were strongly influenced by the works of Cherkaev and Gibiansky [153] and Cherkaev [188]. The papers by Bendsøe *et al.* [154, 155] are evidently the first examples where the arbitrary tensor-valued function representing material properties was treated directly as the design variable. Guedes and Taylor [156] presented an algorithm that provided a way to generate a "0–1" topology design through the usage of a non-uniform unit relative cost. Taylor

[157] reformulated the free material design problem with the cost constraint expressed in a generalized form. In the last decade, Dzierżanowski and Lewiński extended the applications to stress-based approaches [158].

In 1993, a new computational procedure was suggested for solving topology problems by Hajela *et al.* [159]. The *Genetic Algorithms* (GA) is generally a global search strategy for generating near-optimal structural topologies, and may be considered a derivative of the ground structure approach. The search procedure has a philosophical basis in Darwin's postulate of the "survival of the fittest". The application of this approach is particularly potent, as structural members can be both added and removed during the search process. An advantage of the genetic search-based approach over mathematical programming or optimality criteria-based methods is the ability to include general design constraints in the problem and locate global optimum.

Discrete optimization methods also play an important role in topology design. They can be classified as:

- sequential integer programming (Svanberg and Werme [160]),
- neighborhood search (Svanberg and Werme [161]),
- efficient enumeration (Werme [162]),
- branch and cut (Stolpe and Bendsøe [163]) method.

The advantage of these methods is that they may guarantee global optimum, but the disadvantage is that a relatively small number of ground elements can be handled at present. However, the Polish group led by Gutkowski and Bauer played a fundamental role in this topic. They organized several conferences [164–166], a course in CISM [167] and published papers [e.g., 168].

"Hard-kill" or "sudden death" methods provide only black or white ground elements, but they use heuristic criteria for element rejection or admission. The most publicized hard-kill method is called inappropriately "ESO" (Evolutionary Structural Optimization), well over a hundred publications with high citation numbers are dealing with this subject. Some improvements of ESO were suggested by Rozvany and Querin [169–171] under the term SERA (sequential element rejection and admission). Shortcomings of ESO are:

- fully heuristic,
- inefficient (much more computer time) compared to gradient methods,
- lacking rationality (first, a large number of solutions are generated by a heuristic criterion, then the "best" solution is located by enumeration for a different objective function,
- may give vastly non-optimal solutions (Zhou and Rozvany [172]).

The interested readers can find a good review of the established numerical methods of structural topology optimization that have reached the stage of application in industrial software by Rozvany [173] in addition to the review papers mentioned earlier in this section.

Finally, we have to mention again the probabilistic or reliability-based topology optimization procedures that are relatively new in this field. Some publications were already mentioned in the previous section, here just some subjectively selected works are given. Among others, Marti and co-workers [174–176] have published several new procedures to obtain optimal topologies. Lógó and his co-workers [52–55, 177, 178] also proposed several new probability-based algorithms for topology design.

In Poland, IPPT (Institute of Fundamental Technological Research, Polish Academy of Sciences) is the research institution where several new results were reported in topology optimization. Besides the earlier cited works of Mróz, the optimal redesign methods based on topological sensitivity derivatives with application to the design of trusses, beam and plate structures, and support conditions were published in the last decades (e.g., Bojczuk and Mróz [179], Mróz and Bojczuk [180]). In addition, reliability-based optimization was an important aspect of the scientific activity at IPPT. The research group led by Kleiber and Jendo published important papers [181–183] and organized several conferences [e.g., 184] in this field.

Tauzowski, Blachowski and Lógó [185] have opened a new computational direction in structural topology optimization, namely the *functor-oriented programming*. Their study applied basic concepts of functional programming to developing special class into the finite element hierarchy. This class is known among the computer science community as *function object* or *functor*. Functor-based implementation of the finite element method leads to simpler and easily expandable FEM software architecture. This new method has already been applied successfully in elasto-plastic topology optimization [186]. They also published a solution technique in the field of plasticity, where the optimal layout of a truss was calculated for the case of impact loading [187].

5. Conclusions

One can see from this relatively short overview that topology optimization has become a widely applied and important scientific field. In the beginning, "only" some people were involved in the research, but currently hundreds of scholars work in this field. Its applications prove the existence of topology analysis/design in all fields of life.

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