# Homotopy approach for solving two-dimensional integral equations of the second kind 

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#### Abstract

In this paper, the two-dimensional linear and nonlinear integral equations of the second kind is analyzed. The homotopy analysis method (HAM) is used for determining the solution of the investigated equation. In this method, a solution is sought in the series form. It is shown that if this series is convergent, its sum gives the solution of the considered equation. The sufficient condition for the convergence of the series is also presented. Additionally, the error of approximate solution, obtained as partial sum of the series, is estimated. Application of the HAM is illustrated by examples.


Keywords: homotopy analysis method, convergence, error estimation, nonlinear integral equation, linear integral equation.
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## 1. Introduction

The HAM was invented by the Chinese mathematician Shijun Liao [25-29] and is dedicated to the solution of various operator equations (linear and nonlinear ones). For the first time, this method was presented in 1992 in the PhD dissertation of its inventor and since then it has found numerous applications in solving the problems described by differential equations [2, 44, 49], as well as fractional differential equations [4, 51], differential-difference equations [48] and integro-differential equations $[7,40]$. In particular, among the problems solved with the aid of the HAM, one can list the following problems: the nonlinear Cauchy problem of parabolic-hyperbolic type [20], the Toda lattice system (described by differential-difference equations) [48], the nonlinear reaction-diffusion-convection problems [37], a nonlocal initial boundary value problem [31], and the fractional differential equations [4,51]. The HAM was also used for investigating the heat conduction problems $[1,16,33]$, whereas in $[36,38]$ the method was applied for solving the inverse heat conduction problem. The theoretical results concerning, among others, the convergence of the method are included, for example, in [26, 28, 32, 34, 43].

The HAM was also applied to solve integral equations. The paper [22] considers the application of the method to one-dimensional nonlinear and linear integral equations of the second kind. The Fredholm and Volterra integral equations are examples of special cases of such equations. In additions the theoretical results concerning the convergence of the method and the error estimation of approximate solution are also presented in [22].

The application of the HAM to the nonlinear Fredholm integral equations of the second type is also described in [3], while its application to the second-kind Volterra equations is presented in [47]. In papers $[10,18]$ a special class of nonlinear Fredholm and Volterra equations of the second-kind is considered, namely those with power nonlinearity in form of $u^{p}$.

Moreover, in [12] the systems of integral equations, some theoretical results and examples of applying this method are presented. Other examples of using the HAM to solve the system of integral equations are included in [35, 39].

In [24], the HAM in a modified form is used for solving the system of Fredholm integral equations of the second type. The work [21] shows the application of the HAM for solving a given type of linear and nonlinear integral equations, the special case of which is the Volterra-Fredholm integral equation. The authors also provide some theoretical results about the convergence of the HAM and the error estimation.

The classical methods dedicated to the solution of integral equations, one- and two-dimensional, include for instance the Nystrom method [8, 17] and the collocation method [19]. Recently, the Euler-type method [30], differential transform [41], radial basis functions [9], spline functions [13], wavelets [15] and matrix-based method [23] were also developed. The application of the HAM to solve the two-dimensional integral equations is considered in [5, 11]. In these papers, the computing examples are only presented, without any theoretical results.

In this paper, a more general type of nonlinear and linear two-dimensional integral equations of the second kind is considered and some theoretical results are also presented. In the considered method, the solution is sought in the series form. It is shown that if the series converges, its sum gives the solution of the investigated equation. The sufficient condition for convergence of the created series is also given. Additionally, the error of approximate solution obtained by using the partial sum of the series is estimated. The application of the HAM is illustrated with examples.

## 2. EQUATIONS UNDER DISCUSSION

Mathematical models of many physical phenomena and engineering problems are described by means of integral equations. In this paper, we consider a two-dimensional integral equation of the form

$$
\begin{align*}
u(x, y)=F(x, y) & +\int_{f_{1}(x)}^{g_{1}(x)} K_{1}(x, y, t) R_{1}(u(t, y)) d t+ \\
& +\int_{f_{2}(y)}^{g_{2}(y)} K_{2}(x, y, s) R_{2}(u(x, s)) d s+\int_{f_{1}(x)}^{g_{1}(x)} \int_{f_{2}(y)}^{g_{2}(y)} K_{3}(x, y, t, s) R_{3}(u(t, s)) d s d t \tag{1}
\end{align*}
$$

where $(x, y) \in D:=\left[a_{1}, b_{1}\right] \times\left[a_{2}, b_{2}\right]$, and for $k=1,2, f_{k}, g_{k} \in C\left[a_{k}, b_{k}\right], a_{1} \leqslant f_{1}(x) \leqslant g_{1}(x) \leqslant b_{1}$, $a_{2} \leqslant f_{2}(y) \leqslant g_{2}(y) \leqslant b_{2}$. Additionally, the functions $K_{k}, k=1,2,3$, and $F$ are continuous in the appropriate sets, $R_{k}: C(D) \rightarrow C(D)$ are the linear or nonlinear operators, whereas the function $u$ is the sought element.

As a norm in space $C(\Omega)$, where $\Omega$ is a compact set, we take the supremum norm

$$
\begin{equation*}
\|\vartheta\|=\sup _{z \in \Omega}|\vartheta(z)| \tag{2}
\end{equation*}
$$

whereas the norm of operator $R_{k}$ is defined as follows:

$$
\left\|R_{k}\right\|=\sup _{\substack{u \in C(D) \\ u \neq 0}} \frac{\left\|R_{k}(u)\right\|}{\|u\|},
$$

where the norms on the right-hand side of the above equation are defined by formula (2).
As a special case of the equation above, we obtain the classical two-dimensional Fredholm integral equation of the second kind

$$
u(x, y)=F(x, y)+\int_{a_{1}}^{b_{1}} \int_{a_{2}}^{b_{2}} K_{3}(x, y, t, s) R_{3}(u(t, s)) d s d t
$$

for $K_{1}=K_{2}=0, f_{k}(x)=a_{k}, g_{k}(x)=b_{k}$, and the two-dimensional Volterra integral equation of the second kind

$$
u(x, y)=F(x, y)+\int_{a_{1}}^{x} \int_{a_{2}}^{y} K_{3}(x, y, t, s) R_{3}(u(t, s)) d s d t
$$

for $K_{1}=K_{2}=0, f_{k}(x)=a_{k}, g_{k}(x)=x$.
Under appropriate assumptions, the two-dimensional Volterra integral equation of the first kind

$$
\int_{a_{1}}^{x} \int_{a_{2}}^{y} k(x, y, t, s) u(t, s) d s d t=f(x, y)
$$

can be transformed into the equation of form (1) (see [30]).

## 3. BASES OF THE HAM

The HAM serves for solving the operator equations of many kinds, linear and nonlinear ones. This method applies the topological concept of homotopy to create a solution of the investigated operator equation in the form of a convergent series. In topological sense, we say that two continuous functions, acting from one topological space to another one, are homotopic if one of them can be continuously transformed into the other one with the aid of a "continuous deformation" called a homotopy between these two functions.

In the HAM, the idea of homotopy is used in such a way that the considered equation is transformed into the corresponding deformation equation. The new equation depends on a parameter, the variation of which, from one boundary value to the other one, corresponds to the variation of the solution from the known initial solution to the sought one. To determine the form of the components of this deformation equation, the generalized Taylor expansion is involved including an auxiliary parameter controlling and adjusting the convergence region of the series.

Let us present the details of this approach. We seek the solution of the following operator equation:

$$
\begin{equation*}
N(u(z))=0, \quad z \in \Omega \tag{3}
\end{equation*}
$$

where $N$ is the operator (in particular, it can be a nonlinear one), while $u$ is the unknown function and $\Omega$ is any domain of variable $z$.

We start the procedure with defining the so-called zero-order deformation equation

$$
\begin{equation*}
(1-p) L\left(\Phi(z ; p)-u_{0}(z)\right)=p h N(\Phi(z ; p)) \tag{4}
\end{equation*}
$$

where $p \in[0,1]$ is an embedding parameter, $h \neq 0$ denotes the convergence control parameter [28, $29,34,46], u_{0}$ represents the initial approximation of the solution of problem (3), and $L$ describes the auxiliary linear operator (let us notice that certainly $L(0)=0$ ). In the investigated method, the linear operator $L$ can be arbitrarily selected. The most common practice is to choose $L$ so that the equations, obtained in the next stages of the procedure, would be as simple to solve as possible.

Substituting $p=0$, we get $L\left(\Phi(z ; 0)-u_{0}(z)\right)=0$, which means that $\Phi(z ; 0)=u_{0}(z)$. However, when we assume $p=1$, we get $N(\Phi(z ; 1))=0$, which means that $\Phi(z ; 1)$ is the searched solution of Eq. (3) $(u(z)=\Phi(z ; 1))$. Thus, the change of parameter $p$ value from zero to one involves a change from the trivial problem to the original problem (and thus the change of the solutions form $u_{0}$ to the sought solution $u$ ). If the operator $N$ has a one-element kernel, then Eq. (3) has one solution that can be determined by the described method. If the kernel consists of a larger number of elements, the method allows to specify one of the existing solutions of Eq. (3).

Let us consider the function $\Phi(z ; p): \Omega \times[0,1] \rightarrow \mathbb{R}$. Taking the Maclaurin series of function $\Phi(z ; p)$, with respect to the parameter $p$, we obtain

$$
\begin{equation*}
\Phi(z ; p)=u_{0}(z)+\sum_{m=1}^{\infty} u_{m}(z) p^{m}, \tag{5}
\end{equation*}
$$

where $u_{0}(z)=\Phi(z ; 0)$ and

$$
\begin{equation*}
u_{m}(z)=\left.\frac{1}{m!} \frac{\partial^{m} \Phi(z ; p)}{\partial p^{m}}\right|_{p=0}, \quad m=1,2,3, \ldots \tag{6}
\end{equation*}
$$

If the above series converges for $p=1$, we obtain the required solution

$$
\begin{equation*}
u(z)=\sum_{m=0}^{\infty} u_{m}(z) . \tag{7}
\end{equation*}
$$

In order to determine the function $u_{m}$ we differentiate $m$-times, with respect to parameter $p$, the left- and right-hand side of formula (4), then the obtained result is divided by $m$ ! and substituted with $p=0$ which gives the so-called $m$ th-order deformation equation ( $m>0$ ):

$$
\begin{equation*}
L\left(u_{m}(z)-\chi_{m} u_{m-1}(z)\right)=h \bar{R}_{m}\left(\bar{u}_{m-1}, z\right), \tag{8}
\end{equation*}
$$

where $\bar{u}_{m-1}=\left\{u_{0}(z), u_{1}(z), \ldots, u_{m-1}(z)\right\}$,

$$
\chi_{m}= \begin{cases}0 & m \leq 1,  \tag{9}\\ 1 & m>1\end{cases}
$$

and

$$
\begin{equation*}
\bar{R}_{m}\left(\bar{u}_{m-1}, z\right)=\left.\frac{1}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial p^{m-1}} N\left(\sum_{i=0}^{\infty} u_{i}(z) p^{i}\right)\right)\right|_{p=0} \tag{10}
\end{equation*}
$$

If we are not able to determine the sum of series in (7), then as the approximate solution of considered equation we can accept the partial sum of this series

$$
\begin{equation*}
\widehat{u}_{n}(z)=\sum_{m=0}^{n} u_{m}(z) . \tag{11}
\end{equation*}
$$

Choosing in appropriate way the convergence control parameter $h$ we can influence the convergence region of the created series and the rate of this convergence [29, 32, 43] (therefore, the name of parameter $h$ is the convergence control parameter). One of the methods to select the value of this parameter is the so-called "optimization method" $[6,29,50]$. In this method, we define the squared residual of the governing equation

$$
\begin{equation*}
E_{n}(h)=\int_{\Omega}\left(N\left[\widehat{u}_{n}(z)\right]\right)^{2} d z . \tag{12}
\end{equation*}
$$

The optimum value of the convergence control parameter is obtained by determining the minimum of this squared residual. The effective region of the convergence control parameter is additionally defined by

$$
\begin{equation*}
\mathbf{R}_{h}=\left\{h: \lim _{n \rightarrow \infty} E_{n}(h)=0\right\} . \tag{13}
\end{equation*}
$$

Choosing a different value of the convergence control parameter than the optimal one, but still belonging to the effective region, we also obtain the convergent series, only the rate of convergence is lower. A version of the method with the above described selection of optimal value the convergence control parameter is called the basic optimal HAM [29].

To accelerate the calculations Liao includes in [29] a suggestion to approximate the integral in formula (12) by applying the quadrature rules. Liao shows in his examples that the optimal values of the convergence control parameter obtained by using this approximation do not differ much from the values obtained by applying formula (12).

Another way of selecting the value of convergence control parameter is the application of the so-called $h$-curve. In order to determine this curve the behavior of a certain quantity of the exact solution as a function of parameter $h$ is analyzed [26, 45]. This method enables to determine the effective-region of the convergence control parameter; however, it gives no possibility to determine its optimal value ensuring the fastest convergence [29].

## 4. LINEAR INTEGRAL EQUATION

We start with considering Eq. (1), in which $R_{k}: C(D) \rightarrow C(D), k=1,2,3$, are the bounded linear operators, that is, $\left\|R_{k}\right\|<\infty, k=1,2,3$. As we mentioned before, set $D$ is a rectangle $D:=\left[a_{1}, b_{1}\right] \times\left[a_{2}, b_{2}\right]$ and $C(D)$ is the class of continuous functions on $D$. Operators $L$ and $N$ can be defined in the following way:

$$
\begin{aligned}
& L(v)=v, \\
& N(v)=v(x, y)-F(x, y)-\int_{f_{1}(x)}^{g_{1}(x)} K_{1}(x, y, t) R_{1}(v(t, y)) d t \\
& -\int_{f_{2}(y)}^{g_{2}(y)} K_{2}(x, y, s) R_{2}(v(x, s)) d s-\int_{f_{1}(x)}^{g_{1}(x)} \int_{f_{2}(y)}^{g_{2}(y)} K_{3}(x, y, t, s) R_{3}(v(t, s)) d s d t
\end{aligned}
$$

where $f_{k}, g_{k} \in C\left[a_{k}, b_{k}\right]$, for $k=1,2$, and moreover $a_{1} \leqslant f_{1}(x) \leqslant g_{1}(x) \leqslant b_{1}, a_{2} \leqslant f_{2}(y) \leqslant g_{2}(y) \leqslant b_{2}$.
Operator $\bar{R}_{m}$ is of the following form (if only the series converges, which will be discussed later):

$$
\begin{aligned}
& \bar{R}_{m}\left(\bar{u}_{m-1}, x, y\right)=\frac{1}{(m-1)!}\left(\frac{\partial^{m-1}}{\partial p^{m-1}} N\left(\sum_{i=0}^{\infty} u_{i}(x, y) p^{i}\right)\right)_{p=0}=\frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial p^{m-1}}\left[\sum_{i=1}^{\infty} u_{i}(x, y) p^{i}-F(x, y)\right. \\
& -\int_{f_{1}(x)}^{g_{1}(x)} K_{1}(x, y, t) R_{1}\left(\sum_{i=1}^{\infty} u_{i}(t, y) p^{i}\right) d t-\int_{f_{2}(y)}^{g_{2}(y)} K_{2}(x, y, s) R_{2}\left(\sum_{i=1}^{\infty} u_{i}(x, s) p^{i}\right) d s \\
& \left.-\int_{f_{1}(x)}^{g_{1}(x)} \int_{f_{2}(y)}^{g_{2}(y)} K_{3}(x, y, t, s) R_{3}\left(\sum_{i=1}^{\infty} u_{i}(t, s) p^{i}\right) d s d t\right]_{p=0}=\frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial p^{m-1}}\left[\sum_{i=1}^{\infty} u_{i}(x, y) p^{i}-F(x, y)\right. \\
& -\sum_{i=1}^{\infty} \int_{f_{1}(x)}^{g_{1}(x)} K_{1}(x, y, t) R_{1}\left(u_{i}(t, y) p^{i}\right) d t-\sum_{i=1}^{\infty} \int_{f_{2}(y)}^{g_{2}(y)} K_{2}(x, y, s) R_{2}\left(u_{i}(x, s) p^{i}\right) d s \\
& \left.-\sum_{i=1}^{\infty} \int_{f_{1}(x)}^{g_{1}(x)} \int_{f_{2}(y)}^{g_{2}(y)} K_{3}(x, y, t, s) R_{3}\left(u_{i}(t, s) p^{i}\right) d s d t\right]_{p=0}=u_{m-1}(x, y)-\frac{1-\chi_{m}}{(m-1)!} F(x, y) \\
& -\int_{f_{1}(x)}^{g_{1}(x)} K_{1}(x, y, t) R_{1}\left(u_{m-1}(t, y)\right) d t-\int_{f_{2}(y)}^{g_{2}(y)} K_{2}(x, y, s) R_{2}\left(u_{m-1}(x, s)\right) d s \\
& -\int_{f_{1}(x)}^{g_{1}(x)} \int_{f_{2}(y)}^{g_{2}(y)} K_{3}(x, y, t, s) R_{3}\left(u_{m-1}(t, s)\right) d s d t= \begin{cases}N\left(u_{0}(x, y)\right) & \text { for } m=1, \\
N\left(u_{m-1}(x, y)\right)+F(x, y) & \text { for } m \geqslant 2 .\end{cases}
\end{aligned}
$$

Hence, we obtain the following formulae for the functions $u_{m}$ :

$$
\begin{align*}
u_{1}(x)=h N\left(u_{0}(x, y)\right)=h\left(u_{0}(x, y)-F(x, y)-\right. & \int_{f_{1}(x)}^{g_{1}(x)} K_{1}(x, y, t) R_{1}\left(u_{0}(t, y)\right) d t \\
& \left.-\int_{f_{2}(y)}^{g_{2}(y)} K_{2}(x, y, s) R_{2}\left(u_{0}(x, s)\right) d s-\int_{f_{1}(x)}^{g_{1}(x)} \int_{f_{2}(y)}^{g_{2}(y)} K_{3}(x, y, t, s) R_{3}\left(u_{0}(t, s)\right) d s d t\right) \tag{14}
\end{align*}
$$

where $u_{0} \in C(D)$, and for $m \geqslant 2$ :

$$
\begin{align*}
& u_{m}(x)=(1+h) u_{m-1}(x)-h\left(N\left(u_{m-1}(x, y)\right)+F(x, y)\right)=(1+h) u_{m-1}(x, y) \\
& -h\left(\int_{f_{1}(x)}^{g_{1}(x)} K_{1}(x, y, t) R_{1}\left(u_{m-1}(t, y)\right) d t+\int_{f_{2}(y)}^{g_{2}(y)} K_{2}(x, y, s) R_{2}\left(u_{m-1}(x, s)\right) d s\right. \\
& \left.\quad+\int_{f_{1}(x)}^{g_{1}(x)} \int_{f_{2}(y)}^{g_{2}(y)} K_{3}(x, y, t, s) R_{3}\left(u_{m-1}(t, s)\right) d s d t\right) \tag{15}
\end{align*}
$$

Let us prove now that the sum of the series is the solution of the considered integral equation.

Theorem 1. Let $R_{k}$ be the bounded linear operators and let the functions $u_{m}, m \geq 1$, be defined by relations (14) and (15), where $u_{0} \in C(D)$. Then, if the series in (7) converges, its sum is the solution of Eq. (1).

Proof. By using the condition necessary for convergence of the series it occurs for any $(x, y) \in D$ that

$$
\lim _{m \rightarrow \infty} u_{m}(x, y)=0
$$

We have also

$$
\begin{aligned}
\sum_{m=1}^{n} L\left(u_{m}(x, y)-\chi_{m} u_{m-1}(x, y)\right) & =u_{1}(x, y)+\left(u_{2}(x, y)-u_{1}(x, y)\right) \\
+ & \left(u_{3}(x, y)-u_{2}(x, y)\right)+\ldots+\left(u_{n}(x, y)-u_{n-1}(x, y)\right)=u_{n}(x, y)
\end{aligned}
$$

The two last dependencies give us the equality

$$
\sum_{m=1}^{\infty} L\left(u_{m}(x, y)-\chi_{m} u_{m-1}(x, y)\right)=0
$$

Thus, from Eq. (8) we have

$$
\sum_{m=1}^{\infty} \bar{R}_{m}\left(\bar{u}_{m-1}, x, y\right)=0
$$

After some transformations we obtain

$$
\begin{aligned}
& 0=\sum_{m=1}^{\infty} \bar{R}_{m}\left(\bar{u}_{m-1}, x, y\right)=N\left(u_{0}(x, y)\right)+\sum_{m=2}^{\infty}\left(N\left(u_{m-1}(x, y)\right)+F(x, y)\right) \\
&= \sum_{m=1}^{\infty}\left(N\left(u_{m-1}(x, y)\right)+F(x, y)\right)-F(x, y)=\sum_{m=1}^{\infty}\left(u_{m-1}(x, y)-\int_{f_{1}(x)}^{g_{1}(x)} K_{1}(x, y, t) R_{1}\left(u_{m-1}(t, y)\right) d t\right. \\
&\left.-\int_{f_{2}(y)}^{g_{2}(y)} K_{2}(x, y, s) R_{2}\left(u_{m-1}(x, s)\right) d s-\int_{f_{1}(x)}^{g_{1}(x)} \int_{f_{2}(y)}^{g_{2}(y)} K_{3}(x, y, t, s) R_{3}\left(u_{m-1}(t, s)\right) d s d t\right)-F(x, y) \\
&=u(x, y)-F(x, y)-\int_{f_{2}(y)}^{g_{1}(x)} K_{1}(x, y, t) R_{1}(u(t, y)) d t \\
&-\int_{f_{2}(x)}^{g_{2}(y)} K_{2}(x, y, s) R_{2}(u(x, s)) d s-\int_{f_{1}(x)}^{g_{1}(x)} \int_{g_{2}(y)} K_{3}(x, y, t, s) R_{3}(u(t, s)) d s d t
\end{aligned}
$$

Let us now give the condition ensuring the convergence of the series under consideration.
Theorem 2. If the following inequality is satisfied:

$$
\begin{equation*}
\bar{M}:=\left\|K_{1}\right\|\left\|R_{1}\right\|\left(b_{1}-a_{1}\right)+\left\|K_{2}\right\|\left\|R_{2}\right\|\left(b_{2}-a_{2}\right)+\left\|K_{3}\right\|\left\|R_{3}\right\|\left(b_{1}-a_{1}\right)\left(b_{2}-a_{2}\right)<1, \tag{16}
\end{equation*}
$$

then the value of the convergence control parameter can be selected so that the series occurring in (7) converges uniformly in region $D=\left[a_{1}, b_{1}\right] \times\left[a_{2}, b_{2}\right]$.

Proof. Let $u_{0}$ be a function of class $C(D)$. Now, we look for the constraints for the function $u_{m}$, $m \geqslant 1$, in region $D$. Sequentially computing, we get for $m=1$ :

$$
\begin{array}{r}
\left|u_{1}(x, y)\right|=\mid h\left(u_{0}(x, y)-F(x, y)-\int_{f_{1}(x)}^{g_{1}(x)} K_{1}(x, y, t) R_{1}\left(u_{0}(t, y)\right) d t\right. \\
\left.-\int_{f_{2}(y)}^{g_{2}(y)} K_{2}(x, y, s) R_{2}\left(u_{0}(x, s)\right) d s-\int_{f_{1}(x)}^{g_{1}(x)} \int_{f_{2}(y)}^{g_{2}(y)} K_{3}(x, y, t, s) R_{3}\left(u_{0}(t, s)\right) d s d t\right) \mid \\
\leqslant|h|\left(\|F\|+\left\|u_{0}\right\|\left(1+\left\|K_{1}\right\|\left\|R_{1}\right\|\left(b_{1}-a_{1}\right)+\left\|K_{2}\right\|\left\|R_{2}\right\|\left(b_{2}-a_{2}\right)\right.\right. \\
\left.\left.+\left\|K_{3}\right\|\left\|R_{3}\right\|\left(b_{1}-a_{1}\right)\left(b_{2}-a_{2}\right)\right)\right) \leqslant|h|\left(\|F\|+\left\|u_{0}\right\|(1+\bar{M})\right)<\infty
\end{array}
$$

for $m=2$ :

$$
\begin{aligned}
\left|u_{2}(x, y)\right|= & \mid(1+h) u_{1}(x, y)-h\left(\int_{f_{1}(x)}^{g_{1}(x)} K_{1}(x, y, t) R_{1}\left(u_{1}(t, y)\right) d t+\int_{f_{2}(y)}^{g_{2}(y)} K_{2}(x, y, s) R_{2}\left(u_{1}(x, s)\right) d s\right. \\
& \left.+\int_{f_{1}(x)}^{g_{1}(x)} \int_{f_{2}(y)}^{g_{2}(y)} K_{3}(x, y, t, s) R_{3}\left(u_{1}(t, s)\right) d s d t\right)\left|\leqslant|1+h|\left\|u_{1}\right\|+|h|\left(\left\|K_{1}\right\|\left\|R_{1}\right\|\left\|u_{1}\right\|\left(b_{1}-a_{1}\right)\right.\right. \\
& \left.\left.+\left\|K_{2}\right\|\left\|R_{2}\right\|\left\|u_{1}\right\|\left(b_{2}-a_{2}\right)+\left\|K_{3}\right\|\left\|R_{3}\right\|\left\|u_{1}\right\|\left(b_{1}-a_{1}\right)\left(b_{2}-a_{2}\right)\right)\right)=\beta_{h}\left\|u_{1}\right\|
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_{h}:=|1+h|+|h|\left(\left\|K_{1}\right\|\left\|R_{1}\right\|\left(b_{1}-a_{1}\right)+\left\|K_{2}\right\|\left\|R_{2}\right\|\right. & \left(b_{2}-a_{2}\right) \\
& \left.+\left\|K_{3}\right\|\left\|R_{3}\right\|\left(b_{1}-a_{1}\right)\left(b_{2}-a_{2}\right)\right)=|1+h|+|h| \bar{M} .
\end{aligned}
$$

Using the above, it can be inductively shown that for $m \geqslant 1$ we have

$$
\left\|u_{m}\right\| \leqslant \beta_{h}^{m-1}\left\|u_{1}\right\| .
$$

Thus, for the considered series (7) we obtain

$$
\sum_{m=0}^{\infty} u_{m}(x) \leqslant \sum_{m=0}^{\infty}\left|u_{m}(x)\right| \leqslant\left\|u_{0}\right\|+\left\|u_{1}\right\| \sum_{m=1}^{\infty} \beta_{h}^{m-1}=\left\|u_{0}\right\|+\left\|u_{1}\right\|\left(1-\beta_{h}\right)^{-1}
$$

if only $\beta_{h}<1$ (certainly that is $\beta_{h}>0$ ). Therefore, by virtue of the comparative criterion, the discussed series will be uniformly convergent in region $D$.

Let us now consider if it is possible to choose the value of parameter $h$ so that $\beta_{h}<1$, that is,

$$
|1+h|+|h| \bar{M}<1 .
$$

The above inequality is equivalent to the following inequality (because $h \neq 0$ ):

$$
\begin{equation*}
\bar{M}<\frac{1-|1+h|}{|h|} . \tag{17}
\end{equation*}
$$

For the right-hand side of this inequality we have

$$
\frac{1-|1+h|}{|h|}=\left\{\begin{array}{cl}
-1-\frac{2}{h} & \text { for } h<-1 \\
1 & \text { for } h \in[-1,0) \\
-1 & \text { for } h>0
\end{array}\right.
$$

Thus, if condition (16) is satisfied, the value of parameter $h$ can be chosen so that the inequality (17) is true (for this purpose, it is sufficient to choose any $h \in[-1,0)$ ) and thereby $\beta_{h}<1$.

Remark 1. In the case of the Volterra equation which we obtain for $f_{1}(x)=a_{1}, g_{1}(x)=x, f_{2}(y)=$ $a_{2}, g_{2}(y)=y$, the equation has a unique solution (see [30]), and the series (7) is always convergent.

By using the estimations derived in the last proof it is easy to prove the following theorem.
Theorem 3. If the inequality (16) is satisfied and $n \in \mathbb{N}$, then the error of approximate solution can be estimated as follows:

$$
\begin{equation*}
\left\|u-\widehat{u}_{n}\right\| \leqslant \frac{\beta_{h}^{n}}{1-\beta_{h}}\left\|u_{1}\right\|, \tag{18}
\end{equation*}
$$

where $\beta_{h}=|1+h|+|h| \bar{M}$ and

$$
\left\|u_{1}\right\| \leqslant|h|\left(\|F\|+\left\|u_{0}\right\|(1+\bar{M})\right) .
$$

Proof. For any $(x, y) \in D$ we have

$$
\begin{aligned}
\left|u(x, y)-\widehat{u}_{n}(x, y)\right|=\left|\sum_{m=0}^{\infty} u_{m}(x, y)-\sum_{m=0}^{n} u_{m}(x, y)\right| & \leqslant \sum_{m=n+1}^{\infty}\left|u_{m}(x, y)\right| \\
& \leqslant \sum_{m=n+1}^{\infty}\left\|u_{m}\right\| \leqslant\left\|u_{1}\right\| \sum_{m=n+1}^{\infty} \beta_{h}^{m-1}=\frac{\beta_{h}^{n}}{1-\beta_{h}}\left\|u_{1}\right\| .
\end{aligned}
$$

## 5. NONLINEAR INTEGRAL EQUATION

Let us now proceed to the case of nonlinear equations. We assume that the operators $R_{k}, k=1,2,3$, occurring in Eq. (1), are nonlinear and they satisfy the Lipschitz condition

$$
\left\|R_{k}\left(v_{1}\right)-R_{k}\left(v_{2}\right)\right\| \leqslant s_{k}\left\|v_{1}-v_{2}\right\|, \quad \text { for every } \quad v_{1}, v_{2} \in C(D)
$$

for some $s_{k}>0, k=1,2,3$.
Defining, as before, the operators $L$ and $N$ and after using the HAM we get the following formula for the function $u_{m}$ :

$$
\begin{equation*}
u_{m}(x, y)=\chi_{m} u_{m-1}(x, y)+h \bar{R}_{m}\left(\bar{u}_{m-1}, x, y\right) \tag{19}
\end{equation*}
$$

where $\chi_{m}$ is determined by (9) and operator $\bar{R}_{m}$ is defined by relationship (10).
By using the definitions of appropriate operators we obtain

$$
\begin{align*}
u_{1}(x, y)=h( & u_{0}(x, y)-F(x, y)-\int_{f_{1}(x)}^{g_{1}(x)} K_{1}(x, y, t) R_{1}\left(u_{0}(t, y)\right) d t \\
& \left.-\int_{f_{2}(y)}^{g_{2}(y)} K_{2}(x, y, s) R_{2}\left(u_{0}(x, s)\right) d s-\int_{f_{1}(x)}^{g_{1}(x)} \int_{f_{2}(y)}^{g_{2}(y)} K_{3}(x, y, t, s) R_{3}\left(u_{0}(t, s)\right) d s d t\right) \tag{20}
\end{align*}
$$

where $u_{0} \in C(D)$, and for $m \geqslant 2$ :

$$
\begin{align*}
& u_{m}(x, y)=(1+h) u_{m-1}(x, y)-\frac{h}{(m-1)!} \\
&\left(\int_{f_{1}(x)}^{g_{1}(x)} K_{1}(x, y, t)\left(\frac{\partial^{m-1}}{\partial p^{m-1}} R_{1}\left(\sum_{i=0}^{\infty} u_{i}(t, y) p^{i}\right)\right)_{p=0} d t\right. \\
& g_{2}(y)  \tag{21}\\
& g_{2}(x, y, s)\left(\frac{\partial^{m-1}}{\partial p^{m-1}} R_{2}\left(\sum_{i=0}^{\infty} u_{i}(x, s) p^{i}\right)\right)_{p=0} d s \\
&\left.+\int_{f_{1}(x)}^{g_{1}(x)} \int_{f_{2}(y)}^{g_{2}(y)} K_{3}(x, y, t, s)\left(\frac{\partial^{m-1}}{\partial p^{m-1}} R_{3}\left(\sum_{i=0}^{\infty} u_{i}(t, s) p^{i}\right)\right)_{p=0} d s d t\right)
\end{align*}
$$

Let us show that if the created series is convergent then its sum is the searched solution of the nonlinear integral equation under consideration.

Theorem 4. Let $u_{m}, m \geq 1$, be the functions defined by relationships (20) and (21), where $u_{0} \in$ $C(D)$. Then, if $s_{k}<1$ for $k=1,2,3$ and the series in (7) converges, the sum of this series is the solution of Eq. (1).

Proof. Let the series in (7) be convergent. For any $(x, y) \in D$, according to the necessary condition for convergence of the series, we have $\lim _{m \rightarrow \infty} u_{m}(x, y)=0$. Let us introduce the following notation:

$$
H_{k, m}(x, y)=\left.\frac{1}{m!}\left(\frac{\partial^{m}}{\partial p^{m}} R_{k}\left(\sum_{i=0}^{\infty} u_{i}(x, y) p^{i}\right)\right)\right|_{p=0}
$$

If $R_{k}$ are the contraction mappings $\left(s_{k}<1\right)$ and the series in (7) converges to $u(x, y)$, then the series $\sum_{m=0}^{\infty} H_{k, m}(x, y), k=1,2,3$, are respectively convergent to $R_{k}(u(x, y))$ (see [14]).

By applying the definition of operator $L$ we can write

$$
\sum_{m=1}^{n} L\left(u_{m}(x, y)-\chi_{m} u_{m-1}(x, y)\right)=\sum_{m=1}^{n}\left(u_{m}(x v)-\chi_{m} u_{m-1}(x, y)\right)=u_{n}(x)
$$

Hence

$$
\sum_{m=1}^{\infty} L\left(u_{m}(x)-\chi_{m} u_{m-1}(x)\right)=\lim _{n \rightarrow \infty} u_{n}(x)=0
$$

From Eq. (8) we get

$$
h \sum_{m=1}^{\infty} \bar{R}_{m}\left(\bar{u}_{m-1}, x\right)=\sum_{m=1}^{\infty} L\left(u_{m}(x)-\chi_{m} u_{m-1}(x)\right) .
$$

And because $h \neq 0$, we have

$$
\sum_{m=1}^{\infty} \bar{R}_{m}\left(\bar{u}_{m-1}, x\right)=0
$$

After some transformations we obtain

$$
\begin{aligned}
& 0=\sum_{m=1}^{\infty} \bar{R}_{m}\left(\bar{u}_{m-1}, x, y\right)=\sum_{m=1}^{\infty}\left(\frac { 1 } { ( m - 1 ) ! } \frac { \partial ^ { m - 1 } } { \partial p ^ { m - 1 } } \left[\sum_{i=1}^{\infty} u_{i}(x, y) p^{i}-F(x, y)\right.\right. \\
& -\int_{f_{1}(x)}^{g_{1}(x)} K_{1}(x, y, t) R_{1}\left(\sum_{i=1}^{\infty} u_{i}(t, y) p^{i}\right) d t-\int_{f_{2}(y)}^{g_{2}(y)} K_{2}(x, y, s) R_{2}\left(\sum_{i=1}^{\infty} u_{i}(x, s) p^{i}\right) d s \\
& \left.\left.-\int_{f_{1}(x)}^{g_{1}(x)} \int_{f_{2}(y)}^{g_{2}(y)} K_{3}(x, y, t, s) R_{3}\left(\sum_{i=1}^{\infty} u_{i}(t, s) p^{i}\right) d s d t\right]_{p=0}\right)=\sum_{m=1}^{\infty}\left(u_{m-1}(x, y)-\frac{1-\chi_{m}}{(m-1)!} F(x, y)\right. \\
& -\int_{f_{1}(x)}^{g_{1}(x)} K_{1}(x, y, t)\left[\frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial p^{m-1}} R_{1}\left(\sum_{i=1}^{\infty} u_{i}(t, y) p^{i}\right)\right]_{p=0} d t \\
& -\int_{f_{2}(y)}^{g_{2}(y)} K_{2}(x, y, s)\left[\frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial p^{m-1}} R_{2}\left(\sum_{i=1}^{\infty} u_{i}(x, s) p^{i}\right)\right]_{p=0} d s \\
& \left.-\int_{f_{1}(x)}^{g_{1}(x)} \int_{f_{2}(y)}^{g_{2}(y)} K_{3}(x, y, t, s)\left[\frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial p^{m-1}} R_{3}\left(\sum_{i=1}^{\infty} u_{i}(t, s) p^{i}\right)\right]_{p=0} d s d t\right) \\
& =\sum_{m=1}^{\infty}\left(u_{m-1}(x, y)-\frac{1-\chi_{m}}{(m-1)!} F(x, y)-\int_{f_{1}(x)}^{g_{1}(x)} K_{1}(x, y, t) H_{1, m-1}(t, y) d t\right. \\
& \left.-\int_{f_{2}(y)}^{g_{2}(y)} K_{2}(x, y, s) H_{2, m-1}(x, s) d s-\int_{f_{1}(x)}^{g_{1}(x)} \int_{f_{2}(y)}^{g_{2}(y)} K_{3}(x, y, t, s) H_{3, m-1}(t, s) d s d t\right)=\sum_{m=1}^{\infty} u_{m-1}(x, y)-F(x, y) \\
& -\int_{f_{1}(x)}^{g_{1}(x)} K_{1}(x, y, t) \sum_{m=1}^{\infty} H_{1, m-1}(t, y) d t-\int_{f_{2}(y)}^{g_{2}(y)} K_{2}(x, y, s) \sum_{m=1}^{\infty} H_{2, m-1}(x, s) d s \\
& -\int_{f_{1}(x)}^{g_{1}(x)} \int_{f_{2}(y)}^{g_{2}(y)} K_{3}(x, y, t, s) \sum_{m=1}^{\infty} H_{3, m-1}(t, s) d s d t=u(x, y)-F(x, y)-\int_{f_{1}(x)}^{g_{1}(x)} K_{1}(x, y, t) R_{1}(u(t, y)) d t \\
& -\int_{f_{2}(y)}^{g_{2}(y)} K_{2}(x, y, s) R_{2}(u(x, s)) d s-\int_{f_{1}(x)}^{g_{1}(x)} \int_{f_{2}(y)}^{g_{2}(y)} K_{3}(x, y, t, s) R_{3}(u(t, s)) d s d t .
\end{aligned}
$$

Remark 2. In the above proof we used the fact that $R_{k}$ are the contraction mappings in order to ensure the convergence of series $\sum_{m=0}^{\infty} H_{k, m}(x, y)$ to $R_{k}(u(x, y))$, in the case when the series in (7) converges to $u(x, y)$. The same convergence can be obtained under other assumptions, for example, when $R_{k}$ are of class $C^{\infty}$ and they fulfil some additional conditions.

It is easy to prove the sufficient condition for the convergence of the series under consideration.
Theorem 5. If $h$ is chosen so that there exist the constants $\gamma_{h} \in(0,1)$ and $m_{0} \in \mathbb{N}$ such that for every $m \geqslant M_{0}$ the inequality

$$
\begin{equation*}
\left\|u_{m+1}\right\| \leqslant \gamma_{h}\left\|u_{m}\right\| \tag{22}
\end{equation*}
$$

is satisfied, then the series appearing in (7) converges uniformly in region $D$.
Remark 3. The above theorem can be generalized as follows.
If parameter $h$ is chosen so that there exists a constant $m_{0} \in \mathbb{N}$, such that for every $m \geqslant M_{0}$ there exists $\gamma_{h, m} \in(0,1)$ satisfying the conditions

$$
\prod_{m=m_{0}}^{\infty} \gamma_{h, m}=0 \quad \text { and } \quad\left\|u_{m+1}\right\| \leqslant \gamma_{h, m}\left\|u_{m}\right\|
$$

then the series appearing in (7) converges uniformly in region $D$.
We proceed now to estimate the error of approximate solution.
Theorem 6. If the assumptions of Theorem 5 are satisfied and, moreover, $n \in \mathbb{N}$ and $n \geqslant m_{0}$, then we have the following estimation of the error of approximate solution:

$$
\begin{equation*}
\left\|u-\widehat{u}_{n}\right\| \leqslant \frac{\gamma_{h}^{n+1-m_{0}}}{1-\gamma_{h}}\left\|u_{m_{0}}\right\| \tag{23}
\end{equation*}
$$

Proof. Let $n \in \mathbb{N}$ and $n \geqslant m_{0}$. Then we get

$$
\begin{aligned}
\left\|u-\widehat{u}_{n}\right\|=\sup _{(x, y) \in D}\left|u(x, y)-\sum_{m=0}^{n} u_{m}(x, y)\right|=\sup _{(x, y) \in D}\left|\sum_{m=n+1}^{\infty} u_{m}(x, y)\right| \leqslant \sup _{(x, y) \in D}\left(\sum_{m=n+1}^{\infty}\left|u_{m}(x, y)\right|\right) \\
\leqslant \sum_{m=n+1}^{\infty} \sup _{(x, y) \in D}\left|u_{m}(x, y)\right|=\sum_{m=n+1}^{\infty}\left\|u_{m}\right\| \leqslant \sum_{m=n+1}^{\infty} \gamma_{h}^{m-m_{0}}\left\|u_{m_{0}}\right\|=\frac{\gamma_{h}^{n+1-m_{0}}}{1-\gamma_{h}}\left\|u_{m_{0}}\right\| .
\end{aligned}
$$

## 6. EXAMPLES

## Example 1

We begin with the solution of a linear equation for which $K_{1}(x, y, t)=\frac{1}{16}(x+y t), K_{2}(x, y, s)=$ $\frac{1}{16}(x s+y), K_{3}(x, y, t, s)=\frac{1}{16}(x s+y t), R_{i}(u)=u, i=1,2,3, F(x, y)=\frac{1}{192}\left(x(175-24 y)-6 x^{2}+\right.$ $y(175-6 y))$ and $f_{i}(x)=0, g_{i}(x)=1$ for $i=1,2$, where $x, y \in[0,1]$. The solution of Eq. (1) is then the function $u_{e}(x, y)=x+y$. In the discussed equation we have

$$
\left\|K_{i}\right\|=\frac{1}{8}, \quad\left\|R_{i}\right\|=1, \quad i=1,2,3
$$

Hence

$$
\bar{M}=\frac{3}{8}
$$

that is, the sum of generated series is the solution of the equation under consideration.
Assuming the initial approximation $u_{0}(x, y)=F(x, y)$ we get in the first step

$$
\begin{aligned}
u_{1}(x)=\frac{1}{36864}\left(-2855 h x-954 h x^{2}+36 h x^{3}-2855 h y\right. & -3912 h x y \\
& \left.+72 h x^{2} y-954 h y^{2}+72 h x y^{2}+36 h y^{3}\right)
\end{aligned}
$$

Figure 1 presents the graph of the logarithm of squared residual $E_{5}$. Numerically determined, the optimal value of the convergence control parameter is equal to -1.1255 . All the necessary calculations were carried out with the aid of Mathematica software.

Table 1 compiles the errors of reconstruction of the exact solution for the successive approximate solutions $\widehat{u}_{n}, n \in\{1,2, \ldots, 10\}$. As revealed by the above results, together with the increasing number of components in sum (7) the errors quickly decrease. The error decrease is the fastest for the optimal value of parameter $h$. For this value, the approximate solution $\widehat{u}_{5}$ provides the approximation of the sought function with the error not higher than $8.363 \cdot 10^{-8}$. Whereas the solution $\widehat{u}_{10}$ approximates the exact solution with the error not higher than $2.554 \cdot 10^{-13}$. Moreover, the further we move away from that optimal value, the more slowly the errors decrease. This last fact is illustrated in Table 2.


Fig. 1. Logarithm of the squared residual $E_{5}$.
Table 1. Values of errors in the reconstruction of the exact solution $\left(\Delta_{n}=\left\|u_{e}-\widehat{u}_{n}\right\|\right)$.

| $n$ | $\Delta_{n}$ | $\Delta_{(18)}$ |
| :---: | :---: | :---: |
| 1 | $1.915 \cdot 10^{-2}$ | 0.418 |
| 2 | $5.528 \cdot 10^{-4}$ | 0.229 |
| 3 | $3.735 \cdot 10^{-5}$ | 0.125 |
| 4 | $6.410 \cdot 10^{-7}$ | $6.863 \cdot 10^{-2}$ |
| 5 | $8.363 \cdot 10^{-8}$ | $3.758 \cdot 10^{-2}$ |
| 6 | $5.460 \cdot 10^{-9}$ | $2.058 \cdot 10^{-2}$ |
| 7 | $2.311 \cdot 10^{-10}$ | $1.127 \cdot 10^{-2}$ |
| 8 | $4.289 \cdot 10^{-11}$ | $6.170 \cdot 10^{-3}$ |
| 9 | $3.605 \cdot 10^{-12}$ | $3.378 \cdot 10^{-3}$ |
| 10 | $2.554 \cdot 10^{-13}$ | $1.850 \cdot 10^{-4}$ |

Table 2. Values of errors in the reconstruction of the exact solution for various values of the convergence control parameter $\left(\Delta_{n}=\left\|u_{e}-\widehat{u}_{n}\right\|\right)$.

| $h$ | $\Delta_{5}$ | $\Delta_{10}$ |
| :---: | :---: | :---: |
| -0.2 | 0.145 | $5.756 \cdot 10^{-2}$ |
| -0.5 | $2.358 \cdot 10^{-2}$ | $1.506 \cdot 10^{-3}$ |
| -1.0 | $2.902 \cdot 10^{-5}$ | $2.052 \cdot 10^{-9}$ |
| -1.1255 | $8.363 \cdot 10^{-8}$ | $2.554 \cdot 10^{-13}$ |
| -1.2 | $1.499 \cdot 10^{-6}$ | $5.990 \cdot 10^{-11}$ |
| -1.5 | $3.339 \cdot 10^{-4}$ | $4.644 \cdot 10^{-6}$ |
| -1.8 | $2.334 \cdot 10^{-2}$ | $4.989 \cdot 10^{-4}$ |

In Table 1 there is also included the estimation of error resulting from inequality (18). In the considered example, we have $\beta_{h}=0.547563$ and $\left\|u_{1}\right\|=0.345429$ for the optimal value of the convergence control parameter $h=-1.1255$. Hence, the inequality (18) takes the form

$$
\Delta_{n}:=\left\|u-\widehat{u}_{n}\right\| \leqslant \Delta_{(18)}=0.763485 \cdot(0.547563)^{n} .
$$

The above formula presents the estimation of error of the approximate solution (the worst possible case). In fact, the errors of approximate solutions are generally much smaller than the value determined on the right-hand side of this inequality.

The difference $\left|u_{e}(x)-\widehat{u}_{n}(x)\right|$ for $n=3$ and $n=10$ is plotted in Fig. 2. The results show that the method converges rapidly and counting only few first components of the generated series provides a very good approximation of the exact solution.


Fig. 2. Distribution of error of the exact solution approximation for: a) $n=3$ and b) $n=10$.

## Example 2

We consider now the equation defined by functions $K_{1}(x, y, t)=x t, K_{2}(x, y, s)=y s, K_{3}(x, y, t, s)=$ $x s-y t, R_{i}(u)=u, i=1,2,3, F(x, y)=\frac{x y}{3}\left(3-x^{3}-y^{3}\right)$ and $f_{i}(x)=0, g_{i}(x)=x$ for $i=1,2$, where $x, y \in[0,1]$. This time the solution of Eq. (1) is given by function $u_{e}(x, y)=x y$. This is an example of the Volterra equation for which, despite the fact that inequality (16) is not satisfied ( $\bar{M}=3$ ), the series (7) still converges (see Remark 1).

Assuming, as before, the initial approximation $u_{0}(x, y)=F(x, y)$ we get in the first step

$$
u_{1}(x)=\frac{1}{180} h x y\left(10 x^{6}-x^{5} y^{2}+x^{2} y^{5}+10 y^{3}\left(y^{3}-6\right)+20 x^{3}\left(2 y^{3}-3\right)\right)
$$

Figure 3 shows the graph of the logarithm of squared residual $E_{5}$. Numerically determined, the optimal value of convergence control parameter is equal to -1.087 .


Fig. 3. Logarithm of the squared residual $E_{5}$.

In Table 3 the errors of successive approximate solutions $\widehat{u}_{n}, n \in\{1, \ldots, 10\}$, are presented. And similarly as in the previous case, we can state that together with the increasing number of components in sum (7) the errors quickly decrease. The variability of the error size with respect to the changes of the value of parameter controlling the convergence is illustrated in Table 4. The graph of the error $\left|u_{e}(x)-\widehat{u}_{n}(x)\right|$ for $n=3$ and $n=10$ is displayed in Fig. 4. Again, the analysis confirms that the method is rapidly convergent, thanks to which the sum of only few first components of the generated series provides a very good approximation of the exact solution.

Table 3. Values of errors in the reconstruction of the exact solution $\left(\Delta_{n}=\left\|u_{e}-\widehat{u}_{n}\right\|\right)$.

| $n$ | $\Delta_{n}$ |
| :---: | :---: |
| 1 | 0.304 |
| 2 | $8.791 \cdot 10^{-2}$ |
| 3 | $1.597 \cdot 10^{-2}$ |
| 4 | $1.681 \cdot 10^{-3}$ |
| 5 | $6.363 \cdot 10^{-5}$ |
| 6 | $5.383 \cdot 10^{-6}$ |
| 7 | $4.570 \cdot 10^{-7}$ |
| 8 | $2.938 \cdot 10^{-8}$ |
| 9 | $6.819 \cdot 10^{-10}$ |
| 10 | $2.175 \cdot 10^{-10}$ |

Table 4. Values of errors in the reconstruction of the exact solution for various values of the convergence control parameter $\left(\Delta_{n}=\left\|u_{e}-\widehat{u}_{n}\right\|\right)$.

| $h$ | $\Delta_{5}$ | $\Delta_{10}$ |
| :---: | :---: | :---: |
| -0.2 | 0.382 | 0.206 |
| -0.5 | 0.124 | $1.583 \cdot 10^{-2}$ |
| -1.0 | $1.761 \cdot 10^{-3}$ | $9.983 \cdot 10^{-8}$ |
| -1.087 | $6.363 \cdot 10^{-5}$ | $2.175 \cdot 10^{-10}$ |
| -1.2 | $1.160 \cdot 10^{-4}$ | $2.552 \cdot 10^{-8}$ |
| -1.5 | $3.999 \cdot 10^{-3}$ | $5.735 \cdot 10^{-5}$ |
| -1.8 | $2.337 \cdot 10^{-2}$ | $3.298 \cdot 10^{-3}$ |



Fig. 4. Distribution of error of the exact solution approximation for: a) $n=3$ and b) $n=10$.

## Example 3

Let us consider now the two-dimensional Fredholm integral equation of the second kind, in which $K_{1}(x, y, t)=0, K_{2}(x, y, s)=0, K_{3}(x, y, t, s)=\frac{1}{20}(x y-s t), R_{i}(u)=u, i=1,2,3, F(x, y)=x+y+$ $\frac{1}{1920}\left(48 \pi+\pi^{5}-96 x y-12 \pi^{3} x y-96\right)+\cos y \sin x$ and $f_{i}(x)=0, g_{i}(x)=\pi / 2$ for $i=1,2$, where $x, y \in[0, \pi / 2]$. The solution of Eq. (1) is in this case the function $u_{e}(x, y)=\sin x \cos y+x+y$.

Assuming the initial approximation $u_{0}(x, y)=F(x, y)$ we obtain in the first step

$$
\begin{aligned}
u_{1}(x)=-\frac{1}{7372800} h\left(\pi^{9}+32 \pi^{6}-\right. & 3984 \pi^{5}-184320 \pi \\
& \left.+x y\left(12 \pi^{7}+48384 \pi^{3}-4608 \pi^{2}\right)+(1-x y)\left(288 \pi^{4}-368640\right)\right) .
\end{aligned}
$$

Figure 5 presents the graph of the logarithm of squared residual $E_{5}$. Numerically determined, the optimal value of the convergence control parameter is equal to -0.9955 .

Let us compare the obtained approximate solutions $\widehat{u}_{n}$, for few $n$, with the exact solution by taking the difference of these functions. Thus we obtain

$$
\begin{aligned}
& \widehat{u}_{3}-u_{e}=-3.780 \cdot 10^{-5}-5.633 \cdot 10^{-6} x y, \\
& \widehat{u}_{5}-u_{e}=1.980 \cdot 10^{-7}-2.059 \cdot 10^{-8} x y, \\
& \widehat{u}_{6}-u_{e}=-1.240 \cdot 10^{-8}+2.266 \cdot 10^{-8} x y, \\
& \widehat{u}_{8}-u_{e}=4.659 \cdot 10^{-11}-9.894 \cdot 10^{-11} x y, \\
& \widehat{u}_{10}-u_{e}=-1.532 \cdot 10^{-13}+4.124 \cdot 10^{-13} x y .
\end{aligned}
$$



Fig. 5. Logarithm of the squared residual $E_{5}$.
In Table 5 the errors $\left(\left\|u_{e}-\widehat{u}_{n}\right\|=\sup _{(x, y) \in D}\left|u_{e}(x, y)-\widehat{u}_{n}(x, y)\right|\right)$ of approximate solutions $\widehat{u}_{n}$, $n \in\{1,2, \ldots, 10\}$ are presented. Whereas the distributions of error for $n=3$ and $n=10$ in the entire domain $D$ are displayed in Fig. 6. And certainly the error decrease is the fastest for the optimal value of the parameter controlling the convergence (see Table 6).

Table 5. Values of errors in the reconstruction of the exact solution $\left(\Delta_{n}=\left\|u_{e}-\widehat{u}_{n}\right\|\right)$.

| $n$ | $\Delta_{n}$ | $\Delta_{(18)}$ |
| :---: | :---: | :---: |
| 1 | $1.553 \cdot 10^{-2}$ | 0.191 |
| 2 | $1.994 \cdot 10^{-3}$ | $5.861 \cdot 10^{-2}$ |
| 3 | $5.170 \cdot 10^{-5}$ | $1.802 \cdot 10^{-2}$ |
| 4 | $9.409 \cdot 10^{-6}$ | $5.543 \cdot 10^{-3}$ |
| 5 | $1.980 \cdot 10^{-7}$ | $1.705 \cdot 10^{-3}$ |
| 6 | $4.352 \cdot 10^{-8}$ | $5.243 \cdot 10^{-4}$ |
| 7 | $9.992 \cdot 10^{-10}$ | $1.612 \cdot 10^{-4}$ |
| 8 | $1.972 \cdot 10^{-10}$ | $4.958 \cdot 10^{-5}$ |
| 9 | $4.899 \cdot 10^{-12}$ | $1.525 \cdot 10^{-5}$ |
| 10 | $8.630 \cdot 10^{-13}$ | $4.689 \cdot 10^{-6}$ |



Fig. 6. Distribution of error of the exact solution approximation for: a) $n=3$ and b) $n=10$.

Table 6. Values of errors in the reconstruction of the exact solution for various values of the convergence control parameter $\left(\Delta_{n}=\left\|u_{e}-\widehat{u}_{n}\right\|\right)$.

| $h$ | $\Delta_{5}$ | $\Delta_{10}$ |
| :---: | :---: | :---: |
| -0.2 | 0.128 | $3.917 \cdot 10^{-2}$ |
| -0.5 | $9.639 \cdot 10^{-3}$ | $2.030 \cdot 10^{-4}$ |
| -0.7 | $5.090 \cdot 10^{-4}$ | $1.687 \cdot 10^{-6}$ |
| -0.9955 | $1.980 \cdot 10^{-7}$ | $8.630 \cdot 10^{-13}$ |
| -1.3 | $9.565 \cdot 10^{-4}$ | $2.782 \cdot 10^{-6}$ |
| -1.5 | $1.532 \cdot 10^{-2}$ | $1.886 \cdot 10^{-4}$ |
| -1.8 | 0.166 | $3.479 \cdot 10^{-2}$ |

Table 5 includes also the error estimations resulting from the inequality (18). In the considered example, we have $\beta_{h}=0.307534,\left\|u_{1}\right\|=0.429133$ for the optimal value of the convergence control parameter $h=-0.9955$. Hence, the inequality (18) takes the form

$$
\Delta_{n}:=\left\|u-\widehat{u}_{n}\right\| \leqslant \Delta_{(18)}=0.619717 \cdot(0.307534)^{n} .
$$

## Example 4

As the next example we consider the two-dimensional nonlinear Volterra integral equation of the second kind, in which: $K_{1}(x, y, t)=0, K_{2}(x, y, s)=0, K_{3}(x, y, t, s)=t^{2}+s x, R_{3}(u)=u^{2}, F(x, y)=$ $x+y-\frac{1}{180} x^{2} y\left(36 x^{3}+75 x^{2} y+80 x y^{2}+45 y^{3}\right)$ and $f_{i}(x)=0, g_{i}(x)=x$ for $i=1,2$, where $x, y \in[0,1]$. The solution of Eq. (1) is now given by the function $u_{e}(x, y)=x+y$.

Assuming the initial approximation $u_{0}(x, y)=F(x, y)$ we get in the first step

$$
\begin{aligned}
& u_{1}(x)=-\frac{h x^{2} y}{1167566400}\left(518918400 x y^{2}+1197504 x^{11} y^{2}+291891600 y^{3}+5115474 x^{10} y^{3}\right. \\
&+11351340 x^{9} y^{4}-13899600 x^{2} y\left(2 y^{5}-35\right)+16750734 x^{8} y^{5}+54054 x^{4} y^{4}\left(27 y^{5}-2488\right) \\
&-540540 x^{3}\left(161 y^{5}-432\right)+32175 x^{6} y^{2}\left(389 y^{5}-2556\right)+8580 x^{5} y^{3}\left(695 y^{5}-15363\right) \\
&\left.+2860 x^{7} y\left(6059 y^{5}-9072\right)\right) .
\end{aligned}
$$

In Fig. 7 the logarithm of squared residual $E_{5}$ is plotted. Numerically determined, the optimal value of the convergence control parameter is equal to -1.131 . Table 7 presents the errors of the successive approximate solutions $\widehat{u}_{n}, n \in\{1,2, \ldots, 10\}$. And in this case, the theoretical error estimation


Fig. 7. Logarithm of the squared residual $E_{5}$.

Table 7. Values of errors in the reconstruction of the exact solution $\left(\Delta_{n}=\left\|u_{e}-\widehat{u}_{n}\right\|\right)$.

| $n$ | $\Delta_{n}$ |
| :---: | :---: |
| 1 | 0.233 |
| 2 | $3.981 \cdot 10^{-2}$ |
| 3 | $3.142 \cdot 10^{-3}$ |
| 4 | $6.646 \cdot 10^{-4}$ |
| 5 | $1.916 \cdot 10^{-5}$ |
| 6 | $1.367 \cdot 10^{-5}$ |
| 7 | $1.315 \cdot 10^{-6}$ |
| 8 | $4.092 \cdot 10^{-7}$ |
| 9 | $6.829 \cdot 10^{-8}$ |
| 10 | $1.498 \cdot 10^{-8}$ |

resulting from the inequality (23) is not submitted, because the formula defining the constant $\gamma$ is unknown in the nonlinear case. The graph of error $\left|u_{e}(x)-\widehat{u}_{n}(x)\right|$ for $n=5$ and $n=10$ is shown in Fig. 8.


Fig. 8. Distribution of error of the exact solution approximation for: a) $n=5$ and b) $n=10$.

## Example 5

In the last theoretical example, we consider the two-dimensional nonlinear Fredholm integral equation of the second kind defined by the functions $K_{1}(x, y, t)=0, K_{2}(x, y, s)=0, K_{3}(x, y, t, s)=x s / 10$, $R_{3}(u)=u^{3}, F(x, y)=1+\frac{67}{150} x+y$ and $f_{i}(x)=0, g_{i}(x)=1$ for $i=1,2$, where $x, y \in[0,1]$. The solution of Eq. (1) is represented by the function $u_{e}(x, y)=x+y+1$.

Assuming the initial approximation $u_{0}(x, y)=F(x, y)$ we obtain in the first step

$$
u_{1}(x, y)=-\frac{96567263 h x}{270000000},
$$

and

$$
\widehat{u}_{1}(x, y)=u_{0}(x, y)+u_{1}(x, y)=1+\frac{67}{150} x+y-\frac{96567263 h x}{270000000} .
$$

In Fig. 9, the graph of the logarithm of squared residual $E_{20}$ is shown. Numerically determined, the optimal value of the convergence control parameter is equal to -1.50182 .


Fig. 9. Logarithm of the squared residual $E_{20}$.

The obtained approximate solutions $\widehat{u}_{n}$ can be compared with the exact solution by taking the differences of these functions. Thus we get

$$
\begin{array}{ll}
u_{e}-\widehat{u}_{1}=1.620 \cdot 10^{-2} x, & u_{e}-\widehat{u}_{17}=3.193 \cdot 10^{-7} x, \\
u_{e}-\widehat{u}_{3}=2.958 \cdot 10^{-3} x, & u_{e}-\widehat{u}_{18}=3.485 \cdot 10^{-7} x, \\
u_{e}-\widehat{u}_{5}=6.821 \cdot 10^{-4} x, & u_{e}-\widehat{u}_{19}=9.348 \cdot 10^{-8} x \\
u_{e}-\widehat{u}_{10}=8.228 \cdot 10^{-5} x, & u_{e}-\widehat{u}_{20}=9.360 \cdot 10^{-8} x, \\
u_{e}-\widehat{u}_{15}=1.096 \cdot 10^{-6} x, & u_{e}-\widehat{u}_{25}=2.413 \cdot 10^{-9} x .
\end{array}
$$

Table 8 presents the errors of successive approximate solutions $\widehat{u}_{n}, n \in\{1, \ldots, 30\}$. Let us notice that, in this case, at the beginning the error of approximate solution does not decrease monotonically as it occurred in the previous examples. At first, the decrease of error oscillates but starting from $n=21$ it behaves monotonically. The graph of error $\left|u_{e}(x)-\widehat{u}_{n}(x)\right|$ for $n=5$ and $n=20$ is presented in Fig. 10.


Fig. 10. Distribution of error of the exact solution approximation for: a) $n=5$ and b) $n=20$.

Table 8. Values of errors in the reconstruction of the exact solution $\left(\Delta_{n}=\left\|u_{e}-\widehat{u}_{n}\right\|\right)$.

| $n$ | $\Delta_{n}$ | $n$ | $\Delta_{n}$ |
| :---: | :---: | :---: | :---: |
| 1 | $1.620 \cdot 10^{-2}$ | 16 | $1.320 \cdot 10^{-6}$ |
| 2 | $4.824 \cdot 10^{-2}$ | 17 | $3.193 \cdot 10^{-7}$ |
| 3 | $2.958 \cdot 10^{-3}$ | 18 | $3.485 \cdot 10^{-7}$ |
| 4 | $7.923 \cdot 10^{-3}$ | 19 | $9.360 \cdot 10^{-8}$ |
| 5 | $6.821 \cdot 10^{-4}$ | 20 | $9.360 \cdot 10^{-8}$ |
| 6 | $1.585 \cdot 10^{-3}$ | 21 | $2.756 \cdot 10^{-8}$ |
| 7 | $1.750 \cdot 10^{-4}$ | 22 | $2.549 \cdot 10^{-8}$ |
| 8 | $3.502 \cdot 10^{-4}$ | 23 | $8.142 \cdot 10^{-9}$ |
| 9 | $4.746 \cdot 10^{-5}$ | 24 | $7.030 \cdot 10^{-9}$ |
| 10 | $8.228 \cdot 10^{-5}$ | 25 | $2.413 \cdot 10^{-9}$ |
| 11 | $1.329 \cdot 10^{-5}$ | 26 | $1.960 \cdot 10^{-9}$ |
| 12 | $2.016 \cdot 10^{-5}$ | 27 | $7.166 \cdot 10^{-10}$ |
| 13 | $3.792 \cdot 10^{-6}$ | 28 | $5.516 \cdot 10^{-10}$ |
| 14 | $5.095 \cdot 10^{-6}$ | 29 | $2.133 \cdot 10^{-10}$ |
| 15 | $1.096 \cdot 10^{-6}$ | 30 | $1.566 \cdot 10^{-10}$ |

## Example 6

Let us consider the horizontal bar of length $l$, the axis of which lies in the $x$-axis (it matches with segment $[0, l])$. Deflection of point $x$ in the bar at time $t$, in direction perpendicular to the $x$-axis, is expressed by function $z$. The transverse oscillations of this bar is then described by the integro-differential equation of the form (see [42]):

$$
\begin{equation*}
z(x, t)=\int_{0}^{l} G(x, s)\left(p(s)-\mu(s) \frac{\partial^{2} z}{\partial t^{2}}(x, s)\right) d s, \quad 0 \leqslant x \leqslant l, \tag{24}
\end{equation*}
$$

where $p(s) d s$ describes the load acting on the subsegment $(s, s+d s)$ of the bar in direction perpendicular to the $x$-axis and $\mu(s) d s$ denotes the mass of this subsegment. The function $G(x, s)$ in the above equation is called the influence function defining the displacement of the bar at point of coordinate $x$ caused by the unit loading in direction perpendicular to the bar at some other point of coordinate $s$.

By transforming, Eq. (24) can be written in the form

$$
\begin{equation*}
z(x, t)=F(x, t)-\int_{0}^{l} K_{2}(x, t, s) R_{2}(z(x, s)) d s, \quad 0 \leqslant x \leqslant l, \tag{25}
\end{equation*}
$$

where

$$
\begin{aligned}
& F(x, t)=\int_{0}^{l} G(x, s) p(s) d s, \\
& K_{2}(x, t, s)=G(x, s) \mu(s)
\end{aligned}
$$

and

$$
R_{2}(z(x, s))=\frac{\partial^{2} z}{\partial t^{2}}(x, s)
$$

The obtained integro-differential equation must be completed with the initial conditions

$$
z(x, 0)=\varphi_{0}(x), \quad \frac{\partial z}{\partial t}(x, 0)=v_{0}(x), \quad 0 \leqslant x \leqslant l
$$

Taking $l=\pi / 2, \varphi_{0}(x)=0, v_{0}(x)=x / 8, K_{2}(x, t, s)=(x+s) / 10$ and

$$
F(x, t)=\frac{1}{160}\left(2\left(x^{2}+\sin \left(\frac{\pi x}{2}\right)+10 \sin (x t)\right)-x(2 x+\pi) \cos \left(\frac{\pi x}{2}\right)\right)
$$

we get the equation, the exact solution of which is given by function

$$
z(x, t)=\frac{1}{8} \sin (x t)
$$

The best choice is to take as the initial approximation the function satisfying the assumed initial conditions. Therefore we may take

$$
z_{0}(x, t)=\varphi_{0}(x)+t v_{0}(x)=\frac{t x}{8} .
$$

Then, in the first step of the method we obtain

$$
z_{1}(x, t)=\frac{1}{160} h\left(x(\pi+2 x) \cos \left(\frac{\pi x}{2}\right)-2\left(x^{2}-10 x t+\sin \left(\frac{\pi x}{2}\right)+10 \sin (x y)\right)\right)
$$

Figure 11 shows the graph of the logarithm of squared residual $E_{10}$. This time the optimal value of convergence control parameter is equal to -1 . Substituting the obtained value $h=-1$ in the derived formulas we get in turn

$$
\begin{aligned}
& z_{1}(x, t)=\frac{1}{160}\left(-x(\pi+2 x) \cos \left(\frac{\pi x}{2}\right)+2\left(x^{2}-10 x t+\sin \left(\frac{\pi x}{2}\right)+10 \sin (x y)\right)\right) \\
& z_{2}(x, t)=\frac{1}{160}\left(x(\pi+2 x) \cos \left(\frac{\pi x}{2}\right)+2\left(-x^{2}-\sin \left(\frac{\pi x}{2}\right)\right)\right) \\
& z_{m}(x, t)=0, \quad m \geqslant 3
\end{aligned}
$$

By adding the determined functions we receive the exact solution of the discussed equation

$$
z(x, t)=\sum_{m=0}^{\infty} z_{m}(x, t)=z_{0}(x, t)+z_{1}(x, t)+z_{2}(x, t)=\frac{1}{8} \sin (x t)
$$



Fig. 11. Logarithm of the squared residual $E_{10}$.

## 7. CONCLUSION

This paper presents the use of the HAM in solving the two-dimensional linear and nonlinear integral equations of the second kind. In the discussed method, the solution is sought in the form of a series. It is shown that if this series is convergent, its sum is the solution of the considered equation. The sufficient condition for the convergence of this series is also presented. Additionally, the error of approximate solution, taken as the partial sum of generated series, is estimated.

Presented examples show that the investigated method is effective in solving the equations of considered kind. The results indicate that the method converges very rapidly and the optimal selection of the convergence control parameter provides a very good approximation of the sought exact solution by the sum of first few components of the series.

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