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Chosen Possibilities of $\overline{\varepsilon}$-Fuzzy Boundary Elements Method Application in the Analysis of Conductivity Problems with Uncertainties

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This paper presents a methodology for solving boundary problems with uncertainty parameters based on the use of interval perturbation numbers. This methodology allows for the analysis of very complex problems with different uncertain parameters. The fuzzy boundary element method (FBEM) using $\bar{\varepsilon}$-number will be called the $\bar{\varepsilon}$-fuzzy boundary element method ($\bar{\varepsilon}$-FBEM). A detailed discussion of the problems of computing and applications will be presented on the example of the fuzzy boundary integral equation arising from the boundary problem for the potential problems with heterogeneous, fuzzy boundary conditions of Dirichlet and Neumann type, fuzzy internal sources, fuzzy boundary, and fuzzy fundamental solution.

The presented methodology can be used to solve various engineering problems (e.g., in civil engineering [9], power engineering [7, 15] and others) – e.g., to analyze the temperature distribution in structural elements or elements located in the vicinity of objects or devices. In the latter case, the increased temperature may be a symptom of a severe failure (e.g., power transformer overload, core overexcitation, or an internal fault), which cannot be tolerated due to the threat to the object itself as well as the entire power system operation. The proposed method may be used for electrical equipment diagnosis and, consequently, as a power system failure prevention [5, 6, 14].

In this paper, calculation methodology is illustrated with the example of an area bounded by a square, on the left boundary of which a certain temperature is set, while on the rest of the boundaries the conditions are equal to zero. The authors' dedicated computer program written in the Fortran programming environment [13] allows the calculation of the temperature and temperature derivative for any number of boundary elements using $\bar{\varepsilon}$-FBEM.

Keywords: $\bar{\varepsilon}$-fuzzy boundary element method, $\bar{\varepsilon}$-number, fuzzy boundary element method, heat conduction, temperature distribution, objects diagnosis.

1. Introduction

Civil and power engineering requires a very high level of safety (compare, e.g., [5, 14, 15]). It should be noted that when the physical problem is converted to an ordinary (deterministic) boundary problem, it usually cannot be known exactly that the model is perfect – particularly for building structures or for electrical machines and devices. The boundary problem cannot be known exactly, and the displacement, temperature, or related features, such as the stress or heat fluxes obtained on the basis of a number of necessary measurements, may bear errors. Quantitative and qualitative analysis of these errors leads to the use of fuzzy numbers instead of random real variables. This, in turn, leads to problems with fuzzy boundary values and, consequently, to diffuse boundary integral equations.

The method of calculation will be presented on the example of the task of solving the Poisson equation:

$$\nabla^2 u = \xi \quad \text{in} \quad \Omega, \tag{1}$$

where $\nabla^2(\cdot)$ is the Laplace operator, and $\xi(\cdot)$ is a function defined in the area of Ω by the boundary Γ.

The correct formulation of the problem solution of task (1) requires relevant boundary conditions. For definiteness, we assume that data are Dirichlet boundary conditions in the following form:

$$u(\boldsymbol{x}) = u_0(\boldsymbol{x}), \quad \text{for} \quad \boldsymbol{x} \in \Gamma_1 \tag{2}$$

and Neumann (natural) boundary conditions may be written as:

$$q(\boldsymbol{x}) = \partial u(\boldsymbol{x})/\partial \boldsymbol{n} = q_0(\boldsymbol{x}), \quad \text{for} \quad \boldsymbol{x} \in \Gamma_2, \quad \Gamma = \Gamma_1 \cup \Gamma_2. \tag{3}$$

If we use the method of boundary equations to solve this boundary problem of potential theory in the area Ω with known boundary conditions at the boundary of this area Ω, we obtain the appropriate integral equation.

If we assume that the weighting function is a fundamental solution $u^*(\boldsymbol{x})$ of the Laplace Eq. (3) and $q^*(\boldsymbol{x}) = \frac{\partial}{\partial n} u^*(\boldsymbol{x})$, the corresponding boundary equation will obtain the following form:

$$c(\boldsymbol{x})u(\boldsymbol{x}) + \int\limits_{\Gamma} q^*(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{y})\,\mathrm{d}\Gamma(\boldsymbol{y}) = \int\limits_{\Gamma} u^*(\boldsymbol{x},\boldsymbol{y})q(\boldsymbol{y})\,\mathrm{d}\Gamma(\boldsymbol{y}) + \int\limits_{\Omega} u^*(\boldsymbol{x},\boldsymbol{y})\xi(\boldsymbol{y})\,\mathrm{d}\Omega(\boldsymbol{y}), \tag{4}$$

where $\xi(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega$ is a known function of heat source density. Boundary integrals are defined in the sense of the Cauchy principal values.

In this paper, the basic principles of the new concept in the boundary element method, called the fuzzy boundary element method (FBEM), are discussed. The problems of computing and applications are presented on the example of the fuzzy boundary integral equation arising from the boundary problem for the problems of heterogeneous interruption, Dirichlet and Neumann type fuzzy boundary conditions, fuzzy internal sources, fuzzy boundary and fuzzy fundamental solution.

The methodology is proposed to solve boundary problems in conditions of uncertainty parameters based on the use of perturbative interval numbers. The calculation uses newly proposed algebra $\overline{\varepsilon}$-interval numbers, which are discussed in detail in [13]. The proposed methodology allows the analysis of very complex tasks with very diverse uncertain parameters.

2. Fuzzy boundary conditions and internal sources

It may be assumed that the values of some boundary magnitudes and the sources' density function are uncertain and the uncertainty may be modeled by using fuzzy variables [2].

Let $\widetilde{u}_0$, $\widetilde{q}_0$ and $\widetilde{\xi}$ be fuzzy functions, then a set of solutions may be defined as:

$$U_\lambda(\boldsymbol{x}) := \left\{ \begin{array}{l} u : c(\boldsymbol{x})u(\boldsymbol{x}) + \displaystyle\int_\Gamma q^*(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{y})\,\mathrm{d}\Gamma(\boldsymbol{y}) + \int_\Omega u^*(\boldsymbol{x},\boldsymbol{y})\xi(\boldsymbol{y})\,\mathrm{d}\Omega(\boldsymbol{y}) \\ \quad = \displaystyle\int_\Gamma u^*(\boldsymbol{x},\boldsymbol{y})q(\boldsymbol{y})\,\mathrm{d}\Gamma(\boldsymbol{y}), \\ u_0(\boldsymbol{z}) \in \overline{u}_{0\lambda}(\boldsymbol{z})|_{\boldsymbol{z}\in\Gamma_1}, \quad q_0(\boldsymbol{z}) \in \overline{q}_{0\lambda}(\boldsymbol{z})|_{\boldsymbol{z}\in\Gamma_2}, \quad \xi(\boldsymbol{z}) \in \overline{\xi}_\lambda(\boldsymbol{z})|_{\boldsymbol{z}\in\Omega} \end{array} \right\}. \tag{5}$$

The "exact" fuzzy solution $\widetilde{u}_1(\boldsymbol{x})$, $\boldsymbol{x} \in \Gamma$ is defined as follows:

$$\mu(\boldsymbol{y}; \widetilde{u}_1(\boldsymbol{x})) := \sup\{\lambda : \boldsymbol{y} \in U_\lambda(\boldsymbol{x})\}, \quad \boldsymbol{x} \in \Gamma, \quad \boldsymbol{y} \in \mathrm{R}. \tag{6}$$

In this case, we obtain a membership function of the "exact" fuzzy solution $\widetilde{u}_1(\boldsymbol{x})$, $\boldsymbol{x} \in \Gamma$ as the relation (6), which defines the membership function of the first kind of fuzzy solutions for the potential problem defined in the deterministic area.

Since the shape of the set $U_\lambda(\boldsymbol{x})$, for $\lambda \in [0,1]$ may be very complicated, we generally search for its upper approximation in the form of the smallest interval. Let us assume that we are looking for the interval type solutions:

$$\widetilde{u}_\lambda(\boldsymbol{x}) = [\widetilde{u}_\lambda^-(\boldsymbol{x}),\ \widetilde{u}_\lambda^+(\boldsymbol{x})], \quad \boldsymbol{x} \in \Gamma, \tag{7}$$

where $0 \le \lambda \le 1$.

Taking λ-cuts, $0 \le \lambda \le 1$ of the fuzzy Eq. (5) we obtain a formally infinite set of interval boundary integral equations in the following form:

$$c(\boldsymbol{x})\left[\widetilde{u}_\lambda^-(\boldsymbol{x}),\widetilde{u}_\lambda^+(\boldsymbol{x})\right]+\int\limits_\Gamma q^*(\boldsymbol{x},\boldsymbol{y})\left[\widetilde{u}_\lambda^-(\boldsymbol{y}),\widetilde{u}_\lambda^+(\boldsymbol{y})\right]\mathrm{d}\Gamma(\boldsymbol{y}) = \int\limits_\Gamma u^*(\boldsymbol{x},\boldsymbol{y})\left[\widetilde{q}_\lambda^-(\boldsymbol{y}),\widetilde{q}_\lambda^+(\boldsymbol{y})\right]\mathrm{d}\Gamma(\boldsymbol{y}) + \int\limits_\Omega u^*(\boldsymbol{x},\boldsymbol{y})\left[\widetilde{\xi}_\lambda^-(\boldsymbol{y}),\widetilde{\xi}_\lambda^+(\boldsymbol{y})\right]\mathrm{d}\Omega(\boldsymbol{y}), \qquad \boldsymbol{x}\in\Gamma. \tag{8}$$

Let us substitute $\widetilde{u}_0$, $\widetilde{q}_0$ and $\widetilde{\xi}$ for the magnitudes u_0, q_0, ξ, respectively, and assume that all operations are performed by using the fuzzy values. In this way, we will consider a fuzzy version of Eq. (4), known in the literature [9, 13] as the fuzzy boundary integral equation:

$$c(\boldsymbol{x})\widetilde{u}(\boldsymbol{x})+\int\limits_{\widetilde{\Gamma}} q^*(\boldsymbol{x},\boldsymbol{y})\widetilde{u}(\boldsymbol{y})\,\mathrm{d}\Gamma(\boldsymbol{y}) = \int\limits_{\widetilde{\Gamma}} u^*(\boldsymbol{x},\boldsymbol{y})\widetilde{q}(\boldsymbol{y})\,\mathrm{d}\Gamma(\boldsymbol{y}) + \int\limits_{\widetilde{\Omega}} u^*(\boldsymbol{x},\boldsymbol{y})\widetilde{\xi}(\boldsymbol{y})\,\mathrm{d}\Omega(\boldsymbol{y}), \qquad \boldsymbol{x}\in\Gamma. \tag{9}$$

In Eq. (9), all the operators are fuzzy, and the integral is also understood in the sense of the main fuzzy values.

3. Fuzzy boundary conditions, internal sources and boundary

Let us, in addition to fuzzy boundary conditions and internal sources, take into account the fuzzy nature of the boundary of the area.

By the fuzzy boundary $\widetilde{\Gamma}$ of the fuzzy area $\widetilde{\Omega} \in \mathrm{F}(\mathrm{R}^3)$, we understand the fuzzy subset of the variety designated herein as M, i.e., mapping M in [0, 1]. Each such a representation is related to its membership function.

We assume that the values of the boundary magnitudes, internal sources and boundary are vague; therefore, we use fuzzy variables to describe them.

Next, we can completely formally treat the integral equation as in Eq. (9), and substitute the fuzzy values $\widetilde{u}_0$, $\widetilde{q}_0$, $\widetilde{\xi}$ and $\widetilde{\Gamma}$ for u_0, q_0, ξ, Γ, respectively:

$$\widetilde{c}(\boldsymbol{x})\widetilde{u}(\boldsymbol{x})+\int\limits_{\widetilde{\Gamma}} q^*(\boldsymbol{x},\boldsymbol{y})\widetilde{u}(\boldsymbol{y})\,\mathrm{d}\Gamma(\boldsymbol{y}) = \int\limits_{\widetilde{\Gamma}} u^*(\boldsymbol{x},\boldsymbol{y})\widetilde{q}(\boldsymbol{y})\,\mathrm{d}\Gamma(\boldsymbol{y}) + \int\limits_{\widetilde{\Omega}} u^*(\boldsymbol{x},\boldsymbol{y})\widetilde{\xi}(\boldsymbol{y})\,\mathrm{d}\Omega(\boldsymbol{y}), \qquad \boldsymbol{x}\in\widetilde{\Gamma}. \tag{10}$$

All operations are treated in the fuzzy sense. Fuzzy integrals are understood in the sense of the main fuzzy values. The equation of the form (10) was called in the literature the fuzzy boundary integral equation with a fuzzy boundary [8–11].

The conditional set of solutions has the following form:

$$\mathrm{R}_\lambda(\boldsymbol{x}|\Gamma) := \left\{ \begin{aligned} &(u,q) : c(\boldsymbol{x})u(\boldsymbol{x}) + \int_\Gamma q^*(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{y})\,\mathrm{d}\Gamma(\boldsymbol{y}) \\ &\qquad + \int_\Omega u^*(\boldsymbol{x},\boldsymbol{y})\xi(\boldsymbol{y})\,\mathrm{d}\Omega(\boldsymbol{y}) = \int_\Gamma u^*(\boldsymbol{x},\boldsymbol{y})q(\boldsymbol{y})\,\mathrm{d}\Gamma(\boldsymbol{y}), \\ &\qquad u_0(\boldsymbol{z}) \in \overline{u}_{0\lambda}(\boldsymbol{z})|_{\boldsymbol{z}\in\Gamma_1}, \quad q_0(\boldsymbol{z}) \in \overline{q}_{0\lambda}(\boldsymbol{z})|_{\boldsymbol{z}\in\Gamma_2}, \\ &\qquad \xi(\boldsymbol{z}) \in \overline{\xi}_\lambda(\boldsymbol{z})|_{\boldsymbol{z}\in\Omega}, \quad \Gamma \in \mathrm{M} \end{aligned} \right\} \tag{11}$$

provided that $\Gamma \in \mathrm{M}$. Let $r_3(\boldsymbol{x}) := [u_3(\boldsymbol{x}), q_3(\boldsymbol{x})]$, $\boldsymbol{x} \in \Gamma$.

We assume further that the family of sets $\mathrm{R}_\lambda(\boldsymbol{x}|\Gamma)$, $0 < \lambda \le 1$ generates a fuzzy function, which will be called the fuzzy conditional solution with collective values $\widetilde{r}_3(\boldsymbol{x}|\Gamma)$, $\boldsymbol{x} \in \Gamma \in \mathrm{M}$.

It can be proven that it occurs if the following conditions are true, namely $\mathrm{R}_\lambda(\boldsymbol{x}|\Gamma)$, $0 < \lambda \le 1$ represents the λ-cross sections of the fuzzy function then and only then, if:

$$\mathrm{R}_\lambda(\boldsymbol{x}|\Gamma) \text{ is a non-empty, compact and convex subset of } \mathrm{R}_2; \tag{12}$$

$$\mathrm{R}_\lambda(\boldsymbol{x}|\Gamma) \subseteq R_\mu(\boldsymbol{x}|\Gamma) \quad \forall \quad 0 \le \mu \le \lambda \le 1; \tag{13}$$

$$\{\lambda_k\} \text{ is a non-decreasing sequence convergent to } \lambda \text{ then } \mathrm{R}_\lambda(\boldsymbol{x}|\Gamma) = \cap_{k\ge1}\mathrm{R}_{\lambda_k}(\boldsymbol{x}|\Gamma). \tag{14}$$

Note that some difficulties may arise in complying with these conditions, which often occur in applications. In such cases, we can settle for an approximation from the top or bottom of a set of solutions.

The conditional solution with fuzzy values $\widetilde{r}_3(\boldsymbol{x}|\Gamma)$, $\boldsymbol{x} \in \Gamma$, $\Gamma \in \mathrm{M}$ can be defined in accordance with the following formula:

$$\mu(\boldsymbol{y}|\widetilde{r}_3(\boldsymbol{x}|\Gamma)) := \sup\{\lambda : \boldsymbol{y} \in \mathrm{R}_\lambda(\boldsymbol{x}|\Gamma)\}, \quad \boldsymbol{x} \in \Gamma \in \mathrm{M}, \quad \boldsymbol{y} \in \mathrm{R}. \tag{15}$$

Due to the fuzzy nature of the area boundary, to obtain the final solution membership function, the fuzzy values should be "composed". We note here that there are many ways of such a composition since there is no unique definition of the relation between various fuzzy variables (see, e.g., [3, 4]).

In order to determine the considerations, we use the min-max rule of submitting fuzzy relationship and we obtain a function of "exact" fuzzy solutions $\widetilde{r}_3(\boldsymbol{x})$, $\widetilde{\boldsymbol{x}} \in \widetilde{\Gamma}$ as:

$$\mu(\boldsymbol{y};\widetilde{r}_3(\widetilde{\boldsymbol{x}})) := \sup_{\Gamma\in\mathrm{M}}\left(\mu(\Gamma;\widetilde{\Gamma}) \ \wedge \ \mu(\boldsymbol{y};\widetilde{r}_3(\boldsymbol{x}|\Gamma))\right), \quad \boldsymbol{y} \in \mathrm{R}^1, \quad \widetilde{\boldsymbol{x}} \in \widetilde{\Gamma}. \tag{16}$$

Formula (16) describes a fuzzy membership function solution of the boundary problem of the first kind of fuzzy area defined by the potential.

4. Fuzzy fundamental solutions, boundary conditions, internal sources, and boundary

Let us assume, as in Sec. 3, $\widetilde{u}_0$, $\widetilde{q}_0$, $\widetilde{\xi}$ and an area $\widetilde{\Omega}$ with a fuzzy boundary $\widetilde{\Gamma}$, in which we consider our task as the fuzzy functions. In addition to the fuzzy boundary conditions, fuzzy internal sources, and fuzzy area, we want to take into account the fuzzy nature of the fundamental solutions.

Completely formally, we can write down the analyzed fuzzy boundary equation similarly to the Eq. (10), substitute the fuzzy magnitudes $\widetilde{u}_0$, $\widetilde{q}_0$, $\widetilde{\xi}$ for u_0, q_0, ξ, Γ, respectively, and write it down in the form that takes into account the fundamental fuzzy solution $\widetilde{u}^*(\cdot)$:

$$\widetilde{c}(\boldsymbol{x})\widetilde{u}(\boldsymbol{x}) + \int\limits_{\widetilde{\Gamma}} \widetilde{q}^*(\boldsymbol{x},\boldsymbol{y})\widetilde{u}(\boldsymbol{y})\,\mathrm{d}\Gamma(\boldsymbol{y}) = \int\limits_{\widetilde{\Gamma}} \widetilde{u}^*(\boldsymbol{x},\boldsymbol{y})\widetilde{q}(\boldsymbol{y})\,\mathrm{d}\Gamma(\boldsymbol{y}) + \int\limits_{\widetilde{\Omega}} \widetilde{u}^*(\boldsymbol{x},\boldsymbol{y})\widetilde{\xi}(\boldsymbol{y})\,\mathrm{d}\Omega(\boldsymbol{y}), \qquad \boldsymbol{x}\in\widetilde{\Gamma}, \quad (17)$$

and then treat all the operations in the fuzzy sense. Equation (17) will be called a fuzzy boundary integral equation in the area of fuzzy parameters.

By a fuzzy fundamental solution $\widetilde{u}^*(\cdot)$ of the fuzzy Laplace equation for an isotropic medium, we mean a fuzzy solution satisfying the equation of the form:

$$\widetilde{\lambda}\nabla^2\widetilde{u}^* = -\widetilde{\delta}\left(|\boldsymbol{x}-\boldsymbol{y}|\right), \quad \widetilde{\lambda}\in \mathrm{F}(\mathrm{R}), \quad \boldsymbol{x},\boldsymbol{y}\in\mathrm{R}^n, \qquad (18)$$

where $\widetilde{\delta}(\boldsymbol{x})$, $\boldsymbol{x}\in\mathrm{R}^n$ represents a fuzzy δ-Dirac distribution and $n = 2$ or $n = 3$.

In a more general form for the anisotropic medium ($n = 2$) by a fuzzy fundamental solution, we mean a solution of the partial equation:

$$\widetilde{\lambda}_{xx}\frac{\partial^2\widetilde{u}}{\partial x^2} + \widetilde{\lambda}_{yy}\frac{\partial^2\widetilde{u}}{\partial y^2} = -\widetilde{\delta}\left(|x-\xi, y-\eta|\right), \qquad (19)$$

where $\widetilde{\lambda}_{xx}$, $\widetilde{\lambda}_{yy}\in\mathrm{F}(\mathrm{R})$. Similarly, as with the conventional Laplace equation $\widetilde{u}^*(\cdot)$ is a scalar fuzzy function. For simplicity, the point of the heat source is denoted as $\boldsymbol{y} = (\xi,\eta)\in\mathrm{R}^2$.

5. $\overline{\varepsilon}$-fuzzy boundary elements method

In this chapter, the methodology of solving boundary problems under uncertainty parameters based on the use of interval perturbation numbers is described. For the calculations, the algebra of $\overline{\varepsilon}$-interval numbers will be applied [13].

The described methodology allows for the analysis of complex tasks with very diverse uncertain parameters. To illustrate the computational capabilities of this method in the analyzed specific boundary potential (the task of heat conduction), we assume that there exist:

- fuzzy boundary conditions $\widetilde{u}_0$, $\widetilde{q}_0$,
- fuzzy internal sources $\widetilde{\xi}$,
- fuzzy fundamental solution $\widetilde{u}^*(\cdot)$,
- fuzzy boundary $\widetilde{\Gamma}$.

5.1. $\overline{\varepsilon}$-FBEM in a perturbation formulation

Fuzzy Eq. (17) will be analyzed in a conditional manner already described, assuming that the fuzzy boundary $\widetilde{\Gamma} \in \mathrm{M}$. With a fixed Γ, we write down Eq. (17) as a family of λ-sections in the form:

$$\widetilde{c}_\lambda(\boldsymbol{x})\widetilde{u}_\lambda(\boldsymbol{x}) + \int_\Gamma \widetilde{q}^*_\lambda(\boldsymbol{x},\boldsymbol{y})\widetilde{u}_\lambda(\boldsymbol{y})\,\mathrm{d}\Gamma(\boldsymbol{y}) = \int_\Gamma \widetilde{u}^*_\lambda(\boldsymbol{x},\boldsymbol{y})\widetilde{q}_\lambda(\boldsymbol{y})\,\mathrm{d}\Gamma(\boldsymbol{y}) + \int_\Omega \widetilde{u}^*_\lambda(\boldsymbol{x},\boldsymbol{y})\widetilde{\xi}_\lambda(\boldsymbol{y})\,\mathrm{d}\Omega(\boldsymbol{y}), \quad \boldsymbol{x},\boldsymbol{y} \in \Gamma, \quad \lambda \in [0,\ 1]. \tag{20}$$

We define a conditional set of solutions:

$$\mathrm{R}_\lambda(\boldsymbol{x}|\Gamma) := \left\{ \begin{array}{l} (u,q) : c(\boldsymbol{x})u(\boldsymbol{x}) + \int_\Gamma q^*(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{y})\,\mathrm{d}\Gamma(\boldsymbol{y}) \\ \qquad + \int_\Omega u^*(\boldsymbol{x},\boldsymbol{y})\xi(\boldsymbol{y})\,\mathrm{d}\Omega(\boldsymbol{y}) = \int_\Gamma u^*(\boldsymbol{x},\boldsymbol{y})q(\boldsymbol{y})\,\mathrm{d}\Gamma(\boldsymbol{y}), \\ \qquad u_0(\boldsymbol{z}) \in \overline{u}_{0\lambda}(\boldsymbol{z})|_{\boldsymbol{z}\in\Gamma_1}, \quad q_0(\boldsymbol{z}) \in \overline{q}_{0\lambda}(\boldsymbol{z})|_{\boldsymbol{z}\in\Gamma_2}, \\ \qquad \xi(\boldsymbol{z}) \in \overline{\xi}_\lambda(\boldsymbol{z})|_{\boldsymbol{z}\in\Omega}, \\ \qquad u^*(\boldsymbol{x},\boldsymbol{y}) \in \overline{u}^*_\lambda(\boldsymbol{x},\boldsymbol{y})|_{\boldsymbol{x},\boldsymbol{y}\in\mathrm{R}^2\times\mathrm{R}^2}, \\ \qquad q^*(\boldsymbol{x},\boldsymbol{y}) \in \overline{q}^*_\lambda(\boldsymbol{x},\boldsymbol{y})|_{\boldsymbol{x},\boldsymbol{y}\in\mathrm{R}^2\times\mathrm{R}^2} \end{array} \right\} \tag{21}$$

assuming that $\Gamma \in \mathrm{M}$.

5.2. $\overline{\varepsilon}$-fundamental solution

By a fuzzy fundamental solution $\widetilde{u}^*(\cdot)$ of the fuzzy Laplace equation for an isotropic medium we mean a fuzzy solution satisfying the equation of the form:

$$\widetilde{\lambda}\nabla^2\widetilde{u}^* = -\widetilde{\delta}(|\boldsymbol{x}-\boldsymbol{y}|), \qquad \widetilde{\lambda} \in \mathrm{F(R)}, \qquad \boldsymbol{x},\boldsymbol{y} \in \mathrm{R}^n, \tag{22}$$

wherein $\widetilde{\delta}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathrm{R}^n$ represents a fuzzy δ-Dirac distribution and $n = 2$ or $n = 3$.

For simplicity, the point of the heat source is denoted as $y = (\xi, \eta) \in \mathrm{R}^2$ or $y = (\xi, \eta, \zeta) \in \mathrm{R}^3$.

In a more general form for the anisotropic medium ($n = 2$), by a fuzzy fundamental solution we mean a solution of the partial equation:

$$\widetilde{\lambda}_{xx} \frac{\partial^2 \widetilde{u}}{\partial x^2} + \widetilde{\lambda}_{yy} \frac{\partial^2 \widetilde{u}}{\partial y^2} = -\widetilde{\delta}(|x - \xi, y - \eta|), \tag{23}$$

or for $n = 3$ in the form:

$$\widetilde{\lambda}_{xx} \frac{\partial^2 \widetilde{u}}{\partial x^2} + \widetilde{\lambda}_{yy} \frac{\partial^2 \widetilde{u}}{\partial y^2} + \widetilde{\lambda}_{zz} \frac{\partial^2 \widetilde{u}}{\partial z^2} = -\widetilde{\delta}(|x - \xi, y - \eta, z - \zeta|), \tag{24}$$

where $\widetilde{\lambda}_{xx}, \widetilde{\lambda}_{yy}, \widetilde{\lambda}_{zz} \in \mathrm{F(R)}$. Similarly, as with the conventional Laplace equation, $\widetilde{u}^*(\cdot)$ is a scalar fuzzy function. For simplicity, the point of the heat source is denoted as $\boldsymbol{y} = (\xi, \eta) \in \mathrm{R}^2$ or $\boldsymbol{y} = (\xi, \eta, \zeta) \in \mathrm{R}^3$.

5.3. $\overline{\varepsilon}$-interval boundary equations – calculation methodology

Let us recall that the fuzzy Eq. (17) will be analyzed in a conditional manner already described, assuming that the fuzzy boundary $\widetilde{\Gamma} \in \mathrm{F(M)}$. To focus the attention on the proposed methodology assumptions, let us repeat Eq. (20), which for a fixed Γ is treated as a family of λ-sections:

$$\widetilde{c}_\lambda(\boldsymbol{x})\widetilde{u}_\lambda(\boldsymbol{x}) + \int_\Gamma \widetilde{q}^*_\lambda(\boldsymbol{x}, \boldsymbol{y})\widetilde{u}_\lambda(\boldsymbol{y})\,\mathrm{d}\Gamma(\boldsymbol{y}) = \int_\Gamma \widetilde{u}^*_\lambda(\boldsymbol{x}, \boldsymbol{y})\widetilde{q}_\lambda(\boldsymbol{y})\,\mathrm{d}\Gamma(\boldsymbol{y}) + \int_\Omega \widetilde{u}^*_\lambda(\boldsymbol{x}, \boldsymbol{y})\widetilde{\xi}_\lambda(\boldsymbol{y})\,\mathrm{d}\Omega(\boldsymbol{y}), \quad \boldsymbol{x} \in \Gamma, \quad \lambda \in [0,\ 1]. \tag{25}$$

Therefore, the main problem is the solution of the family of boundary Eq. (26). The previously described mathematical difficulties mean that a family of interval equations for $\Gamma \in \mathrm{M}$, $0 \le \lambda \le 1$ will be considered in the sense of ε-intervals.

Let us assume that we are looking for an ε-interval type solution in the form of a family of λ-ε-sections

$$\widetilde{u}_\lambda(\boldsymbol{x}) = \breve{u}_\lambda(\boldsymbol{x}) + \mathrm{rad}(\widetilde{u}_\lambda(\boldsymbol{x}))\overline{\varepsilon}, \quad \boldsymbol{x} \in \Gamma, \quad \Gamma \in \mathrm{M}, \tag{26}$$

where $0 \le \lambda \le 1$.

Let:

$$\widetilde{u}_{0\lambda}(\boldsymbol{x}) = \breve{u}_{0\lambda}(\boldsymbol{x}) + \mathrm{rad}(\widetilde{u}_{0\lambda}(\boldsymbol{x}))\overline{\varepsilon}, \qquad \boldsymbol{x} \in \Gamma_1, \tag{27}$$

$$\widetilde{q}_{0\lambda}(\boldsymbol{x}) = \breve{q}_{0\lambda}(\boldsymbol{x}) + \mathrm{rad}(\widetilde{q}_{0\lambda}(\boldsymbol{x}))\overline{\varepsilon}, \qquad \boldsymbol{x} \in \Gamma_2, \tag{28}$$

$$\widetilde{\xi}_\lambda(\boldsymbol{x}) = \breve{\xi}_\lambda(\boldsymbol{x}) + \mathrm{rad}(\widetilde{\xi}_\lambda(\boldsymbol{x}))\overline{\varepsilon}, \qquad \boldsymbol{x} \in \Omega, \tag{29}$$

$$\widetilde{c}_\lambda(\boldsymbol{x}) = \breve{c}_\lambda(\boldsymbol{x}) + \mathrm{rad}(\widetilde{c}_\lambda(\boldsymbol{x}))\overline{\varepsilon}, \qquad \boldsymbol{x} \in \Omega, \tag{30}$$

and for λ-ε-sections of $\overline{\varepsilon}$-fundamental solutions:

$$\widetilde{u}^*_\lambda(\boldsymbol{x}) = \breve{u}^*_\lambda(\boldsymbol{x}) + \mathrm{rad}(\widetilde{u}^*_\lambda(\boldsymbol{x}))\overline{\varepsilon}, \qquad \boldsymbol{x} \in \mathrm{R}^k, \quad k = 2, 3, \tag{31}$$

$$\widetilde{q}^*_\lambda(\boldsymbol{x}) = \breve{q}^*_\lambda(\boldsymbol{x}) + \mathrm{rad}(\widetilde{q}^*_\lambda(\boldsymbol{x}))\overline{\varepsilon}, \qquad \boldsymbol{x} \in \mathrm{R}^k, \quad k = 2, 3. \tag{32}$$

Bearing in mind that λ-sections of fuzzy numbers are intervals, we write Eq. (25) in the ε-interval form:

$$\overline{c}_\lambda(\boldsymbol{x})\overline{u}_\lambda(\boldsymbol{x}) + \int\limits_\Gamma \overline{q}^*_\lambda(\boldsymbol{x},\boldsymbol{y})\overline{u}_\lambda(\boldsymbol{y})\,\mathrm{d}\Gamma(\boldsymbol{y}) = \int\limits_\Gamma \overline{u}^*_\lambda(\boldsymbol{x},\boldsymbol{y})\overline{q}_\lambda(\boldsymbol{y})\,\mathrm{d}\Gamma(\boldsymbol{y}) + \int\limits_\Omega \overline{u}^*_\lambda(\boldsymbol{x},\boldsymbol{y})\overline{\xi}_\lambda(\boldsymbol{y})\,\mathrm{d}\Omega(\boldsymbol{y}), \quad \boldsymbol{x} \in \Gamma, \quad \lambda \in [0,\ 1], \tag{33}$$

where all operations are in the ε-interval sense, whereas integrals are $\overline{\varepsilon}$-extensions of ε-surface integrals.

6. The problem of conductivity with the uncertainties of interval type

Let us consider the case of the restricted area in the shape of a square, whose boundary is divided into 16 fixed elements. On the left edge of the boundary, we set the temperature equal to 300°C, while on the remaining edges, the boundary conditions are equal to zero (see Fig. 1). Values u and q are continuous; however, for the numerical calculations they are approximated by the node values on each boundary element Γ_j.

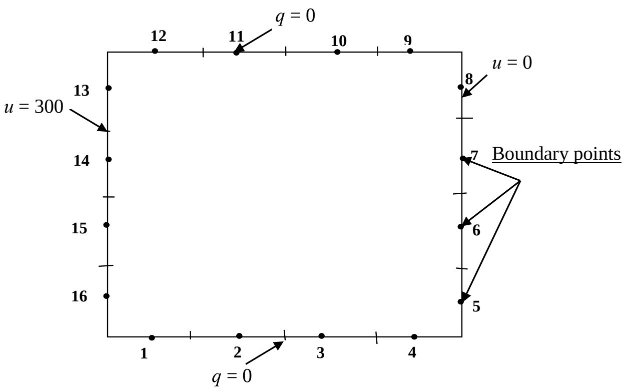

Fig. 1. A diagram showing the shape of the tested area with marked examined boundary points (nodes) and pre-defined boundary conditions.

The above data correspond to deterministic boundary conditions and do not contain any uncertain elements. The program to which they were introduced makes it possible to calculate the temperature and the temperature derivative for any number of boundary elements using $\overline{\varepsilon}$-FBEM. At the same time, the program compares these results with the theoretical results and calculates the resulting calculation error, which is the difference between the theoretical results and those obtained from $\overline{\varepsilon}$-FBEM. Data regarding the boundary geometry and boundary values of temperature and flow (a derivative of temperature) are all variants $2\overline{\varepsilon}$-perturbation values. The results have been developed graphically and shown in Fig. 2.

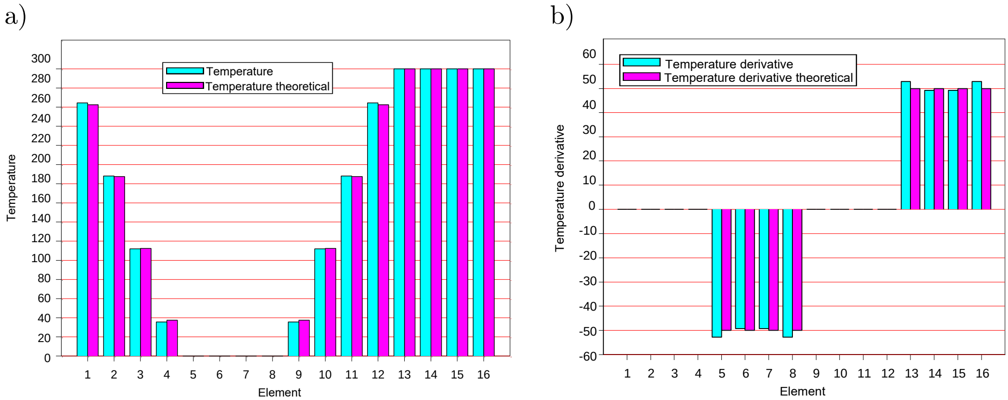

Fig. 2. Comparison of the temperature (a) and the temperature derivative (b) calculated by $\overline{\varepsilon}$-FBEM with the theoretical result.

Another procedure of the same program (see the Appendix) was used to estimate the temperature and its derivative at the boundary points with the uncertainties of the boundary conditions (Subsecs. 6.2 and 6.3), the uncertainty of the boundary (Subsec. 6.4) and the occurrence of uncertainty for both the boundary and the boundary conditions (Subsec. 6.5). A graphical interpretation of the results is presented for each example.

6.1. Numerical solution without a disorder

The calculations were performed for a case wherein, on the left boundary of the area (see Fig. 1), the temperature was set at 300°C (the other boundary conditions are zero), then the temperature and temperature derivative consecutively were calculated for 12, 16 and 40 nodes at the boundary of the area. The results of calculations carried out by $\overline{\varepsilon}$-FBEM were compared with the theoretical results. The maximum error for the calculated temperature on the boundary is 8.414669E-01 at 40 nodes, at 16 nodes, this error is: 1.861275, and at 12 nodes: 2.249790. Similarly, the calculation results were compared to the temperature derivative. Here, the errors are successively at 40 boundary elements: 2.555412, with 16 elements: 2.785511, and with 12 elements: 2.969486. It may be seen that the solution error decreases with an increasing number of nodes on the boundary.

The results of calculations for 16 nodes on the boundary are shown in Fig. 2.

6.2. Perturbation of boundary conditions

Recall that the interval number $\bar{a}$ can be an ordered pair of real numbers $(\check{a},\,\Delta a)_r$, where Δa is the radius of the interval. Suppose further that the radius of the interval is an n-perturbation number (pure n-perturbation), i.e.:

$$\mathrm{rad}(\overline{a}) = \Delta a = \varepsilon_1\,\delta a_1 + \varepsilon_2\,\delta a_2 + \ldots + \varepsilon_n\,\delta a_n, \qquad \delta a_1, \delta a_2, \ldots, \delta a_n \geq 0. \tag{34}$$

If we define n symbolic independent perturbation intervals $\overline{\varepsilon}_i := [-\varepsilon_i, \varepsilon_i]$, $i = 1, 2, \ldots, n$, then an interval number of type $\overline{a}$, which is defined by the relation (34), will be called interval number n-perturbation ($n\varepsilon$-interval number) and may be written as:

$$\overline{a} = \check{a} + \delta a_1 \overline{\varepsilon}_1 + \delta a_2 \overline{\varepsilon}_2 + \ldots + \delta a_n \overline{\varepsilon}_n. \tag{35}$$

More information on perturbation numbers can be found in [10].

Let us now consider the case as in Subsec. 6.1, and assume the possibility of temperature disturbance on the boundary of 10°C (temperature perturbations were given on the boundary: (300, 10, 0)). Calculations were carried out as before for 12, 16 and 40 boundary elements.

The maximum error for 40 boundary elements: 8.414669E-01 + $\varepsilon_1 \cdot$ 2.805507E-02, for 16 elements: 1.861275 + $\varepsilon_1 \cdot$ 6.204832E-02 (see Fig. 3), and for 12 elements: 2.249790 + $\varepsilon_1 \cdot$ 7.499290E-02. The maximum error for the temperature deriva-

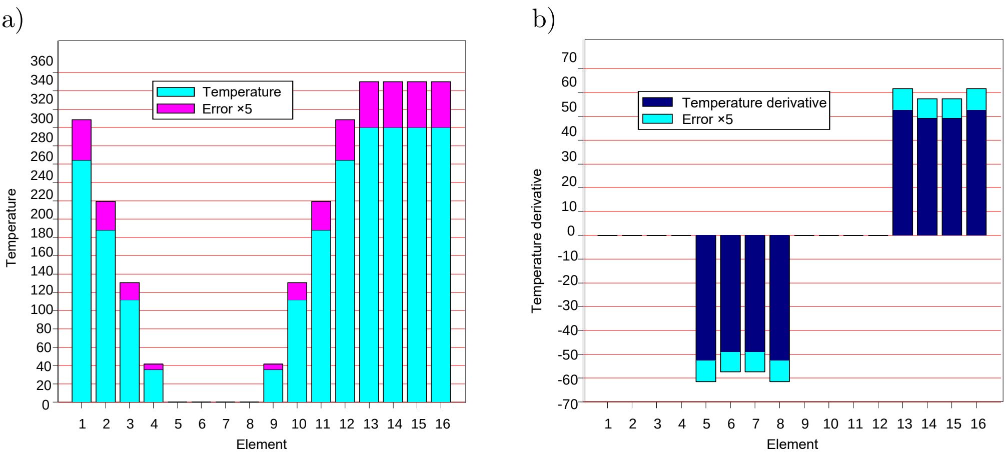

Fig. 3. The results of $\overline{\varepsilon}$-FBEM calculations: a) temperature and b) temperature derivative, with marked boundary solution perturbation for perturbation of boundary conditions.

tive of 40 boundary elements is $2.555412 + \varepsilon_1 \cdot 8.518851\text{E-}02$, for 16 elements: $2.785511 + \varepsilon_1 \cdot 9.284532\text{E-}02$ (see Fig. 3), and for 12 elements, this error is $2.969486 + \varepsilon_1 \cdot 9.898412\text{E-}02$.

The results of the calculations for 16 nodes located on the boundary are shown in Fig. 3.

6.3. Perturbation of boundary conditions depending on two groups of parameters

Let us again consider the case as in Subsec. 6.1, assuming the possibility of dual disturbance of temperature on the boundary, i.e., for all elements on the left boundary of the area uncertainty of temperature in the range of 10°, and further temperature uncertainty for subsequent elements on the left boundary of values: 20°, 15°, 15° and 20° (temperature perturbation (double) on the boundary: (300, 10, (20, 15, 15, 20)) are established. The calculations were performed for 16 boundary elements.

Figure 4 shows the results of $\bar{\varepsilon}$-FBEM calculations for the temperature and temperature derivative in the subsequent nodes with marked perturbations. The maximum error for the temperature is $1.861275 + \varepsilon_1 \cdot 6.204832\text{E-}02 + \varepsilon_2 \cdot 2.113348$, and for the temperature derivative the maximum error is $2.785511 + \varepsilon_1 \cdot 9.284532\text{E-}02 + \varepsilon_2 \cdot 5.889031$.

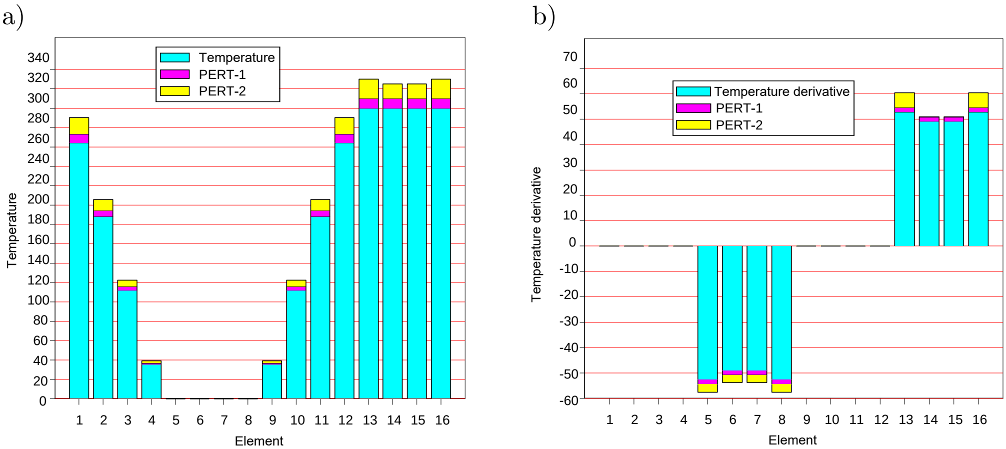

Fig. 4. The results of $\bar{\varepsilon}$-FBEM calculations for a) temperature and b) temperature derivative with marked boundary solution perturbation for perturbation of boundary conditions depending on two groups of parameters.

6.4. Area boundary perturbation

Above, we dealt with the cases where the uncertainty is associated with the boundary conditions (in the tested examples, the uncertainty refers to the temperature). It is clear, however, that the uncertainty may also relate to the tested boundary. Therefore, calculations with the interval uncertainty of the area boundary were performed (type I boundary perturbation). This is illustrated in Fig. 5, wherein the uncertainties here are related to the coordinates of vertices of the boundary and are selected arbitrarily.

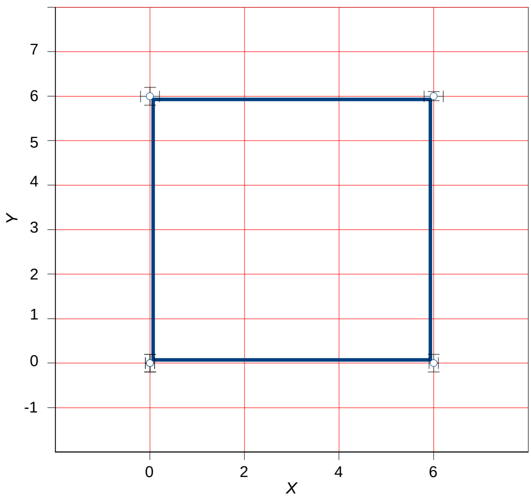

Fig. 5. Boundary with perturbations located at the vertices.

For the square with uncertainties in the vertices, as in Fig. 5, the boundary conditions were given as in Subsec. 6.1 and the temperature calculations were carried out for 16 boundary elements, using $\overline{\varepsilon}$-FBEM. The maximum error of calculations performed here is equal to $1.861275+\varepsilon_1\cdot 3.352861$ (Fig. 6). Similarly, in 16 nodes for this case, the temperature derivative is calculated. For these results, the maximum error is $2.785511 + \varepsilon_1 \cdot 6.593236\text{E-}01$ (Fig. 6).

6.5. The complex state of uncertainty: disturbance of area boundary and boundary conditions

In this example, the uncertainty was introduced for boundary conditions as well as the uncertainty of the area boundary. The temperature perturbation (indicated in the following figures as PERT-2) is the same as in Subsec. 6.2, and the boundary perturbation (designated in the following figures as PERT-1) has the character described in Subsec. 6.4. The results of the computation of the temperature in 16 boundary nodes are shown in Fig. 7.

The maximum error, which here is the sum of errors for both perturbations, is $1.861275 + \varepsilon_1 \cdot 3.352861 + \varepsilon_2 \cdot 1.187952$. Similarly, in 16 computational points for the temperature derivative, the maximum error, which is the sum of errors for both perturbations, is $2.785511 + \varepsilon_1 \cdot 6.593236\text{E-}01 + \varepsilon_2 \cdot 1.759512$ (Fig. 7).

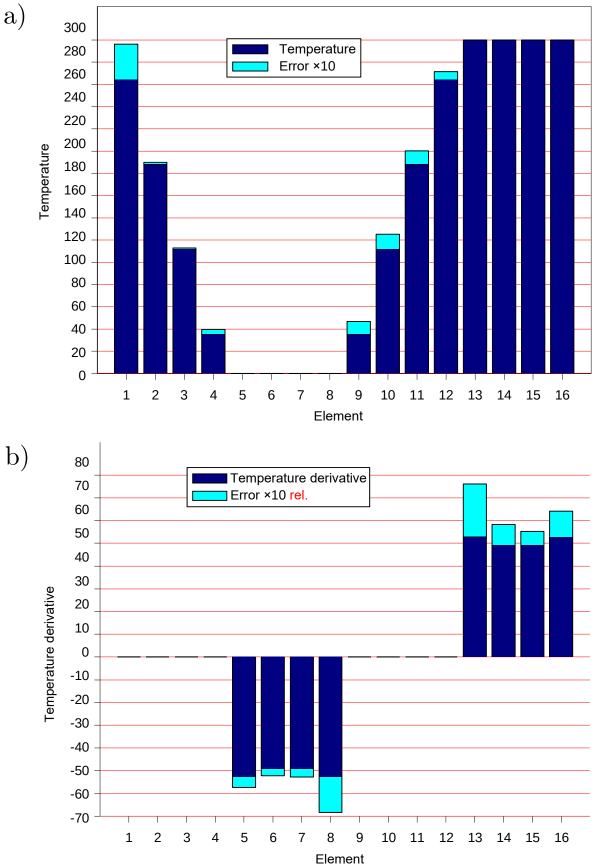

FIG. 6. The results of $\overline{\varepsilon}$-FBEM calculations for a) temperature and b) temperature derivative with a marked boundary solution perturbation for area boundary perturbation.

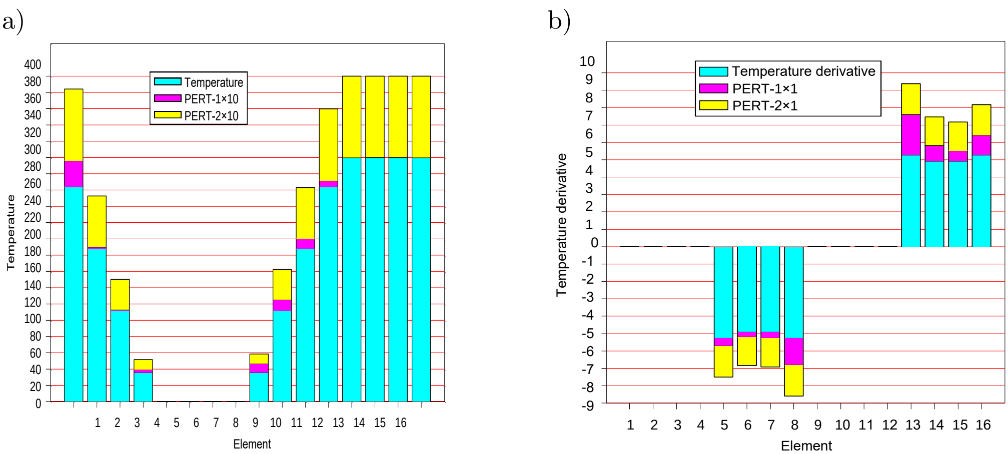

FIG. 7. The results of $\overline{\varepsilon}$-FBEM calculation calculations for a) temperature and b) temperature derivative, with marked boundary solution perturbation for Subsec. 6.5.

6.6. The task of conductivity with the uncertainties of fuzzy type

Let us consider again the case of the restricted area in the shape of a square, whose edge is divided into 16 fixed elements. On the left boundary edge, the temperature was assumed to be equal to 300°C, while at the remaining boundaries, the conditions are equal to zero (see Fig. 1). For such an area, uncertainties of a fuzzy type were introduced for a preset temperature (Subsec. 6.6.1), and for a given area boundary (Subsec. 6.6.2). In the following examples, the results of the achieved temperature and temperature derivative under the influence of these uncertainties, depending on the assumed shape of the membership function, are shown. For clear visualization of the results, we refer only to their graphic interpretation for the selected nodes.

6.6.1. Uncertainty of boundary conditions. In this example, for a given boundary of the area, a fuzzy type uncertainty of boundary conditions with a membership function in the shape of a symmetrical triangle (as in Fig. 8) was introduced. The temperature values at nodes 1 and 12 and the temperature derivative at nodes 5 and 16 for the subsequent sections α are shown in Figs. 9 and 10.

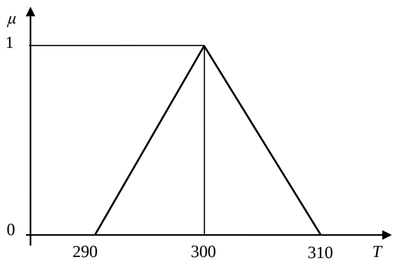

Fig. 8. Membership function assumed as a symmetrical triangle.

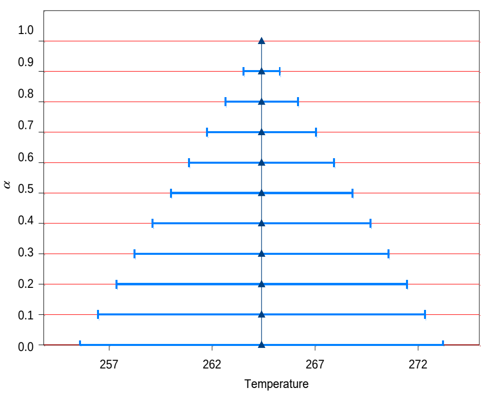

Fig. 9. Temperature values for membership function in the shape of a symmetrical triangle.

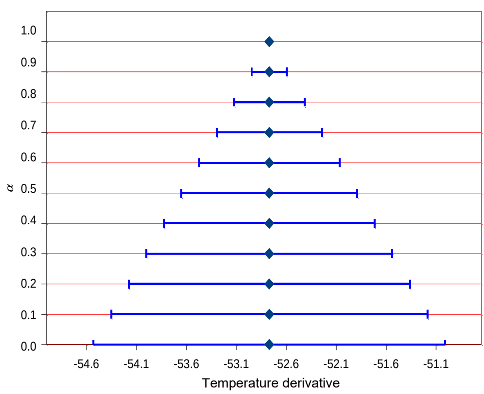

Fig. 10. Temperature derivative values for membership function in the shape of a symmetrical triangle.

Next, the same task as in the previous case was performed for the membership function of an asymmetric triangle shape (see Fig. 11). The calculation results for temperature are shown in Fig. 12 and for temperature derivative in Fig. 13.

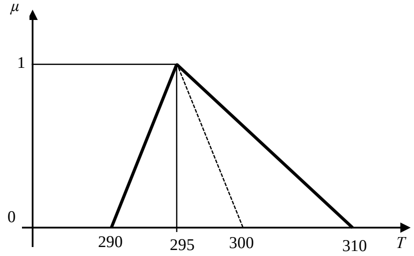

FIG. 11. Membership function in the shape of an asymmetrical triangle.

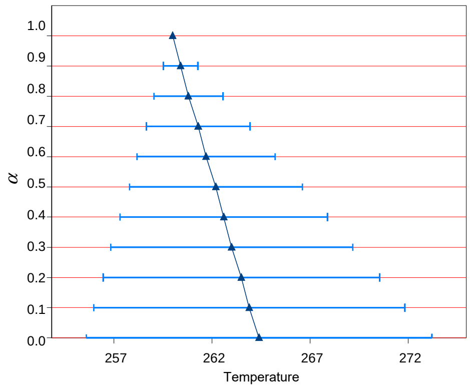

FIG. 12. Temperature values for membership function in the shape of an asymmetrical triangle.

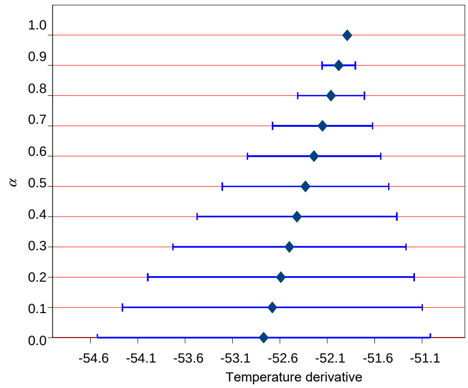

FIG. 13. Temperature derivative values for membership function in the shape of an asymmetrical triangle.

Analogously, the same task as the previous one, has been performed for the membership function in an asymmetric trapezoidal shape (see Fig. 14). The results of calculations for temperature are shown in Fig. 15 and for temperature derivative in Fig. 16.

6.6.2. Uncertainty of the fuzzy type of boundary conditions and the boundary of the area. In the example analyzed here, for the given area boundary, the fuzzy type uncertainty was introduced for both the boundary conditions and the area boundary (similarly to Subsec. 6.5), with the membership function in the shape of a symmetrical triangle (as in Fig. 11). The temperature and temperature derivative values for successive α sections in selected nodes of the area boundary are shown in Figs. 17 and 18. It can be observed that due to the imposed "double" uncertainty, the solution changes its shape from the symmetrical triangle to the symmetrical trapezoid.

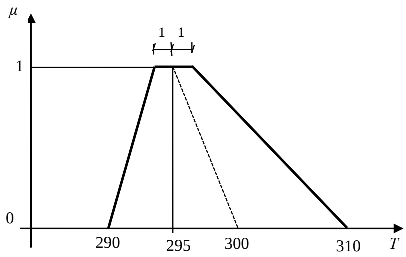

FIG. 14. Membership function in the shape of an asymmetrical trapezoid.

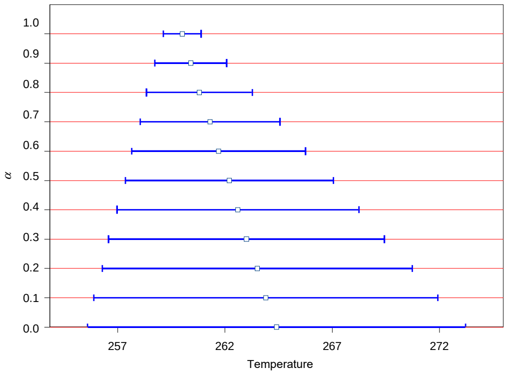

FIG. 15. Temperature values for membership function in the shape of an asymmetrical trapezoid.

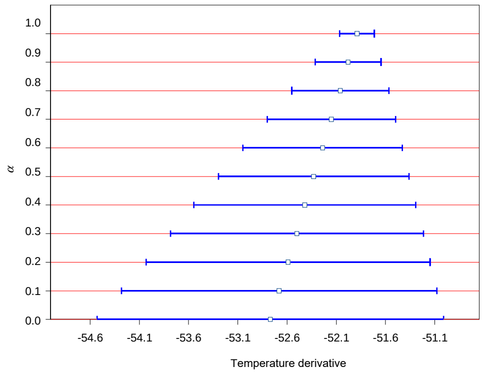

FIG. 16. Temperature derivative values for membership function in the shape of an asymmetrical trapezoid.

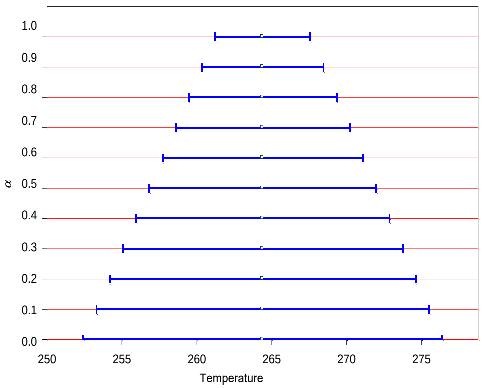

FIG. 17. Temperature values for Subsec. 6.2.2.

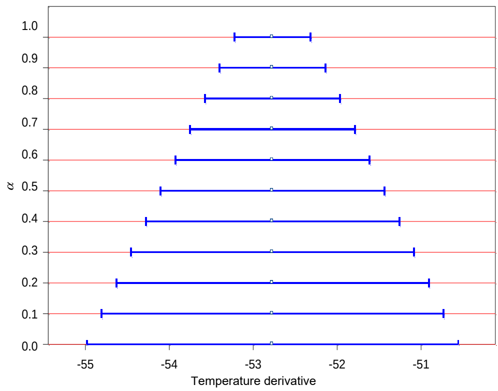

FIG. 18. Temperature derivative values for Subsec. 6.2.2.

7. Final remarks and conclusion

Temperature monitoring and analysis are essential issues in many branches of technology and engineering. This paper shows the achievements of the boundary element method and is a continuation of the work described, among others, in [1, 2]. The paper presents the concept of the compartmental algebraic system constructed on the basis of new perturbation numbers introduced in [10] and [12], which may be applied for temperature distribution and heat conduction analysis, which, in turn, may be applied to real object (e.g., buildings or electrical equipment) diagnostics.

The advantages of the proposed $\bar{\varepsilon}$-algebraic system are:

- simplification of the activities carried out so far by way of analysis,
- most of the numerical algorithms can be easily adapted to the new algebraic system.

The methodology allows for the analysis of complex tasks with very varied uncertainties. The computational capabilities of the new method have been tested in the potential boundary analysis task (heat conduction task). The use of the $\bar{\varepsilon}$-algebraic system has allowed the development of a new variant of the fuzzy boundary elements method called $\bar{\varepsilon}$-FBEM.

Calculations for different examples illustrating the use of $\bar{\varepsilon}$-FBEM were carried out with the use of the Fortran PowerStation computer program. For this purpose, the calculation procedures were developed by the authors. The applied programming methodology in the proposed algebraic system allows one to calculate two-dimensional heat flows with the following assumptions:

- boundary conditions $\widetilde{u}_0$, $\widetilde{q}_0$ may be fuzzy, with any membership function,
- internal sources $\widetilde{\xi}$ may be fuzzy, with any membership function,
- fundamental solution $\widetilde{u}^*(\cdot)$ may be fuzzy, with any membership function, for an isotropic medium,
- task definition area must be single-sided, of any area $\widetilde{\Omega}$ that may be fuzzy at the boundary $\widetilde{\Gamma}$.

All of the above elements of the fuzzy boundary task may occur simultaneously, and their independence or interactions may be freely programmed. One can include any dependency of a bounded function on fuzzy parameters. Moreover, the generalization of the problem to the three-dimensional problem and the various types of orthotropy does not pose much of a problem as is in the classical BEM. The proposed variant of the $\bar{\varepsilon}$-FBEMs can be used to consider technical tasks with high computational complexity under uncertainty of all parameters. Operations on perturbation numbers are equivalent to activities performed in perturbation methods with approximations limited to first order, so one can try to extend the proposed method to higher-order approximations.

Appendix

Fortran script (fragment) for the case: disturbance of the area boundary and boundary conditions perturbation (for 16 nodes).

```
HEAT FLOW EXAMPLE (16 CONSTANT ELEMENTS)
DATA
   NUMBER OF BOUNDARY ELEMENTS = 16
   NUMBER OF INTERNAL POINTS WHERE THE FUNCTION IS CALCULATED = 5
   COORDINATES OF THE EXTREME POINTS OF THE BOUNDARY ELEMENTS
   POINT X Y
    1 .00E+00 .10E+00 .00E+00 .00E+00 .20E+00 .00E+00
    2 .15E+01 .10E+00 .00E+00 .00E+00 .20E+00 .00E+00
    3 .30E+01 .10E+00 .00E+00 .00E+00 .20E+00 .00E+00
    4 .45E+01 .10E+00 .00E+00 .00E+00 .20E+00 .00E+00
    5 .60E+01 .10E+00 .00E+00 .00E+00 .20E+00 .00E+00
    6 .60E+01 .13E+00 .00E+00 .15E+01 .17E+00 .00E+00
    7 .60E+01 .15E+00 .00E+00 .30E+01 .15E+00 .00E+00
    8 .60E+01 .18E+00 .00E+00 .45E+01 .12E+00 .00E+00
    9 .60E+01 .20E+00 .00E+00 .60E+01 .10E+00 .00E+00
   10 .45E+01 .20E+00 .00E+00 .60E+01 .12E+00 .00E+00
   11 .30E+01 .20E+00 .00E+00 .60E+01 .15E+00 .00E+00
   12 .15E+01 .20E+00 .00E+00 .60E+01 .17E+00 .00E+00
   13 .00E+00 .20E+00 .00E+00 .60E+01 .20E+00 .00E+00
   14 .00E+00 .18E+00 .00E+00 .45E+01 .20E+00 .00E+00
   15 .00E+00 .15E+00 .00E+00 .30E+01 .20E+00 .00E+00
   16 .00E+00 .13E+00 .00E+00 .15E+01 .20E+00 .00E+00
   ...
   BOUNDARY NODES
    X    Y    DERIVATIVE   POTENTIAL    ERROR
    .75  .00   .00000E+00   .00000E+00   .00000E+00
   2.25  .00   .00000E+00   .00000E+00   .00000E+00
   3.75  .00   .00000E+00   .00000E+00   .00000E+00
   5.25  .00   .00000E+00   .00000E+00   .00000E+00
   6.00  .75   .27678E+01  -.21209E+01   .17589E+01
   6.00 2.25  -.78430E+00  -.19532E+01   .16405E+01
   6.00 3.75  -.78444E+00  -.20237E+01   .16405E+01
   6.00 5.25   .27678E+01  -.32122E+01   .17589E+01
   5.25 6.00   .00000E+00   .00000E+00   .00000E+00
   3.75 6.00   .00000E+00   .00000E+00   .00000E+00
   2.25 6.00   .00000E+00   .00000E+00   .00000E+00
    .75 6.00   .00000E+00   .00000E+00   .00000E+00
    .00 5.25  -.27855E+01  -.65932E+00  -.17595E+01
    .00 3.75   .81195E+00   .25660E+01  -.16396E+01
    .00 2.25   .81174E+00   .22815E+01  -.16396E+01
    .00  .75  -.27852E+01   .27973E+01  -.17595E+01
   ERROR DFI_MAX = 2.785511+eps1* 6.593236E-01+eps2* 1.759512
```

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