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Coupling techniques of Trefftz methods

Hung-Tsai Huang
Department of Applied Mathematics, I-Shou University, Kaohsiung, Taiwan 840

Zi-Cai Li
Department of Applied Mathematics and
Department of Computer Science and Engineering
National Sun Yat-sen University, Kaohsiung, Taiwan 80424

Alexander H.-D. Cheng
Department of Civil Engineering, University of Mississippi, University, MS 38677

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The Trefftz method pioneered by Trefftz [71] in 1926 is described as follows: The particular solutions or the fundamental solutions are chosen, a linear combination of those functions is regarded as an approximate solution of partial differential equations (PDEs), and their expansion coefficients are sought by satisfying the interior and exterior boundary conditions. When the solution domain is not rectangular or sectors, the piecewise particular solutions may be chosen in different subdomains, and some coupling techniques must be employed along their interior boundary conditions. In Li *et al.* [49], the collocation method is used for the Trefftz method, to lead to the collocation Trefftz method (i.e., the indirect Trefftz method). In this paper, we will also discuss other four coupling techniques: (1) the simplified hybrid techniques, (2) the hybrid plus penalty techniques, (3) the Lagrange multiplier techniques for the direct Trefftz method, and (4) the hybrid Trefftz method of Jirousek [23] and Qin [62]. Error bounds are derived in detail for these four couplings, to achieve exponential convergence rates. Numerical experiments are carried out, and comparisons are also made.

1. Introduction

The Trefftz method (TM) pioneered by Trefftz [71] in 1926 is described as follows: The particular solutions or the fundamental solutions are chosen, a linear combination of those functions is regarded as an approximate solution of partial differential equations (PDEs), and their expansion coefficients are sought by satisfying the interior and exterior boundary conditions. The Trefftz method (TM) has been applied to solve many engineering problems since the important work by Zienkiewicz *et al.* [75] and Jirousek and Leon [24] in 1977. References on the TM include [8, 66, 68, 73, 74], before 1995, and [1, 9, 10, 16, 19, 21, 28–30, 33, 35, 61, 65, 67, 72], after 1995. In 1995, there was a journal special issue on the Trefftz method [31, 32]. The error analysis of the TM is reported in the monograph [49] in 2008.

By the Green formulas, the Trefftz method can be extended to some elliptic equations, which framework was given by the algebraic approaches in Herrera [17] in 1984. Since the algebraic notations and operations are simple and easily understood, the Trefftz method (or called the Trefftz–Herrera approaches) has been applied to many engineering problems, in particular, the coupling problems where there exist the jumps of both the solutions and their derivatives along the interior boundary Γ_0. A great progress has been made by Herrera and his colleagues, and numerous papers have been published. Here we only mention a few important works, Herrera [17–19], Herrera and

Diaz [20, 22] and Herrera and Solan [21], and a complete list of references for the Trefftz–Herrera approaches can be found in [18, 19].

For Laplace's equation or other second order partial differential equations (PDE), using piecewise particular solutions is important for the Trefftz method (TM) to solve the complicated problems, in particular those with multiple singularities. Let the solution domain S be divided by Γ_0 into two subdomains S^+ and S^- without overlaps,[1] $S = S^+ \cup S^-$. We may choose different particular solutions in S^+ and S^-, denoted by

$$v = \begin{cases} v^+ = \sum_{i=0}^{L} a_i \Phi_i & \text{in } S^+, \\ v^- = \sum_{i=0}^{N} b_i \Psi_i & \text{in } S^-, \end{cases} \tag{1}$$

where $v^\pm$ are the particular solutions satisfying the PDE in S^+ and S^- respectively. Since the admissible functions v in Eq. (1) are not continuous along the interior boundary Γ_0, some coupling techniques must be chosen to link v^+ and v^- to satisfy the interior continuity conditions,

$$v^+ = v^-, \qquad p^+ \frac{\partial v^+}{\partial n} = p^- \frac{\partial v^-}{\partial n} \qquad \text{on } \Gamma_0, \tag{2}$$

where $p^\pm$ are the positive coefficients, and n is the outward normal of ∂S^+. The direct collocation for Eq. (2) is given by

$$v^+(Q_k) = v^-(Q_k), \qquad p^+ \frac{\partial v^+}{\partial n}(Q_k) = p^- \frac{\partial v^-}{\partial n}(Q_k), \quad Q_k \in \Gamma_0. \tag{3}$$

This leads to the collocation Trefftz method (CTM), error analysis has been provided in Li *et al.* [49]. The CTM is also called the indirect TM in [20, 32]. Note that the boundary approximation method and the TM are called in our previous study and in engineering journals [19, 32], respectively.

In this paper, we pursue other efficient techniques to couple v^+ and v^- on Γ_0. Consider the Laplace or the Debye–Huckel equation,

$$\mathcal{L}u = -\Delta u + cu = 0 \qquad \text{in } S, \tag{4}$$

where $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$, and constant $c = 0$ or $c = 1$. For simplicity in exposition, we may assume that the particular solutions Φ_i and Ψ_i also satisfy the exterior boundary condition on ∂S. Denote by $V_{L,N}$ the finite dimensional collection of v in Eq. (1).

The following four coupling techniques on Γ_0 will be discussed in this paper.

I. The simplified hybrid techniques. To seek $u_{L,N} \in V_{L,N}$ such that

$$A_{Hyb}(u_{L,N}, v) = 0, \qquad \forall v \in V_{L,N}, \tag{5}$$

where

$$A_{Hyb}(u, v) = \iint_{S^+} \nabla u \nabla v + \iint_{S^-} \nabla u \nabla v + I_{Hyb}^{(\alpha,\beta)}(u, v). \tag{6}$$

In Eq. (6), the simplified hybrid coupling is

$$I_{Hyb}^{(\alpha,\beta)}(u, v) = \alpha \int_{\Gamma_0} \frac{\partial u^+}{\partial n} v^- - \beta \int_{\Gamma_0} \frac{\partial u^-}{\partial n} v^+ - \alpha \int_{\Gamma_0} \frac{\partial v^+}{\partial n} u^- + \beta \int_{\Gamma_0} \frac{\partial v^-}{\partial n} u^+, \tag{7}$$

[1] The Schwarz alternating methods can be employed for the TM, where S^+ and S^- may or may not have overlaps. Along the interior boundary, different interior boundary conditions are explored in Li *et al.* [46], such as the Dirichlet, the Neumann, the Robin conditions and their mixed types, to speed the convergence of the iterative algorithms.

where n is the unit outnormal of ∂S^+, $u^\pm = u|_{\partial S^\pm \cap \Gamma_0}$, and α and β are real. For convergence of the solutions, choose $\alpha + \beta = 1$, and for symmetry, $\alpha = \beta = \frac{1}{2}$.

In our previous study, we always choose $\alpha\beta = 0$. Such couplings are called the simplified hybrid combinations of the Trefftz method and FEM, and reported in Li and Liang [47], Li and Bui [40, 42], Li [38], and Li and Huang [45]. The "simplified" means that no extra-variables such as the multiplier as in IV is needed. The bias derivatives $\frac{\partial u^+}{\partial n}$ and $\frac{\partial v^+}{\partial n}$ in $I_{Hyb}^{(1,0)}$ are easily formulated in the stiffness matrix, since the particular solutions used in S^+ are explicit. Hence, the simplified hybrid techniques are very efficient. However, when both particular solutions are used in S^+ and S^-, the symmetric hybrid techniques with $\alpha = \beta = \frac{1}{2}$ should also be studied. Since our previous analysis can not be applied to the case $\alpha\beta \neq 0$, new error analysis for the simplified hybrid method (5) is imperative. Note that the original Trefftz method by Trefftz in 1926 [71] is just the special case of Eq. (5) with $S^+ = S$, $S^- = \emptyset$, $\alpha = 1$ and $\beta = 0$.

II. The penalty plus hybrid techniques. To seek $u^*_{L,N} \in V_{L,N}$ such that

$$B_{PH}(u^*_{L,N}, v) = 0, \quad \forall v \in V_{L,N}, \tag{8}$$

where

$$B_{PH}(u,v) = \iint_{S^+} \nabla u \nabla v + \iint_{S^-} \nabla u \nabla v + I_{PH}^{(\alpha,\beta,P_c)}(u,v). \tag{9}$$

In Eq. (9) the penalty plus hybrid coupling is

$$I_{PH}^{(\alpha,\beta,P_c)}(u,v) = -\int_{\Gamma_0} \left(\alpha \frac{\partial u^+}{\partial n} + \beta \frac{\partial u^-}{\partial n}\right)(v^+ - v^-) - \int_{\Gamma_0} \left(\alpha \frac{\partial v^+}{\partial n} + \beta \frac{\partial v^-}{\partial n}\right)(u^+ - u^-)$$
$$+ P_c(L^\sigma + N^\tau) \int_{\Gamma_0} (u^+ - u^-)(v^+ - v^-), \tag{10}$$

where the parameters $\alpha + \beta = 1$ for better convergence of the solutions, $\sigma \geq 1$ and $\tau \geq 1$ are two constants independent of L and N, and P_c is a large enough constant but still independent of L and N. The penalty plus hybrid techniques have been used for the combinations of the Trefftz method and FEM in [37, 41, 43], and in this paper for the Trefftz method using piecewise particular solutions.

III. The Lagrange multiplier techniques for the direct TM. Consider the Debye–Huckel equation,

$$\mathcal{L}u = -\Delta u + u = 0 \quad \text{in } S. \tag{11}$$

Suppose that Φ_i and Ψ_i satisfy Eq. (11) in S^+ and S^-, respectively. We also choose Eq. (1) as the admissible functions, and denote by $V_{L,N}$ their finite dimensional collection. Also denote V_h the piecewise k–order polynomials. We choose the Lagrange multiplier $\lambda_h \in V_h$, to couple the displacement continuity on Γ_0. The Lagrange multiplier technique is to seek $(u^{\#}_{L,N}, \lambda_h) \in V_{L,N} \times V_h$ such that

$$A_{Lag}\left(u^{\#}_{L,N}, \lambda_h; v, \mu\right) = 0, \quad \forall (v,\mu) \in V_{L,N} \times V_h, \tag{12}$$

where

$$A_{Lag}(u,\lambda; v,\mu) = \iint_{S^+} \nabla u \nabla v + \iint_{S^-} \nabla u \nabla v + D_{Lag}(u,\lambda; v,\mu). \tag{13}$$

In Eq. (13) the Lagrange multiplier coupling is given by

$$D_{Lag}(u,\lambda; v,\mu) = -\int_{\Gamma_0} \lambda(v^+ - v^-) - \int_{\Gamma_0} \mu(u^+ - u^-). \tag{14}$$

The error analysis for Eq. (12) can be obtained by following Li [36, 37]. Note that the direct Trefftz method called in engineering is just the Lagrange multiplier Trefftz method, see [30, 33].

IV. The Hybrid Trefftz method of Jirousek [23] and Qin [62]. When the particular solutions are chosen, the interior flux condition $u_n^+ = u_n^-$ on Γ_0 is constrained *a priori*, also by means of the Lagrange multiplier λ, and the displacement continuity condition $u^+ = u^-$ on Γ_0 is a natural consequence. In this case, the true Lagrange multiplier is the solution u on Γ_0. Consider the Dirichlet condition $u = f$ on ∂S. Define the energy,

$$I(v) = \frac{1}{2}\left\{\iint_{S^+}(v_x^2 + v_y^2 + v^2) + \iint_{S^-}(v_x^2 + v_y^2 + v^2)\right\} - \int_{\Gamma_0}(v_n^+ - v_n^-)\lambda - \int_{\Gamma_D} f v_n\,, \tag{15}$$

where n on Γ_0 is the exterior normal of ∂S^+. Let the true solutions $u \in H^1(S)$ and the Lagrange multiplier $\lambda(= u) \in H^{\frac{1}{2}}(\Gamma_0)$ satisfy the following Galerkin problem:

$$A_h(u, v) + B_{HT}(u, \lambda; v, \mu) = f(v), \quad \forall v \in H^1(S),\ \mu \in H^{\frac{1}{2}}(\Gamma_0), \tag{16}$$

where

$$A_h(u, v) = \iint_{S^+}(\nabla u \nabla v + uv)\,\mathrm{d}s + \iint_{S^-}(\nabla u \nabla v + uv)\,\mathrm{d}s,$$

$$B_{HT}(u, \lambda; v, \mu) = -\int_{\Gamma_0}\lambda(v_n^+ - v_n^-)\,\mathrm{d}\ell - \int_{\Gamma_0}\mu(u_n^+ - u_n^-)\,\mathrm{d}\ell,$$

$$f(v) = \int_{\Gamma} f v_n.$$

Also let V_h the piecewise $k-$order polynomials on Γ_0, then $V_h \subset H^{\frac{1}{2}}(\Gamma_0)$. The Hybrid Trefftz method reads: To seek $(u_{L,N}, \lambda_h) \in V_{L,N} \times V_h$ such that

$$A_h(u_{L,N}, v) + B_{HT}(u_{L,N}, \lambda_h; v, \mu) = f(v), \quad \forall (v, \mu) \in V_{L,N} \times V_h\,. \tag{17}$$

Note that there exist the extra-variables: Lagrange multiplier in III and IV, but not in I and II. The simplified hybrid method is the simplest among I–IV.

For III, the Lagrange multiplier used to couple the Dirichlet condition and the interior continuity $u^+ = u^-$. However, for elasticity problem, when satisfying the stress equilibrium equations in S and the interior traction continuity condition, the interior displacement continuity is a natural consequence. Hence, the Lagrange multiplier is employed to couple the traction (i.e. Neumann) condition and the interior traction continuity $u_n^+ = u_n^-$. The Lagrange multiplier is regarded as the true solution u on Γ_N and Γ_0, which is easy and simple in numerical computation, and the hybrid Trefftz method (HTM) is called in Jirousek [23]. An error analysis is given in Li [39]. We may also use the simplified hybrid techniques, to remove the Lagrange multiplier for the interior flux conditions, to obtain exactly the same algorithms as in I for the interior boundary.

This paper is organized as follows. In Section 2, the simplified hybrid techniques are described, and in Section 3, new error analysis is made. In Section 4, the penalty plus hybrid techniques are studied, and in Section 5, the Lagrange multiplier techniques are explored. In Section 6, numerical experiments are carried out for Motz's problem by the CTM and the simplified hybrid TM. In the last section, some remarks are given.

2. The simplified hybrid techniques

Consider the Laplace equation with the Dirichlet and Neumann boundary conditions, see Fig. 1,

$$\triangle u = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \qquad \text{in } S, \tag{18}$$

$$u = g_1 \quad \text{on} \quad \Gamma_D\,, \qquad \frac{\partial u}{\partial n} = g_2 \quad \text{on} \quad \Gamma_N\,,$$

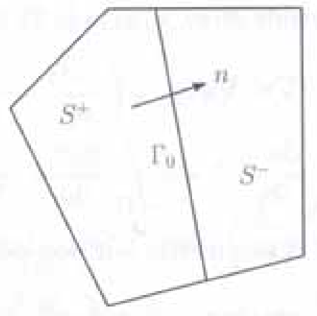

Fig. 1. Partition of S into S^+ and S^-

where S is a polygon, and ∂S is its boundary with $\partial S = \Gamma_D \cup \Gamma_N$. Let S be divided by the piecewise straight sections Γ_0 into two subdomains S^+ and S^- without overlaps. Suppose that the admissible functions are given by

$$v = \begin{cases} v^+ = \Phi_0 + \sum_{i=0}^{L} a_i \Phi_i & \text{in } S^+, \\ v^- = \Psi_0 + \sum_{i=0}^{N} b_i \Psi_i & \text{in } S^-, \end{cases} \tag{19}$$

where a_i and b_i are the coefficients to be sought, and Φ_i and Ψ_i are the particular solutions

$$\triangle \Phi_i = 0 \quad \text{in} \quad S^+, \qquad \triangle \Psi_i = 0 \quad \text{in} \quad S^-. \tag{20}$$

$\{\Phi_i\}$ and $\{\Psi_i\}$ are two complete and linearly independent bases in S^+ and S^- respectively. For simplicity in exposition, we let the partial solution $v^\pm$ just satisfy the exterior boundary conditions

$$v^\pm \Big|_{\partial S^\pm \cap \Gamma_D} = g_1, \qquad \frac{\partial}{\partial n} v^\pm \Big|_{\partial S^\pm \cap \Gamma_N} = g_2 . \tag{21}$$

Otherwise, the coupling techniques can be discussed as those for the interior boundary conditions. Denote by $V_{L,N}$ the finite dimensional collection of v in Eq. (19). Then we design the simplified hybrid Trefftz method (SHTM) as follows. To seek $u_{L,N} \in V_{L,N}$ to satisfy Eq. (5). When $u = v$, the integral

$$I_{Hyb}^{(\alpha,\beta)}(u, v) = 0. \tag{22}$$

Then for the nontrivial $v \in V_{L,N}$ we have

$$A_{Hyb}(v, v) = \iint_{S^+} \nabla v \nabla v + \iint_{S^-} \nabla v \nabla v > 0. \tag{23}$$

This may imply that the bilinear functional $A_{Hyb}(u, v)$ defined in Eq. (6) is positive definite. However, since $A_{Hyb}(u, v)$ is non-symmetric, more computational efforts and more computer storage are needed. In computation, we may seek $u_{L,N}$ differently. Since v^+ and v^- are independent to each other, we obtain from Eq. (5),

$$\iint_{S^+} \nabla u^+ \nabla v^+ - \alpha \int_{\Gamma_0} \frac{\partial v^+}{\partial n} u^- - \beta \int_{\Gamma_0} \frac{\partial u^-}{\partial n} v^+ = 0, \tag{24}$$

$$\iint_{S^-} \nabla u^- \nabla v^- + \alpha \int_{\Gamma_0} \frac{\partial u^+}{\partial n} v^- + \beta \int_{\Gamma_0} \frac{\partial v^-}{\partial n} u^+ = 0. \tag{25}$$

Subtracting Eq. (25) from Eq. (24) yields $B(u_{L,N}, v) = 0, \forall v \in V_{L,N}$, where

$$B(u,v) = \iint_{S^+} \nabla u^+ \nabla v^+ - \iint_{S^-} \nabla u^- \nabla v^- \\ - \alpha \int_{\Gamma_0} \frac{\partial v^+}{\partial n} u^- - \beta \int_{\Gamma_0} \frac{\partial u^-}{\partial n} v^+ - \alpha \int_{\Gamma_0} \frac{\partial u^+}{\partial n} v^- - \beta \int_{\Gamma_0} \frac{\partial v^-}{\partial n} u^+. \tag{26}$$

The new bilinear function $B(u,v)$ is symmetric but non-definite. Denote a function,

$$T(v) = \frac{1}{2} \iint_{S^+} \nabla v^+ \nabla v^+ - \frac{1}{2} \iint_{S^-} \nabla v^- \nabla v^- - \alpha \int_{\Gamma_0} \frac{\partial v^+}{\partial n} v^- - \beta \int_{\Gamma_0} \frac{\partial v^-}{\partial n} v^+. \tag{27}$$

The variational equation of the function $T(v)$

$$\frac{\partial T(v)}{\partial v} = 0 \tag{28}$$

leads to exactly Eq. (5). Equation (27) can be expressed as the matrix and vectors

$$T(v) = \frac{1}{2} \vec{X}^T \mathbf{A} \vec{X} - \vec{X}^T \vec{b}, \tag{29}$$

where $\vec{X} = (\vec{y}, \vec{z})^T$, $\vec{y} = (a_1, \ldots, a_L)^T$, $\vec{z} = (b_1, \ldots, b_N)^T$. The stiffness matrix is

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^T & -\mathbf{A}_{22} \end{pmatrix}, \tag{30}$$

where $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ resulting from $\iint_{S^+} \nabla v^+ \nabla v^+$ and $\iint_{S^-} \nabla v^- \nabla v^-$ are positive definite and symmetric. The variational equation (28) gives the linear algebraic equations

$$\mathbf{A}\vec{X} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^T & -\mathbf{A}_{22} \end{pmatrix} \begin{pmatrix} \vec{y} \\ \vec{z} \end{pmatrix} = \vec{b}. \tag{31}$$

Since the matrix $\mathbf{A}$ is symmetric and nonsingular, we may use the symmetric Gaussian elimination without pivoting, to obtain the coefficients a_i and b_i easily. Note that the solution from Eq. (31) is easier than that directly from Eq. (5), and that the running CPU time and computer storage are also saved.

Let us link the simplified hybrid techniques to the interior continuity conditions (2). We assume that $\alpha + \beta = 1$. From the Green formulas and $\Delta u^{\pm} = \Delta v^{\pm} = 0$ in $S^{\pm}$, we have

$$\iint_{S^+} \nabla u^+ \nabla v^+ = \int_{\Gamma_0} \frac{\partial u^+}{\partial n} v^+ = \int_{\Gamma_0} \frac{\partial v^+}{\partial n} u^+, \tag{32}$$

$$\iint_{S^-} \nabla u^- \nabla v^- = - \int_{\Gamma_0} \frac{\partial u^-}{\partial n} v^- = - \int_{\Gamma_0} \frac{\partial v^-}{\partial n} u^-. \tag{33}$$

Hence when $\alpha + \beta = 1$,

$$\iint_{S^+} \nabla u^+ \nabla v^+ = \alpha \int_{\Gamma_0} \frac{\partial v^+}{\partial n} u^+ + \beta \int_{\Gamma_0} \frac{\partial u^+}{\partial n} v^+, \tag{34}$$

$$\iint_{S^-} \nabla u^- \nabla v^- = -\alpha \int_{\Gamma_0} \frac{\partial u^-}{\partial n} v^- - \beta \int_{\Gamma_0} \frac{\partial v^-}{\partial n} u^-. \tag{35}$$

From Eq. (34), Eq. (24) leads to

$$\alpha \int_{\Gamma_0} \frac{\partial v^+}{\partial n} (u^+ - u^-) + \beta \int_{\Gamma_0} \left(\frac{\partial u^+}{\partial n} - \frac{\partial u^-}{\partial n} \right) v^+ = 0. \tag{36}$$

Also from Eq. (35), Eq. (25) leads to

$$\beta \int_{\Gamma_0} \frac{\partial v^-}{\partial n}(u^+ - u^-) + \alpha \int_{\Gamma_0} \left(\frac{\partial u^+}{\partial n} - \frac{\partial u^-}{\partial n} \right) v^- = 0. \tag{37}$$

Combining Eqs. (36) and (37) gives

$$\int_{\Gamma_0} \left(\alpha \frac{\partial v^+}{\partial n} + \beta \frac{\partial v^-}{\partial n} \right) (u^+ - u^-) + \int_{\Gamma_0} (\beta v^+ + \alpha v^-) \left(\frac{\partial u^+}{\partial n} - \frac{\partial u^-}{\partial n} \right) = 0. \tag{38}$$

Since v^+ and v^- are arbitrary, Eq. (38) implies the condition (2).

3. Error analysis for the simplified hybrid Trefftz method

Denote the norm

$$\|v\|_1 = \left\{ \|v\|^2_{1,S^+} + \|v\|^2_{1,S^-} \right\}^{\frac{1}{2}}, \qquad |v|_1 = \left\{ |v|^2_{1,S^+} + |v|^2_{1,S^-} \right\}^{\frac{1}{2}}, \tag{39}$$

where $\|v\|_{1,S^+}$ and $|v|_{1,S^+}$ are the Sobolev norms.

3.1. Basic error analysis

First we assume that there is no integration errors involved. We have the following lemmas.

Lemma 1. *Let the constant be excluded in $V_{L,N}$, and α and β be bounded. Then for the simplified hybrid Trefftz method (5), there exist two constants C_0 and C_1 independent of u and v such that*

$$|A_{Hyb}(u,v)| \le C_1 \|u\|_1 \|v\|_1, \qquad \forall v \in V_{L,N}, \tag{40}$$

$$C_0 \|v\|_1^2 \le A_{Hyb}(v,v), \qquad \forall v \in V_{L,N}. \tag{41}$$

When the particular solutions Φ_i and Ψ_i exclude the constant, their linear combination also exclude constant. If both

$$S^+ \cap \Gamma_D \neq \emptyset, \quad S^- \cap \Gamma_D \neq \emptyset, \tag{42}$$

we obtain from the Poincaré inequality,

$$C_0 \|v\|^2_{1,S^+} \le |v|^2_{1,S^+}, \qquad C_0 \|v\|^2_{1,S^-} \le |v|^2_{1,S^-}, \tag{43}$$

where C_0 is a constant independent of v. We also obtain from Eq. (43)

$$C_0 \|v\|_1^2 \le A_{Hyb}(v,v). \tag{44}$$

Lemma 2. *Let u and $u_{L,N}$ be the true solution and the solution from the simplified hybrid Trefftz method (5), respectively. Suppose that $\alpha + \beta = 1$. Then there exists the equality*

$$A_{Hyb}(u - u_{L,N}, v) = 0, \quad \forall v \in V_{L,N}. \tag{45}$$

Proof: For the true solution u, we have $u^+ = u^- = u$ on Γ_0. Then we have from Eq. (7)

$$I^{(\alpha,\beta)}_{Hyb}(u,v) = \alpha \int_{\Gamma_0} \frac{\partial u}{\partial n} v^- - \beta \int_{\Gamma_0} \frac{\partial u}{\partial n} v^+ - \alpha \int_{\Gamma_0} \frac{\partial v^+}{\partial n} u + \beta \int_{\Gamma_0} \frac{\partial v^-}{\partial n} u. \tag{46}$$

Since u and v^+ satisfy $\triangle u = 0$ in S^+, we have from the Green theory

$$\int_{\Gamma_0} \frac{\partial v^+}{\partial n} u = \iint_{S^+} \nabla u \nabla v = \int_{\Gamma_0} \frac{\partial u}{\partial n} v^+. \tag{47}$$

Similarly, since u and v^- satisfy $\triangle u = 0$ in S^-, we also have

$$\int_{\Gamma_0} \frac{\partial v^-}{\partial n} u = \int_{\Gamma_0} \frac{\partial u}{\partial n} v^- . \tag{48}$$

Combining Eqs. (46)–(48) gives

$$I_{Hyb}^{(\alpha,\beta)}(u,v) = -(\alpha+\beta)\int_{\Gamma_0} \frac{\partial u}{\partial n}(v^+ - v^-). \tag{49}$$

Hence for the true solution u, we obtain

$$\begin{aligned} A_{Hyb}(u,v) &= \int_{\partial S^+} \frac{\partial u}{\partial n} v^+ - \int_{\partial S^-} \frac{\partial u}{\partial n} v^- + I_{Hyb}^{(\alpha,\beta)}(u,v) \\ &= \int_{\Gamma_0} \frac{\partial u}{\partial n}(v^+ - v^-) + I_{Hyb}^{(\alpha,\beta)}(u,v) = [1-(\alpha+\beta)]\int_{\Gamma_0} \frac{\partial u}{\partial n}(v^+ - v^-) = 0, \end{aligned} \tag{50}$$

where we have used the assumption $\alpha + \beta = 1$. The desired result (45) follows from Eqs. (5) and (50). ∎

Now we give a main theorem.

Theorem 1. *Let the constant be excluded in $V_{L,N}$ and $\alpha + \beta = 1$. Then the solutions from the simplified hybrid Trefftz method (5) have the optimal error bounds,*

$$\|u - u_{L,N}\|_1 \le C \inf_{v \in V_{L,N}} \|u - v\|_1 , \tag{51}$$

where C is a constant independent of L, N, u and v.

Proof: When $w = u_{L,N} - v$, where $v \in V_{L,N}$. Then $w \in V_{L,N}$. From Lemmas 1 and 2, we obtain

$$C_0\|w\|_1^2 \le A_{Hyb}(u_{L,N} - v, w) = A_{Hyb}(u - v, w) \le C_1\|u - v\|_1\|w\|_1 . \tag{52}$$

Hence we have

$$\|u_{L,N} - v\|_1 \le \frac{C_1}{C_0}\|u - v\|_1 . \tag{53}$$

Moreover,

$$\|u - u_{L,N}\|_1 \le \|u - v\|_1 + \|v - u_{L,N}\|_1 \le \left(1 + \frac{C_1}{C_0}\right)\|u - v\|_1 , \tag{54}$$

the desired result (51) follows by letting $C = \left(1 + \frac{C_1}{C_0}\right)$. This completes the proof of Theorem 1. ∎

Suppose that there exist the solutions

$$u = \sum_{i=1}^{\infty} \bar{a}_i \Phi_i = \bar{u}_L + R_L \quad \text{in } S^+, \tag{55}$$

where $\bar{a}_i$ are the true expansion coefficients, and

$$\bar{u}_L = \sum_{i=1}^{L} \bar{a}_i \Phi_i, \qquad R_L = \sum_{i=L+1}^{\infty} \bar{a}_i \Phi_i . \tag{56}$$

Moreover, suppose that the convergence rates of Eq. (55) are exponential, i.e., there exists a constant γ_1 with $0 < \gamma_1 < 1$ such that

$$|R_L| \le \bar{C}_1 \gamma_1^L = \bar{C}_1 \exp(-\theta_1 L), \tag{57}$$

where $\gamma_1 = \exp(-\theta_1)$ with $\theta_1 > 0$, and $\bar{C}_1$ is a constant independent of L. Similarly, we also suppose that

$$u = \sum_{i=1}^{\infty} \bar{b}_i \Psi_i = \bar{u}_N + R_N \quad \text{in } S^-, \tag{58}$$

where $\bar{b}_i$ are the true expansion coefficients, and

$$\bar{u}_N = \sum_{i=1}^{N} \bar{b}_i \Psi_i, \qquad R_N = \sum_{i=N+1}^{\infty} \bar{b}_i \Psi_i, \tag{59}$$

and

$$|R_N| \leq \bar{C}_2 \gamma_2^L = \bar{C}_2 \exp(-\theta_2 N), \tag{60}$$

where $\gamma_2 = \exp(-\theta_2) < 1$ with $\theta_2 > 0$, and $\bar{C}_2$ is a constant independent of N. We have the following corollary from Theorem 1.

Corollary 1. *Let all the conditions in Theorem 1, and Eqs. (55)–(60) hold. Then for the simplified hybrid Trefftz method (5) there exist the exponential convergence rates.*

$$\|u - u_{L,N}\|_1 \leq C \left\{ \sqrt{L} \exp(-\theta_1 L) + \sqrt{N} \exp(-\theta_2 N) \right\}, \tag{61}$$

where $\theta_1 > 0$ and $\theta_2 > 0$, and C is a constant independent of L and N.

Remark 1. There exist some limitations for applying the simplified hybrid TM to Laplace's equation[2]. By noting that a constant solution may be included into the piecewise particular solutions, which violates the condition in Lemma 1. Let S be split into S_i, i.e., $S = \cup_i S_i$ and $S_i \cap S_j = \emptyset$, $i \neq j$. First the condition (43) leads to $\partial S_i \cap \Gamma_D \neq \emptyset$, where Γ_D is given in Eq. (18). This condition excludes any interior subdomains S_i with $S_i \cap \Gamma = \emptyset$. Next, let us consider the simplified hybrid TMs from Eq. (5) with $\alpha = 1$ and $\beta = 0$, to seek $u_{L,N} \in V_{L,N} (= V_L \times V_N)$ such that

$$A^*_{Hyb}(u_{L,N}, v) = 0, \quad \forall v \in V_{L,N}, \tag{62}$$

where

$$A^*_{Hyb}(u, v) = \iint_{S^+} \nabla u \nabla v + \iint_{S^-} \nabla u \nabla v + \int_{\Gamma_0} \frac{\partial u^+}{\partial n} v^- - \int_{\Gamma_0} \frac{\partial v^+}{\partial n} u^-. \tag{63}$$

Note that a constant in u^+ and v^+ does not make any difference in Eq. (63), so the constant must be excluded into the particular solution v^+ in V_L. Moreover, if a constant is involved into the particular solutions $v^- \in V_N$, the parameter $\beta = 0$ in Eq. (6) and then $\alpha = 1$ are also required to give Eq. (62). This is just the computational model for Motz's problem in Section 6. Therefore, the condition in Lemma 1 may be relaxed as that a constant solution must not be included into the piecewise particular solutions $v^+ \in V_L$.

Remark 2. The limitation of the simplified hybrid Trefftz method in Remark 1 can be removed by the *a posteriori* (see Qin [62]), to seek a suitable constant. For simplicity, let $\Gamma_D \neq \emptyset$, and $\Gamma_D \cap S_1 \neq \emptyset$. When the solutions u_N^+ in S_1 and u_N^- in S_2 have been obtained from Eq. (5) or Eq. (62), the general solutions are given by adding a constant $c_0^\pm$: $u_N^\pm + c_0^\pm$. From the Dirichlet condition

$$u_N^+ + c_0^+ = g_1 \qquad \text{on } \Gamma_D \cap S^+, \tag{64}$$

[2]The simplified hybrid TM can be applied similarly to the Debye–Huckel equation, $-\Delta u + u = 0$. The limitations described in Remark 1 are removed.

we obtain the constant

$$c_0^+ = \frac{1}{|\Gamma_D \cap S^+|} \int_{\Gamma_D \cap S^+} (g_1 - u_N^+), \tag{65}$$

where $|\Gamma_D \cap S^+|$ is the length of $\Gamma_D \cap S^+$. Also from the interior continuity condition (2), we obtain the other constant

$$c_0^- = \frac{1}{|\Gamma_0|} \int_{\Gamma_0} (u_N^+ - u_N^-) + c_0^+. \tag{66}$$

Equations (65) and (66) can also be evaluated by numerical approximation:

$$c_0^+ = \frac{1}{|\Gamma_D \cap S^+|} \widehat{\int}_{\Gamma_D \cap S^+} (g_1 - u_N^+), \qquad c_0^- = \frac{1}{|\Gamma_0|} \widehat{\int}_{\Gamma_0} (u_N^+ - u_N^-) + c_0^+. \tag{67}$$

3.2. Integration approximation for the hybrid Trefftz method

In this subsection, we will follow the traditional FEM analysis, to derive the error bounds of the simplified hybrid Trefftz solution when the integration approximation is involved. The simplified hybrid Trefftz method involving integration approximation is expressed as: To solution $\tilde{u}_{L,N} \in V_{L,N}$ such that

$$\widehat{A}_{Hyb}(\tilde{u}_{L,N}, v) = 0, \quad \forall v \in V_{L,N}, \tag{68}$$

where $V_{L,N}$ is the same collection of Eq. (19) satisfying Eq. (21), and

$$\widehat{A}_{Hyb}(u,v) = \widehat{\iint}_{S^+} \nabla u \nabla v + \widehat{\iint}_{S^-} \nabla u \nabla v + \hat{I}_{Hyb}^{(\alpha,\beta)}(u,v). \tag{69}$$

In Eq. (6), the simplified hybrid coupling with integration approximation is

$$\hat{I}_{Hyb}^{(\alpha,\beta)}(u,v) = \alpha \widehat{\int}_{\Gamma_0} \frac{\partial u^+}{\partial n} v^- - \beta \widehat{\int}_{\Gamma_0} \frac{\partial u^-}{\partial n} v^+ - \alpha \widehat{\int}_{\Gamma_0} \frac{\partial v^+}{\partial n} u^- + \beta \widehat{\int}_{\Gamma_0} \frac{\partial v^-}{\partial n} u^+. \tag{70}$$

The notations $\widehat{\iint}_{S^+}$ and $\widehat{\int}_{\Gamma_0}$ denote the approximations of $\iint_{S^+}$ and $\int_{\Gamma_0}$ by some quadrature rules. Since

$$\iint_{S^+} \nabla u \nabla v = \int_{\Gamma_0} \frac{\partial u^+}{\partial n} v^+, \qquad \iint_{S^-} \nabla u \nabla v = \int_{\Gamma_0} \frac{\partial u^-}{\partial n} v^-,$$

the integration approximation of $\widehat{\iint}_{S^\pm}$ can be carried out as that of $\widehat{\int}_{\Gamma_0}$,

$$\widehat{\iint}_{S^+} \nabla u \nabla v = \widehat{\int}_{\Gamma_0} \frac{\partial u^+}{\partial n} v^+, \qquad \widehat{\iint}_{S^-} \nabla u \nabla v = \widehat{\int}_{\Gamma_0} \frac{\partial u^-}{\partial n} v^-.$$

Suppose the following inequalities hold,

$$\widehat{A}_{Hyb}(u,v) \le C_1 \|u\|_1 \|v\|_1, \qquad \forall v \in V_{L,N}, \tag{71}$$

$$C_0 \|v\|_1^2 \le \widehat{A}_{Hyb}(v,v), \qquad \forall v \in V_{L,N}, \tag{72}$$

where C_0 and C_1 are two positive constants independent of L and N. We have the following theorem.

Theorem 2. *Let Eqs. (71) and (72) hold. Suppose that $\alpha+\beta=1$. Then the solutions $\tilde{u}_{L,N}$ for the simplified hybrid Trefftz method involving integration approximation (68) have the following error bounds,*

$$\|u-\tilde{u}_{L,N}\|_1 \le C\left\{ \inf_{v\in V_{L,N}} \|u-v\|_1 + \sup_{w\in V_{L,N}} \frac{1}{\|w\|_1}\left[\left|\left(\int_{\Gamma_0}-\widehat{\int}_{\Gamma_0}\right)\frac{\partial u}{\partial n}w^+\right| + \left|\left(\int_{\Gamma_0}-\widehat{\int}_{\Gamma_0}\right)\frac{\partial u}{\partial n}w^-\right| + \left|\left(\int_{\Gamma_0}-\widehat{\int}_{\Gamma_0}\right)\frac{\partial w^+}{\partial n}u\right| + \left|\left(\int_{\Gamma_0}-\widehat{\int}_{\Gamma_0}\right)\frac{\partial w^-}{\partial n}u\right|\right]\right\}. \tag{73}$$

Proof: Since $\alpha+\beta=1$ we have from Eq. (50)

$$A_{Hyb}(u,v)=0, \quad \forall v\in V_{L,N}. \tag{74}$$

Then

$$\widehat{A}_{Hyb}(u,v)=A_{Hyb}(u,v)+\left(\widehat{A}_{Hyb}-A_{Hyb}\right)(u,v)=(\widehat{A}_{Hyb}-A_{Hyb})(u,v). \tag{75}$$

From Eqs. (68) and (75)

$$\widehat{A}_{Hyb}(u-\tilde{u}_{L,N},v)=\left(\widehat{A}_{Hyb}-A_{Hyb}\right)(u,v). \tag{76}$$

Let $w=\tilde{u}_{L,N}-v$, where $v\in V_{L,N}$. Then $w\in V_{L,N}$. We obtain from Eqs. (71), (72) and (76)

$$\begin{aligned} C_0\|w\|_1^2 &\le \widehat{A}_{Hyb}(\tilde{u}_{L,N}-v,w)=\hat{A}_{Hyb}(u-v,w)-\left(\widehat{A}_{Hyb}-A_{Hyb}\right)(u,w) \\ &\le C_1\|u-v\|_1\|w\|_1+\left|\left(\widehat{A}_{Hyb}-A_{Hyb}\right)(u,w)\right|. \end{aligned}$$

Hence we have

$$\|\tilde{u}_{L,N}-v\|_1 \le \frac{C_1}{C_0}\|u-v\|_1+\frac{1}{C_0\|w\|_1}\left|\left(\widehat{A}_{Hyb}-A_{Hyb}\right)(u,w)\right|.$$

Moreover, since $\|u-\tilde{u}_{L,N}\|_1 \le \|u-v\|_1+\|v-\tilde{u}_{L,N}\|_1$

$$\|u-\tilde{u}_{L,N}\|_1 \le \left(1+\frac{C_1}{C_0}\right)\|u-v\|_1+\frac{1}{C_0\|w\|_1}\left|\left(\widehat{A}_{Hyb}-A_{Hyb}\right)(u,w)\right|, \tag{77}$$

where

$$\left(\widehat{A}_{Hyb}-A_{Hyb}\right)(u,w) \le \left|\left(\int_{\Gamma_0}-\widehat{\int}_{\Gamma_0}\right)\frac{\partial u}{\partial n}w^+\right| + \left|\left(\int_{\Gamma_0}-\widehat{\int}_{\Gamma_0}\right)\frac{\partial u}{\partial n}w^-\right| + \left|\left(\int_{\Gamma_0}-\widehat{\int}_{\Gamma_0}\right)\frac{\partial w^+}{\partial n}u\right| + \left|\left(\int_{\Gamma_0}-\widehat{\int}_{\Gamma_0}\right)+\frac{\partial w^-}{\partial n}u\right|. \tag{78}$$

Then the desired result (73) follows from Eqs. (77) and (78), and this completes the proof of Theorem 2. ∎

Suppose that $v^+\in V_{L,N}$ satisfies the following inequalities

$$|v^+|_{k,\Gamma_0} \le C_1L^{\sigma k}|v^+|_{0,\Gamma_0}, \qquad \left|\frac{\partial v^+}{\partial n}\right|_{k,\Gamma_0} \le C_1L^{\sigma(k+1)}|v^+|_{0,\Gamma_0}, \tag{79}$$

where $\sigma \geq 1$ and C_1 are two constants independent of L and v^+. Also suppose that $v^- \in V_{L,N}$ satisfies the following inequalities

$$|v^-|_{k,\Gamma_0} \leq C_2 N^{\tau k}|v^-|_{0,\Gamma_0}\,, \qquad \left|\frac{\partial v^-}{\partial n}\right|_{k,\Gamma_0} \leq C_2 N^{\tau(k+1)}|v^-|_{0,\Gamma_0}\,, \tag{80}$$

where $\tau \geq 1$ and C_2 are two constants independent of N and v^-. We have the following lemma.

Lemma 3. *Let v^+ satisfy Eq. (79). For the quadrature rules with the accuracy of order k. Then for the simplified hybrid Trefftz method involving integration approximation (68) there exist the error bounds,*

$$\left|\left(\int_{\Gamma_0} - \widehat{\int}_{\Gamma_0}\right)\frac{\partial u}{\partial n}w^+\right| \leq CL^{\sigma(k+1)}h^{k+1}\left|\frac{\partial u}{\partial n}\right|_{k+1,\Gamma_0}\|w^+\|_{1,S^+}\,, \quad w^+ \in V_{L,N}\,, \tag{81}$$

$$\left|\left(\int_{\Gamma_0} - \widehat{\int}_{\Gamma_0}\right)\frac{\partial w^+}{\partial n}u\right| \leq CL^{\sigma(k+2)}h^{k+1}|u|_{k+1,\Gamma_0}\|w^+\|_{1,S^+}\,, \quad w^+ \in V_{L,N}\,, \tag{82}$$

where h is the maximal interval of integration rule.

We have the following corollary from Theorem 2 and Lemma 3.

Corollary 2. *Let all the conditions in Theorem 2 and Lemma 3. Then for the simplified hybrid Trefftz method involving integration approximation (68) there exist the error bounds,*

$$\begin{aligned}\|u-\tilde{u}_{L,N}\|_1 \leq C\Big\{&\sqrt{L}\exp(-\theta_1 L)+\sqrt{L}\exp(-\theta_2 L)\\ &+\left(L^{\sigma(k+2)}+N^{\tau(k+2)}\right)h^{k+1}\left(\left|\frac{\partial u}{\partial n}\right|_{k+1,\Gamma_0}+|u|_{k+1,\Gamma_0}\right)\Big\}\,,\end{aligned} \tag{83}$$

where C is a constant independent of L, N and h.

Below let us examine the inequalities (71) and (72). Denote

$$\overline{|v|}_1 = \left\{\widehat{\iint_{S^+}}|\nabla v|^2 + \widehat{\iint_{S^-}}|\nabla v|^2\right\}^{\frac{1}{2}}, \qquad \overline{\|v\|}_1 = \left\{\overline{|v|}_1^2 + \|v\|_{0,S}^2\right\}^{\frac{1}{2}}.$$

Suppose that there exists the norm equivalence

$$|v|_1 \asymp \overline{|v|}_1\,, \quad \forall v \in V_{L,N}\,, \tag{84}$$

i.e., there exist two positive constants C_0 and C_1 such that

$$C_0|v|_1 \leq \overline{|v|}_1 \leq C_1|v|_1\,, \quad \forall v \in V_{L,N}\,. \tag{85}$$

Then we have the following lemma.

Lemma 4. *If $\overline{|v|}_1 \asymp |v|_1$, then $\overline{\|v\|}_1 \asymp \|v\|_1$.*

Proof: From Eq. (85), we have

$$C_0^2|v|_1 + \|v\|_{0,S}^2 \leq \overline{\|v\|}_1^2 = \overline{|v|}_1^2 + \|v\|_{0,S}^2 \leq C_1^2|v|_1^2 + \|v\|_{0,S}^2\,, \quad v \in V_{L,N}\,. \tag{86}$$

Then

$$\min\{1, C_0^2\}\|v\|_1^2 \leq \overline{\|v\|}_1^2 \leq \max\{1, C_1^2\}\|v\|_1^2\,, \quad v \in V_{L,N}\,. \tag{87}$$

This displays $\overline{\|v\|}_1 \asymp \|v\|_1$, and completes the proof of Lemma 4. ∎

Lemma 5. *Let all the conditions in Lemma 1, Eqs. (79) and (80) hold. Suppose that the integral interval h in the quadrature rule of order k is chosen so small that* [3]

$$h^{k+1}L^{\sigma(k+2)} = o(1), \qquad h^{k+1}N^{\tau(k+2)} = o(1). \tag{88}$$

Then the uniformly $V_{L,N}$-elliptic inequality (72) holds.

Proof: We have

$$\widehat{A}_{Hyb}(v,v) = \widehat{\iint_{S^+}} |\nabla v|^2 + \widehat{\iint_{S^-}} |\nabla v|^2 = \widehat{\int_{\Gamma_0}} \frac{\partial v^+}{\partial n} v^+ + \widehat{\int_{\Gamma_0}} \frac{\partial v^-}{\partial n} v^- . \tag{89}$$

For the quadrature rule of order k, we have

$$\left| \left(\int_{\Gamma_0} - \widehat{\int_{\Gamma_0}} \right) \frac{\partial v^+}{\partial n} v^+ \right| \le Ch^{k+1} \left| \frac{\partial v^+}{\partial n} v^+ \right|_{k+1,\Gamma_0} . \tag{90}$$

From Eq. (79) we obtain

$$\begin{aligned} \left| \frac{\partial v^+}{\partial n} v^+ \right|_{k+1,\Gamma_0} &\le C \sum_{i=0}^{k+1} \left| \frac{\partial v^+}{\partial n} \right|_{k+1-i,\Gamma_0} |v^+|_{i,\Gamma_0} \le C \sum_{i=0}^{k+1} L^{\sigma(k+2-i)} \times L^{\sigma i} \|v^+\|^2_{0,\Gamma_0} \\ &\le CL^{\sigma(k+2)} \|v^+\|^2_{0,\Gamma_0} \le CL^{\sigma(k+2)} \|v\|^2_{1,S^+} , \end{aligned} \tag{91}$$

where C is a constant independent of L and v. Combining Eqs. (90) and (91) gives

$$\left| \left(\int_{\Gamma_0} - \widehat{\int_{\Gamma_0}} \right) \frac{\partial v^+}{\partial n} v^+ \right| \le Ch^{k+1} L^{\sigma(k+2)} \|v\|^2_{1,S^+} , \quad \forall v \in V_{L,N} . \tag{92}$$

Similarly, we have from Eq. (80)

$$\left| \left(\int_{\Gamma_0} - \widehat{\int_{\Gamma_0}} \right) \frac{\partial v^-}{\partial n} v^- \right| \le Ch^{k+1} N^{\tau(k+2)} \|v\|^2_{1,S^-} , \quad \forall v \in V_{L,N} . \tag{93}$$

Hence we have from Eq. (89)

$$\widehat{A}_{Hyb}(v,v) = \iint_{S^+} |\nabla v|^2 + \iint_{S^-} |\nabla v|^2 + \left(\widehat{\int_{\Gamma_0}} - \int_{\Gamma_0} \right) \frac{\partial v^+}{\partial n} v^+ + \left(\widehat{\int_{\Gamma_0}} - \int_{\Gamma_0} \right) \frac{\partial v^-}{\partial n} v^- . \tag{94}$$

From Lemma 1, the uniformly $V_{L,N}-$inequality holds,

$$\bar{C}_0 \|v\|_1^2 \le A_{Hyb}(v,v), \ \forall v \in V_{L,N} , \tag{95}$$

where $\bar{C}_0$ is a positive constant independent of L, N and h. Hence we obtain from Eqs. (94) and (95)

$$\begin{aligned} \widehat{A}_{Hyb}(v,v) &\ge \bar{C}_0 \|v\|_1^2 - \left| \left(\int_{\Gamma_0} - \widehat{\int_{\Gamma_0}} \right) \frac{\partial v^+}{\partial n} v^+ \right| - \left| \left(\widehat{\int_{\Gamma_0}} - \int_{\Gamma_0} \right) \frac{\partial v^-}{\partial n} v^- \right| \\ &\ge \left\{ \bar{C}_0 - Ch^{k+1} (L^{\sigma(k+2)} + L^{\tau(k+2)}) \right\} \|v\|_1^2 . \end{aligned} \tag{96}$$

When the conditions (88) hold, we have

$$\widehat{A}_{Hyb}(v,v) \ge \{\bar{C}_0 - o(1)\} \|v\|_1^2 \ge \frac{\bar{C}_0}{2} \|v\|_1^2 . \tag{97}$$

This is the desired result (72) by letting $C_0 = \frac{\bar{C}_0}{2}$, and competes the proof of Lemma 5. ∎

[3]Equation (88) implies that $h = \min \left\{ o\left(L^{-\sigma(1+\frac{1}{k+1})} \right), o\left(N^{-\tau(1+\frac{1}{k+1})} \right) \right\}$.

4. The penalty plus hybrid techniques

In this section, we pursue the penalty plus hybrid techniques,

$$I_{PH}^{(\alpha,\beta,P_c)}(u,v) = -\int_{\Gamma_0}\left(\alpha\frac{\partial u^+}{\partial n}+\beta\frac{\partial u^-}{\partial n}\right)(v^+-v^-)-\int_{\Gamma_0}\left(\alpha\frac{\partial v^+}{\partial n}+\beta\frac{\partial v^-}{\partial n}\right)(u^+-u^-)$$
$$+P_c(L^\sigma+N^\tau)\int_{\Gamma_0}(u^+-u^-)(v^+-v^-), \tag{98}$$

where $\alpha+\beta=1$, P_c a large enough but still independent of L, N, and $\sigma\geq 1$ and $\tau\geq 1$ are two constants given in Eqs. (79) and (80).

Denote by $V_{L,N}$ the finite dimensional collection of v in Eq. (19). The penalty plus hybrid techniques is expressed by: To seek $u^*_{L,N}\in V_{L,N}$ such that

$$B_{PH}(u^*_{L,N},v)=0,\quad \forall v\in V_{L,N}\,, \tag{99}$$

where

$$B_{PH}(u,v)=\iint_{S^+}\nabla u\nabla v+\iint_{S^-}\nabla u\nabla v+I_{PH}^{(\alpha,\beta,P_c)}(u,v). \tag{100}$$

Denote the norms

$$\|v\|_p=\left\{\|v\|^2_{1,S^+}+\|v\|^2_{1,S^-}+P_c(L^\sigma+N^\tau)\|v^+-v^-\|^2_{0,\Gamma_0}\right\}^{\frac12}. \tag{101}$$

Suppose the following inequalities hold,

$$B_{PH}(u,v)\leq C_1\|u\|_p\|v\|_p\,,\quad \forall v\in V_{L,N}\,, \tag{102}$$
$$C_0\|v\|_p^2\leq B_{PH}(v,v),\quad \forall v\in V_{L,N}\,, \tag{103}$$

where C_0 and C_1 are two positive constants independent of L and N. We have the following theorem.

Theorem 3. *Let Eqs. (102) and (103) hold. Suppose that $\alpha+\beta=1$. Then the solutions $u^*_{L,N}$ from Eq. (99) have the following error bound,*

$$\|u-u^*_{L,N}\|_p\leq C\inf_{v\in V_{L,N}}\|u-v\|_p\,, \tag{104}$$

where C is a constant independent of L and N.

Proof: For the true solution u, we have

$$B_{PH}(u,v)=\{1-(\alpha+\beta)\}\int_{\Gamma_0}\frac{\partial u^+}{\partial n}(v^+-v^-)=0. \tag{105}$$

Then from Eqs. (99) and (105)

$$B_{PH}(u-u^*_{L,N},v)=0,\quad \forall v\in V_{L,N}\,. \tag{106}$$

Let $w=u^*_{L,N}-v$, where $v\in V_{L,N}$. Then $w\in V_{L,N}$. From Eqs. (102), (103) and (106)

$$C_0\|w\|_p^2\leq B_{PH}(u^*_{L,N}-v,w)=B_{PH}(u-v,w)\leq C_1\|u-v\|_p\|w\|_p\,.$$

Thus gives

$$\|u^*_{L,N}-v\|_p\leq\frac{C_1}{C_0}\|u-v\|_p\,.$$

Moreover, since

$$\|u-u^*_{L,N}\|_p\leq\|u-v\|_p+\|v-u^*_{L,N}\|_p\leq\left(1+\frac{C_1}{C_0}\right)\|u-v\|_p\,,$$

the desired result (104) follows by letting $C=(1+\frac{C_1}{C_0})$. This completes the proof of Theorem 3. ∎

Corollary 3. *Let all the conditions in Theorem 3, and Eqs. (55)–(60) hold. Then there exist the exponential convergence rates.*

$$\|u-u^*_{L,N}\|_p \le C\Big\{\sqrt{L}\exp(-\theta_1 L)+\sqrt{N}\exp(-\theta_2 N) + \sqrt{P_c}\left(L^{\frac{\sigma}{2}}+N^{\frac{\tau}{2}}\right)\left(\exp(-\theta_1 L)+\exp(-\theta_2 N)\right)\Big\}, \tag{107}$$

where $\theta_1>0$ and $\theta_2>0$, and C is a constant independent of L and N.

Below, let us study Eq. (103), and give the following lemma.

Lemma 6. *Let the following bounds for $v\in V_{L,N}$ be given*

$$\left|\frac{\partial v^+}{\partial n}\right|_{0,\Gamma_0} \le CL^{\frac{\sigma}{2}}\|v\|_{1,S^+}, \tag{108}$$

$$\left|\frac{\partial v^-}{\partial n}\right|_{0,\Gamma_0} \le CN^{\frac{\tau}{2}}\|v\|_{1,S^-}, \tag{109}$$

where $\sigma\ge 1$ and $\tau\ge 1$ are two constants independent of L and N. Suppose that α and β are bounded. When P_c is chosen large enough but still independent of L and N. The the uniformly $V_{L,N}$-inequality (103) holds.

Proof: We have

$$B_{PH}(v,v)=\iint_{S^+}|\nabla v|^2+\iint_{S^-}|\nabla v|^2 - \int_{\Gamma_0}\left(\alpha\frac{\partial v^+}{\partial n}+\beta\frac{\partial v^-}{\partial n}\right)(v^+-v^-)+P_c(L^\sigma+N^\tau)\int_{\Gamma_0}(v^+-v^-)^2. \tag{110}$$

There exists a positive constant C_0 independent of L and N such that

$$C_0^2(\|v\|^2_{1,S^+}+\|v\|^2_{1,S^-})\le |v|^2_{1,S^+}+|v|^2_{1,S^-}. \tag{111}$$

Moreover, from Eq. (108) we have

$$\left|\int_{\Gamma_0}\frac{\partial v^+}{\partial n}(v^+-v^-)\right|\le C\left\|\frac{\partial v^+}{\partial n}\right\|_{0,\Gamma_0}\|v^+-v^-\|_{0,\Gamma_0}\le CL^{\frac{\sigma}{2}}\|v^+\|_{1,S^+}\|v^+-v^-\|_{0,\Gamma_0}. \tag{112}$$

From $2xy\le x^2+y^2$ we obtain

$$\left|\int_{\Gamma_0}\frac{\partial v^+}{\partial n}(v^+-v^-)\right|\le \frac{C_0^2}{2}\|v\|^2_{1,S^+}+\frac{C^2L^\sigma}{2C_0^2}\|v^+-v^-\|^2_{0,\Gamma_0}. \tag{113}$$

Similarly, from Eq. (109) we have

$$\left|\int_{\Gamma_0}\frac{\partial v^-}{\partial n}(v^+-v^-)\right|\le \frac{C_0^2}{2}\|v\|^2_{1,S^-}+\frac{C^2N^\tau}{2C_0^2}\|v^+-v^-\|^2_{0,\Gamma_0}. \tag{114}$$

Hence we obtain from Eqs. (110)–(114) and $0\le\alpha,\beta\le 1$

$$\begin{aligned} B_{PH}(v,v) &\ge C_0^2\left(\|v\|^2_{1,S^+}+\|v\|^2_{1,S^-}\right)+P_c(L^\sigma+N^\tau)\|v^+-v^-\|^2_{0,\Gamma_0} \\ &\quad -\left|\int_{\Gamma_0}\left(\alpha\frac{\partial v^+}{\partial n}+\beta\frac{\partial v^-}{\partial n}\right)(v^+-v^-)\right| \\ &\ge \frac{C_0^2}{2}\left(\|v\|^2_{1,S^+}+\|v\|^2_{1,S^-}\right)+\left(P_c-\frac{C^2}{2C_0^2}\right)(L^\sigma+N^\tau)\|v^+-v^-\|^2_{0,\Gamma_0}. \end{aligned}$$

If we choose P_c to be large enough such that $P_c - \frac{C^2}{2C_0^2} \geq \frac{P_c}{2}$, i.e. $P_c \geq \frac{C^2}{C_0^2}$. Then we obtain

$$B_{PH}(v,v) \geq \frac{C_0^2}{2}\left(\|v\|_{1,S^+}^2 + \|v\|_{1,S^-}^2\right) + \frac{P_c}{2}(L^\sigma + N^\tau)\|v^+ - v^-\|_{0,\Gamma_0}^2 \geq \bar{C}_0\|v\|_p^2 ,$$

where $\bar{C}_0 = \min\left\{\frac{C_0^2}{2}, \frac{1}{2}\right\}$. This completes the proof of Lemma 6. ∎

Next, consider the integration approximation. The penalty plus hybrid techniques involving integration approximation is expressed by: To seek $\tilde{u}_{L,N}^* \in V_{L,N}$ such that

$$\widehat{B}_{PH}(u_{L,N}^*, v) = 0, \quad \forall v \in V_{L,N}\,, \tag{115}$$

where

$$\widehat{B}_{PH}(u,v) = \widehat{\iint_{S^+}} \nabla u \nabla v + \widehat{\iint_{S^-}} \nabla u \nabla v + \widehat{I}_{PH}^{(\alpha,\beta,P_c)}(u,v)$$

where

$$\widehat{I}_{PH}^{(\alpha,\beta,P_c)}(u,v) = -\widehat{\int}_{\Gamma_0}\left(\alpha\frac{\partial u^+}{\partial n} + \beta\frac{\partial u^-}{\partial n}\right)(v^+ - v^-) - \widehat{\int}_{\Gamma_0}\left(\alpha\frac{\partial v^+}{\partial n} + \beta\frac{\partial v^-}{\partial n}\right)(u^+ - u^-)$$
$$+ P_c(L^\sigma + N^\tau)\widehat{\int}_{\Gamma_0}(u^+ - u^-)(v^+ - v^-),$$

where $\widehat{\int}_{\Gamma_0}$ is the approximation of $\int_{\Gamma_0}$ by some rules.

In computation, define the energy

$$T_{PH}^{(\alpha,\beta,P_c)}(v,v) = \frac{1}{2}\widehat{\int}_{\Gamma_0}\frac{\partial v^+}{\partial n}v^+ + \frac{1}{2}\widehat{\int}_{\Gamma_0}\frac{\partial v^-}{\partial n}v^- - \widehat{\int}_{\Gamma_0}\left(\alpha\frac{\partial v^+}{\partial n} + \beta\frac{\partial v^-}{\partial n}\right)(v^+ - v^-)$$
$$+ \frac{P_c}{2}(L^\sigma + N^\tau)\widehat{\int}_{\Gamma_0}(v^+ - v^-)^2.$$

The minimum of $T_{PH}^{(\alpha,\beta,P_c)}(v,v)$,

$$\frac{\partial T_{PH}^{(\alpha,\beta,P_c)}(v,v)}{\partial v} = 0,$$

yields Eq. (115), which can be expressed in the matrix and vectors

$$\mathrm{A}\vec{X} = \vec{b},$$

where $\vec{X} = \{a_1, \ldots, a_L, b_1, \ldots b_M\}^T$, $\vec{b}$ is a known vector, and the matrix $\mathbf{A}$ is positive definite and symmetric.

Suppose the following inequalities hold,

$$\widehat{B}_{PH}(u,v) \leq C_1\|u\|_p\|v\|_p\,, \quad \forall v \in V_{L,N}\,, \tag{116}$$

$$C_0\|v\|_p^2 \quad \leq \widehat{B}_{PH}(v,v), \qquad \forall v \in V_{L,N}\,, \tag{117}$$

where C_0 and C_1 are two positive constants independent of L and N. We have the following theorem.

Theorem 4. *Let Eqs. (116) and (117) hold. Suppose that $\alpha+\beta=1$. Then the solutions $\tilde{u}^*_{L,N}$ from Eq. (115) have the following error bound,*

$$\|u-\tilde{u}^*_{L,N}\|_p \le C\Big\{ \inf_{v\in V_{L,N}} \|u-v\|_p + \sup_{w\in V_{L,N}} \frac{1}{\|w\|_p}\Big[\Big|\Big(\int_{\Gamma_0}-\widehat{\int}_{\Gamma_0}\Big)\frac{\partial u}{\partial n}w^+\Big| + \Big|\Big(\int_{\Gamma_0}-\widehat{\int}_{\Gamma_0}\Big)\frac{\partial u}{\partial n}w^-\Big| + \Big|\Big(\int_{\Gamma_0}-\widehat{\int}_{\Gamma_0}\Big)\frac{\partial u}{\partial n}(w^+-w^-)\Big|\Big]\Big\}. \tag{118}$$

Proof: For the true solution u, we have

$$\widehat{B}_{PH}(u,v) = \Big(\widehat{\iint}_{S^+} - \iint_{S^+}\Big)\nabla u\nabla v + \Big(\widehat{\iint}_{S^-} - \iint_{S^-}\Big)\nabla u\nabla v - \Big(\widehat{\int}_{\Gamma_0} - \int_{\Gamma_0}\Big)\frac{\partial u}{\partial n}(v^+-v^-).$$

Hence we have

$$\widehat{B}_{PH}(u-\tilde{u}^*_{L,N},v) = \Big(\widehat{\iint}_{S^+} - \iint_{S^+}\Big)\nabla u\nabla v + \Big(\widehat{\iint}_{S^-} - \iint_{S^-}\Big)\nabla u\nabla v - \Big(\widehat{\int}_{\Gamma_0} - \int_{\Gamma_0}\Big)\frac{\partial u}{\partial n}(v^+-v^-).$$

The rest of proof is similar to that in Theorem 2. ∎

To close this section, we explore the relation between the hybrid techniques in Section 3 and the penalty plus hybrid techniques in this section. We have the following lemma.

Lemma 7. *Let $\alpha=1$ and $\beta=0$. The simplified hybrid method in Section 3 is equivalent to the special case of $P_c=0$ in Eq. (98).*

Proof: For $\alpha=1$ and $\beta=0$, the simplified hybrid method in Section 3

$$A^*_{Hyb}(u_{L,N},v)=0, \quad \forall v\in V_{L,N}, \tag{119}$$

where

$$A^*_{Hyb}(u,v) = \iint_{S^+}\nabla u\nabla v + \iint_{S^-}\nabla u\nabla v + \int_{\Gamma_0}\frac{\partial u^+}{\partial n}v^- - \int_{\Gamma_0}\frac{\partial v^+}{\partial n}u^-.$$

On the other hand, when $P_c=0$, $\alpha=1$ and $\beta=0$, Eq. (8) leads to

$$B_{PH}(u^*_{L,N},v)=0, \quad \forall v\in V_{L,N}, \tag{120}$$

where

$$B_{PH}(u,v) = \iint_{S^+}\nabla u\nabla v + \iint_{S^-}\nabla u\nabla v - \int_{\Gamma_0}\frac{\partial u^+}{\partial n}(v^+-v^-) - \int_{\Gamma_0}\frac{\partial v^+}{\partial n}(u^+-u^-). \tag{121}$$

Since v^+ and v^- are independent to each other, we have from Eq. (119)

$$\iint_{S^+}\nabla u^+_L\nabla v^+ - \int_{\Gamma_0}\frac{\partial v^+}{\partial n}u^-_N = 0, \tag{122}$$

$$\iint_{S^-}\nabla u^-_N\nabla v^- + \int_{\Gamma_0}\frac{\partial u^+_L}{\partial n}v^- = 0. \tag{123}$$

Note that the solution u_L^+ and v^+ satisfy the Laplace equation, there exist the equalities

$$\iint_{S^+} \nabla u_L^+ \nabla v^+ = \int_{\Gamma_0} \frac{\partial u_L^+}{\partial n} v^+ = \int_{\Gamma_0} \frac{\partial v^+}{\partial n} u_L^+ . \tag{124}$$

We obtain from Eqs. (120) and (124)

$$\iint_{S^-} \nabla u \nabla v + \int_{\Gamma_0} \frac{\partial u_L^+}{\partial n} v^- - \left(\int_{\Gamma_0} \frac{\partial u_L^+}{\partial n} v^+ - \int_{\Gamma_0} \frac{\partial v^+}{\partial n} u_N^- \right) = 0. \tag{125}$$

Also since v^+ and v^- are independent to each other, we have from Eq. (125)

$$\int_{\Gamma_0} \frac{\partial u_L^+}{\partial n} v^+ - \int_{\Gamma_0} \frac{\partial v^+}{\partial n} u_N^- \qquad = 0, \tag{126}$$

$$\iint_{S^-} \nabla u_N^- \nabla v^- + \int_{\Gamma_0} \frac{\partial u_L^+}{\partial n} v^- = 0. \tag{127}$$

Equation (126) is just Eq. (122) by noting Eq. (124), and Eq. (127) is exactly Eq. (123). This proves the equivalence of the two methods, (119) and (120). ∎

Similarly, the equivalence for the two methods for $\beta = 1$ and $\alpha = 0$. Two cases can be rewritten as $\alpha\beta = 0$. Note that Lemma 7 is also valid for the combinations of the Trefftz method in S^+ and the other methods such as FEM, FDM, FVM, etc. in S^-,

$$v = \begin{cases} v^+ & \text{in } S^+, \\ v_k^- & \text{in } S^-, \end{cases} \tag{128}$$

where v_k^- are the piecewise $k-$order polynomials, because the key equalities (124) of particular solutions hold true for the Trefftz method in S^+. It is worthy noting that when $\alpha = \beta = \frac{1}{2}$, the simplified hybrid-Trefftz method is not equivalent to the special case of $P_c = 0$ and $\alpha = \beta = \frac{1}{2}$ of the penalty plus hybrid method.

Remark 3. For the CTM and the penalty plus hybrid TM, there are no such limitations as in Remark 1, because the following lemma cites from [37], p. 105.

Lemma 8. *For the Laplace equation (18) with the mixed type of boundary conditions, let S be split by Γ_0 into S^+ and S^-. Suppose that $\Gamma_D \neq \emptyset$. Then there exists a constant C independent of u and v such that*

$$\|v\|_1 \leq C \left\{ |v|_1 + |v|_{0,\Gamma_D} + |v^+ - v^-|_{0,\Gamma_0} \right\}, \quad \forall v \in V_{L,N} . \tag{129}$$

By means of $Pc > 0$, Eq. (103) holds. When $S = \cup_i S_i$, an interior subdomain S_i may be allowed, and the constant solution is also permitted into piecewise particular solutions. Besides, the Lagrange multiplier TM in the next section shares the limitations as in the simplified hybrid TM as in Remark 1. But a remedy is given in Remark 2.

5. Lagrange Multiplier Techniques

The Lagrange multiplier method was first introduced by Babuška [2] to treat the constraint Dirichlet boundary condition as a natural boundary condition, and to relax the limitation on the admissible functions used. Such techniques have been adopted to mixed and hybrid methods, see Brezzi and Fortin [7] and Raviart and Thomas [64]. The boundary condition using Lagrange multipliers is also extended to that involving flux in [6]. More analysis and applications are given in [5, 11, 34, 59, 60], and particularly for domain decomposition methods in [54, 55, 57].

In this section we adopt Lagrange multiplier to couple the different particular solutions, called the Lagrange multiplier Trefftz method, see [30, 33]. Let us consider the Debye–Huckel equation

$$\mathcal{L}u = \Delta u + u = 0 \quad \text{in } S. \tag{130}$$

Suppose that Φ_i and Ψ_i satisfy Eq. (130) in S^+ and S^- respectively. We also choose

$$v = \begin{cases} v^+ = \sum_{i=0}^{M} a_i \Phi_i & \text{in } S^+, \\ v^- = \sum_{i=0}^{N} b_i \Psi_i & \text{in } S^-, \end{cases} \tag{131}$$

where a_i and b_i are the coefficients to be sought. We have for the true solution u

$$\iint_{S^+} (\nabla u \nabla v + uv) + \iint_{S^-} (\nabla u \nabla v + uv) - \int_{\Gamma_0} \lambda (v^+ - v^-) = 0. \tag{132}$$

Denote $\lambda = \frac{\partial u}{\partial n}$ as a new variable. Let λ_L be the $L-$order polynomials on Γ_0, for example,

$$\lambda_L = \sum_{i=0}^{L} d_i T_i \,, \tag{133}$$

where T_i are the orthogonal polynomials of order i. Denote by $V_{M,N} \times V_L$ the collection of finite dimensions of Eqs. (131) and (133). We may design the Lagrange multiplier method: To seek $(u_{M,N}\,, \lambda_L) \in V_{M,N} \times V_L$ such that

$$A_{Lag}(u^{\#}_{M,N}\,, v) + D_{Lag}(u^{\#}_{M,N}\,, \lambda_L\,; v, \mu) = 0, \quad \forall (v, \mu) \in V_{M,N} \times V_L\,, \tag{134}$$

where

$$A_{Lag}(u, v) = \iint_{S^+} \nabla u \nabla v + \iint_{S^-} \nabla u \nabla v, \tag{135}$$

$$D_{Lag}(u, \lambda; v, \mu) = -\int_{\Gamma_0} \lambda (v^+ - v^-) - \int_{\Gamma_0} \mu (u^+ - u^-). \tag{136}$$

Note that the direct Trefftz method called in engineering is just the Lagrange multiplier Trefftz method.

Define the error norms

$$\|(v, \mu)\|_H = \left(\|v\|_1 + \|\mu\|_{-\frac{1}{2}, \Gamma_0} \right)^{1/2}, \qquad \|v\|_1 = \left(\|v\|^2_{1,S_1} + \|v\|^2_{1,S_2} \right)^{1/2},$$

$$\|\mu\|_{-\frac{1}{2}, \Gamma_0} = \sup_v \frac{\left| \int_{\Gamma_0} \mu (v^+ - v^-) \, d\ell \right|}{|v^+|_{\frac{1}{2}, \Gamma_0} + |v^-|_{\frac{1}{2}, \Gamma_0}},$$

$$\|v\|_{\frac{1}{2}, \Gamma_0} = \left(\int_{\Gamma_0} \int_{\Gamma_0} \frac{(v(P) - v(Q))^2}{(P - Q)^2} \, d\ell(Q) \, d\ell(P) + \frac{1}{2} (d_1^{-1} + d_2^{-1}) \|v\|^2_{0, \Gamma_0} \right)^{\frac{1}{2}},$$

where d_i is roughly the diameter of S_i.

We need the following assumptions.

(A1) For $A_{Lag}(u,v)$ there exist the bounds

$$C_0\|v\|_1^2 \le A_{Lag}(v,v), \qquad |A_{Lag}(u,v)| \le C_1\,\|u\|_1\,\|v\|_1, \quad \forall v \in V_{M,N}\,.$$

(A2) For $\int_{\Gamma_0} \mu(v^+ - v^-)\,\mathrm{d}\ell$, the Ladyzhenskaya–Babuška–Brezzi (LBB) condition holds: For all $\mu_L \in V_L$, there exists $v \in V_{M,N}$, $v \neq 0$ such that

$$\left|\int_{\Gamma_0} \mu_L(v^+ - v^-)\,\mathrm{d}\ell\right| \ge \beta\,\|v\|_1\,\|\mu_L\|_{-\frac{1}{2},\Gamma_0}. \tag{137}$$

(A3) Also the following bounds hold

$$\left|\int_{\Gamma_0} \lambda(v^+ - v^-)\,\mathrm{d}\ell\right| \le C\,\|\lambda\|_{-\frac{1}{2},\Gamma_0}\,\|v\|_1\,, \quad \forall v \in V_{M,N}\,. \tag{138}$$

Now we cite the following theorem and Lemma from [37].

Theorem 5. *Let* **(A1)**–**(A3)** *hold. There exist the error bounds,*

$$\|\lambda - \lambda_L\|_{-\frac{1}{2},\Gamma_0} + \|u - u_{M,N}\|_1 \le C\left\{\inf_{v\in V_{M,N}} \|u - v\|_1 + \inf_{\eta\in V_L} \|\lambda - \eta\|_{-\frac{1}{2},\Gamma_0}\right\}. \tag{139}$$

Lemma 9. *There exists a constant $\beta(>0)$ independent of u and v such that $\forall\mu \in H^{-\frac{1}{2}}(\Gamma_0)$, $\exists v \in H^1(S)$, $v \neq 0$ such that*

$$\int_{\Gamma_0} \mu(v^+ - v^-)\,\mathrm{d}\ell \ge \beta\,\|\mu\|_{-\frac{1}{2},\Gamma_0}\,\|v\|_1\,. \tag{140}$$

Below we prove a new lemma.

Lemma 10. *For all $\mu_L \in V_L$ there exists $v \in V_{M,N}$, where $v = v_{M,N} + r_{M,N}$, i.e.,*

$$v^+ = v_M + r_M\,, \qquad v^- = v_N + r_N\,. \tag{141}$$

Suppose that the following bound exists,

$$\frac{\|r_{M,N}\|_1}{\|v_{M,N}\|_1} = o(1). \tag{142}$$

Then the LBB condition holds: For all $\mu_L \in V_L$ there exists $v \in V_{M,N}$ such that

$$\int_{\Gamma_0} \mu_L(v^+ - v^-)\,\mathrm{d}\ell \ge \beta\,\|\mu_L\|_{-\frac{1}{2},\Gamma_0}\,\|v\|_1\,, \tag{143}$$

where β is a positive constant independent of μ_L and v.

Proof: We have from **(A2)** and Eq. (141)

$$\begin{aligned}\int_{\Gamma_0} \mu_L(v_M^+ - v_N^-)\,\mathrm{d}\ell &= \int_{\Gamma_0} \mu_L(v^+ - v^-)\,\mathrm{d}\ell - \int_{\Gamma_0} \mu_L(r_M^+ - r_N^-)\,\mathrm{d}\ell \\ &\ge \beta\|\mu_L\|_{-\frac{1}{2},\Gamma_0}\|v\|_1 - C\|\mu_L\|_{-\frac{1}{2},\Gamma_0}\,\|r_{M,N}\|_1 \\ &\ge \beta\,\|\mu_L\|_{-\frac{1}{2},\Gamma_0}\,\|v_{M,N}\|_1 - (\beta + C)\|\mu_L\|_{-\frac{1}{2},\Gamma_0}\,\|r_{M,N}\|_1\,. \end{aligned} \tag{144}$$

When Eq. (142) holds, we have

$$\int_{\Gamma_0} \mu_L(v_M^+ - v_N^-)\,d\ell \geq (\beta - (\beta + C)o(1))\,\|\mu_L\|_{-\frac{1}{2},\Gamma_0}\|v_{M,N}\|_1 \geq \frac{\beta}{2}\|\mu_L\|_{-\frac{1}{2},\Gamma_0}\|v_{M,N}\|_1 . \quad (145)$$

This completes the proof of Lemma 10. ∎

The error analysis of Lagrange multiplier to couple the Dirichlet condition also is given in Babuška [2], Babuška *et al.* [3] and Pitkaranta [59].

Remark 4. In Eq. (134), the Lagrange multiplier is applied to couple the interior continuity condition $u^+ = u^-$ on Γ_0. If the Lagrange multiplier is applied to couple the interior flux condition, $\frac{\partial u^+}{\partial n} = \frac{\partial u^-}{\partial n}$ on Γ_0, we obtain the hybrid-Trefftz method (16). When the three conditions similar to **(A1)**–**(A3)** are satisfied, there exists the error bound

$$\|\lambda - \lambda_h\|_{\frac{1}{2},\Gamma_0} + \|u - u_{M,N}\|_1 \leq C\left\{\inf_{v\in V_{M,N}} \|u - v\|_1 + \inf_{\eta\in V_h} \|\lambda - \eta\|_{\frac{1}{2},\Gamma_0}\right\}.$$

This proof is given in [39].

6. Numerical Experiments

In this section, first we consider Motz's problem (see Fig. 2)

$$\begin{aligned} &\triangle u = 0 \quad \text{in } S,\\ &u = 0 \quad \text{on } \overline{OD}, \qquad u = 500 \quad \text{on } \overline{AB},\\ &\frac{\partial u}{\partial n} = 0 \quad \text{on } \overline{BC} \cup \overline{CD} \cup \overline{OA}, \end{aligned} \quad (146)$$

where $S = \{(x, y) | -1 < x < 1, 0 < y < 1\}$. The admissible functions are found as in [37]

$$v = \sum_{i=0}^{L} d_i r^{i+\frac{1}{2}} \cos\left(i + \frac{1}{2}\right)\theta \quad \text{in } S. \quad (147)$$

We may use the CTM and simplified hybrid TM involving integration approximation, to investigate the convergence rates. In Li, *et al.* [50], for the entire domain S, the uniform particular solutions (147) are chosen, and the numerical experiments have been carried out by four TMs: the CTM, the original (i.e. the simplified hybrid) TM, the penalty plus hybrid TM and the direct TM. Since the uniform particular solutions may not be always found, using the piecewise particular solutions is important for the TMs to solve general PDEs.

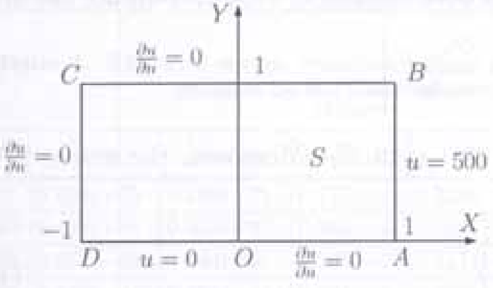

Fig. 2. Motz's problem

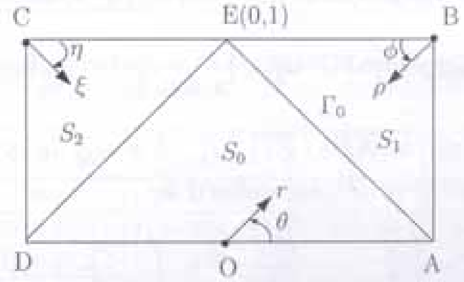

Fig. 3. Partition of the rectangle

We consider the partition in Fig. 3, where the solution domain S is divided into three subdomains, S_0, S_1 and S_2. For Eq. (146) the piecewise particular solutions are expressed as

$$v_0 = \sum_{i=0}^{L} \bar{D}_i r^{i+\frac{1}{2}} \cos\left(i+\frac{1}{2}\right)\theta \quad \text{in } S_0, \tag{148}$$

$$v_1 = 500 + \sum_{i=0}^{M} \bar{A}_i \rho^{2i+1} \cos(2i+1)\phi \quad \text{in } S_1, \tag{149}$$

$$v_2 = b_0 + \sum_{i=1}^{N} \bar{B}_i \xi^{2i} \cos 2i\eta \quad \text{in } S_2, \tag{150}$$

where $\bar{D}_i$, $\bar{A}_i$ and $\bar{B}_i$ are the coefficients to be sought, (r,θ), (ρ,ϕ) and (ξ,η) are the polar coordinates shown in Fig. 3.

First we use the CTM from Eq. (3)

$$v^+(P_i) = v^-(P_i), \qquad w\frac{\partial v^+}{\partial n}(P_i) = w\frac{\partial v^-}{\partial n}(P_i), \qquad P_i \in \overline{AE}, \tag{151}$$

$$v^+(Q_i) = v^-(Q_i), \qquad w\frac{\partial v^+}{\partial n}(Q_i) = w\frac{\partial v^-}{\partial n}(Q_i), \qquad Q_i \in \overline{DF}, \tag{152}$$

where w is a suitable positive weight. The linear algebraic equations can be obtained from Eqs. (151) and (152),

$$\mathbf{F}\mathbf{x} = \mathbf{b}, \tag{153}$$

where $\mathbf{x}$ is the unknown vector consisting of the coefficients $\bar{D}_i$, $\bar{A}_i$ and $\bar{B}_i$, $\mathbf{b}$ is the known vector, and $\mathbf{A}$ is the stiffness matrix. Denote by $\bar{M}$ the number of the integration nodes along $\overline{AE}$, then the dimensions of matrix $\mathbf{F}$ are $m \times n$, where $m = 4\bar{M}$ and $n = L+M+N+3$. In computation, we always choose $m > n$, then Eq. (153) is an over-determined system, and the least squares method is used to seek $\mathbf{x}$.

Equations (151) and (152) can be interpreted as the boundary approximation method (BAM) in [37], and the CTM in [56] and this paper. Denote by $V_{L,M,N}(= V_L \times V_M \times V_N)$ the finite dimensional collection of functions (148)–(150). Then the CTM is designed to seek the solution $u_{L,M,N} \in V_{L,M,N}$ such that

$$I(u_{L,M,N}) = \min_{v \in V_{L,M,N}} I(v), \tag{154}$$

where

$$I(v) = \widehat{\int_{\Gamma_0}} (v^+ - v^-)^2 + w^2 \widehat{\int_{\Gamma_0}} \left(\frac{\partial v^+}{\partial n} - \frac{\partial v^-}{\partial n}\right)^2. \tag{155}$$

In Eq. (155), $\Gamma_0 = \overline{AE} \cup \overline{ED}$, $v^+ = v_0$ in S_0, $v^- = v_1$ in S_1 and $v^- = v_2$ in S_2. The $\widehat{\int_{\Gamma_0}}$ is the approximation of $\int_{\Gamma_0}$ by some rule of integration. The weight w in Eq. (155) can be obtained, based on the analysis in [44, 52]. After the solution $u_{L,M,N}$ has been obtained, we will compute the errors

$$|\varepsilon|_{\infty,\Gamma_0} = \left|u_L^+ - u_{M,N}^-\right|_{0,\infty,\Gamma_0}, \qquad |\varepsilon_n|_{\infty,\Gamma_0} = \left|\frac{\partial u_L^+}{\partial n} - \frac{\partial u_{M,N}^-}{\partial n}\right|_{0,\infty,\Gamma_0},$$

where $\Gamma_0 = \overline{AE} \cup \overline{ED}$, $u_{M,N}^- = u_M$ in S_1 and $u_{M,N}^- = u_N$ in S_2. Moreover, the errors in the semi-norm of H^1 are defined by

$$|\varepsilon|_1 = \left\{\int_{\overline{AE}\cup\overline{DE}} \frac{\partial \varepsilon^+}{\partial n}\varepsilon^+ + \int_{\overline{AE}} \frac{\partial \varepsilon^-}{\partial n}\varepsilon^- + \int_{\overline{DE}} \frac{\partial \varepsilon^-}{\partial n}\varepsilon^-\right\}^{\frac{1}{2}}, \tag{156}$$

where $\varepsilon = u - u_{L,M,N}$.

Since the accuracy of the leading coefficients is important in applications, we also compute the relative errors

$$\left|\frac{\Delta D_i}{D_i}\right|, \quad i = 0, 1, 2, 3, \tag{157}$$

where $\Delta D_i = D_i - \tilde{D}_i$, and D_i are the true coefficients given in [53]. The condition number is defined by

$$\mathrm{Cond} = \frac{\max_i \lambda_i(\mathbf{F}^T\mathbf{F})}{\min_i \lambda_i(\mathbf{F}^T\mathbf{F})}, \tag{158}$$

where $\lambda_i(\mathbf{F}^T\mathbf{F})$ are the eigenvalues of the matrix $\mathbf{F}^T\mathbf{F}$.

In Lu *et al.* [56], the CTM is used for Motz's problems by using the uniform particular solutions (147), and the Gaussian rule of six nodes are used to raise the accuracy of the leading coefficients $\tilde{D}_i$. Hence, in this paper, we always choose the Gaussian rule of six nodes. The solutions $u_{L,M,N}$ are obtained, and the errors and the condition number are listed in Tables 1–3. Note that when

$$L = 34, \quad M = N = 26, \quad \bar{M} = 36, \tag{159}$$

Table 1. The error norms, condition number and errors of leading coefficients from the CTM for Motz's problem by the Gaussian rule of six nodes rule as $L = 34$, $M = N$ and $\bar{M} = 36$

M	$\|\varepsilon\|_B$	$\|\varepsilon\|_{\infty,\Gamma_0}$	$\|\varepsilon_n\|_{\infty,\Gamma_0}$	$\|\varepsilon\|_1$	$\|\frac{\Delta D_0}{D_0}\|$	$\|\frac{\Delta D_1}{D_1}\|$	$\|\frac{\Delta D_2}{D_2}\|$	$\|\frac{\Delta D_3}{D_3}\|$	Cond
10	0.305(−4)	0.523(−4)	0.153(−2)	0.371(−3)	0.490(−11)	0.781(−10)	0.208(−8)	0.438(−7)	0.219(4)
12	0.298(−5)	0.525(−5)	0.155(−3)	0.285(−4)	0.334(−13)	0.489(−12)	0.252(−10)	0.617(−9)	0.220(4)
14	0.300(−6)	0.480(−6)	0.175(−4)	0.420(−5)	0.893(−14)	0.180(−13)	0.196(−12)	0.832(−11)	0.224(4)
16	0.306(−7)	0.399(−7)	0.178(−6)	0.427(−6)	0.112(−13)	0.613(−13)	0.118(−12)	0.220(−12)	0.236(4)
18	0.286(−8)	0.251(−8)	0.197(−6)	0.454(−7)	0.822(−14)	0.331(−13)	0.736(−13)	0.920(−13)	0.574(4)
20	0.228(−9)	0.315(−9)	0.302(−7)	0.572(−8)	0.439(−14)	0.144(−13)	0.196(−13)	0.623(−13)	0.152(5)
22	0.158(−10)	0.566(−10)	0.678(−8)	0.112(−8)	0.298(−14)	0.600(−14)	0.173(−13)	0.156(−13)	0.446(5)
24	0.122(−11)	0.158(−10)	0.187(−8)	0.256(−9)	0.425(−15)	0.162(−15)	0.948(−14)	0.176(−14)	0.146(6)
26	0.420(−12)	0.401(−11)	0.454(−9)	0.498(−10)	0*	0.648(−15)	0.412(−14)	0.462(−14)	0.208(6)
28	0.370(−11)	0.139(−10)	0.100(−8)	0.291(−9)	0.850(−15)	0.211(−14)	0.247(−14)	0.352(−14)	0.104(7)
30	0.281(−11)	0.728(−11)	0.111(−8)	0.275(−9)	0.850(−15)	0.373(−14)	0.227(−14)	0.169(−13)	0.354(7)

Table 2. The error norms, condition number and errors of leading coefficients from the CTM for Motz's problem by the Gaussian rule of six nodes rule as $L = 34$ and $M = N = 26$

$\bar{M}$	$\|\varepsilon\|_B$	$\|\varepsilon\|_{\infty,\Gamma_0}$	$\|\varepsilon_n\|_{\infty,\Gamma_0}$	$\|\varepsilon\|_1$	$\|\frac{\Delta D_0}{D_0}\|$	$\|\frac{\Delta D_1}{D_1}\|$	$\|\frac{\Delta D_2}{D_2}\|$	$\|\frac{\Delta D_3}{D_3}\|$	Cond
24	0.137(−10)	0.523(−10)	0.335(−8)	0.143(−8)	0.171(−13)	0.708(−13)	0.148(−12)	0.314(−12)	0.760(6)
30	0.857(−12)	0.659(−11)	0.746(−9)	0.865(−10)	0.567(−15)	0.195(−14)	0.907(−14)	0.141(−13)	0.623(6)
36	0.420(−12)	0.401(−11)	0.454(−9)	0.498(−12)	0*	0.648(−15)	0.412(−14)	0.642(−14)	0.208(6)
42	0.455(−12)	0.131(−11)	0.149(−9)	0.209(−10)	0.850(−15)	0.130(−14)	0.115(−13)	0.242(−13)	0.344(6)
48	0.455(−12)	0.824(−12)	0.644(−10)	0.970(−11)	0.142(−15)	0.243(−14)	0.103(−13)	0.123(−13)	0.278(6)

Table 3. The error norms, condition number and errors of leading coefficients from the CTM for Motz's problem by the Gaussian rule of six nodes rule as $\bar{M} = 36$ and $M = N$

L	M	$\|\varepsilon\|_B$	$\|\varepsilon\|_{\infty,\Gamma_0}$	$\|\varepsilon_n\|_{\infty,\Gamma_0}$	$\|\varepsilon\|_1$	$\|\frac{\Delta D_0}{D_0}\|$	$\|\frac{\Delta D_1}{D_1}\|$	$\|\frac{\Delta D_2}{D_2}\|$	$\|\frac{\Delta D_3}{D_3}\|$	Cond
10	8	0.824(−3)	0.106(−2)	0.171(−1)	0.822(−2)	0.546(−8)	0.191(−7)	0.373(−8)	0.625(−4)	49.2
18	14	0.437(−6)	0.430(−6)	0.352(−4)	0.669(−5)	0.506(−13)	0.452(−12)	0.426(−11)	0.191(−8)	828
26	20	0.271(−9)	0.444(−9)	0.245(−7)	0.711(−8)	0.425(−14)	0.162(−13)	0.192(−13)	0.224(−13)	0.150(5)
34	26	0.420(−12)	0.401(−11)	0.454(−9)	0.498(−10)	0*	0.648(−15)	0.412(−14)	0.462(−14)	0.208(6)
40	30	0.281(−11)	0.171(−10)	0.860(−9)	0.182(−9)	0.128(−14)	0.308(−14)	0.453(−13)	0.143(−13)	0.424(7)

Table 4. The leading coefficients $\bar{D}_i$ by the collation TM for Motz's problem at $L = 34$ and $M = N = 26$ by the Gaussian rule of six nodes with $\bar{M} = 36$ along $\overline{AB}$

i	$\bar{D}_i$	i	$\bar{D}_i$
0	.401162453745234416(3)	18	.115352471443778851(−4)
1	.876559201950879725(2)	19	−.529572415268499927(−5)
2	.172379150794468821(2)	20	.229123737707347517(−5)
3	−.807121525969817100(1)	21	.106323020872833201(−5)
4	.144027271702286663(1)	22	.531249576038153798(−6)
5	.331054885920768371	23	−.247431076339611956(−6)
6	.275437344509163740	24	.109928975890681925(−6)
7	−.869329945256786807(−1)	25	.516695735614091014(−7)
8	.336048784266203271(−1)	26	.257457708954766480(−7)
9	.153843744826868429(−1)	27	−.120317413747487121(−7)
10	.730230167385251521(−2)	28	.540803145614940174(−8)
11	−.318411391615677731(−2)	29	.260150547393998650(−8)
12	.122064610942808609(−2)	30	.131774021431445718(−8)
13	.530965480203339111(−3)	31	−.716717640212716954(−9)
14	.271512182358950110(−3)	32	.340026463417246646(−9)
15	−.120046375387525815(−3)	33	.145637472345680654(−9)
16	.505398334499953102(−4)	34	.646319941944504533(−10)
17	.231668222198537339(−4)		

the leading coefficients are listed in Tables 4 and 5, where the coefficient $\bar{D}_0$ has 17 significant digits, the same accuracy of $\bar{D}_0$ in [56]. For the case of Eq. (159) the errors and condition number are given by

$$|\varepsilon|_1 = 0.498(-10), \qquad \text{Cond} = 0.208(6),$$

$$|\varepsilon|_1 = 0.175(-6), \qquad \text{Cond} = 0.786(6),$$

from Table 1 and [56] respectively. Obviously, by using the piecewise particular solutions, the accuracy of the solutions from the CTM will be improved, while the condition number retain almost the same. This is also a new discovery from [37, 52].

Next, consider the simplified hybrid TM. First for the partition of Fig. 3, condition (42) is satisfied by noting that $\partial S_2 \cap \Gamma_D \neq \emptyset$ due to

$$u|_D = u(\xi,\eta)|_{\xi=1,\eta=\frac{\pi}{2}} = 0 \qquad \text{in } S_2\,. \tag{160}$$

Hence the particular solutions v_2 in Eq. (150) satisfying (160) lead to an additional condition

$$\bar{B}_0 + \sum_{i=1}^{N} \bar{B}_i \cos(i\pi) = \bar{B}_0 + \sum_{i=1}^{N} (-1)^i \bar{B}_i = 0.$$

This gives

$$\bar{B}_0 = \sum_{i=1}^{N} (-1)^{i+1} \bar{B}_i\,. \tag{161}$$

By removing the coefficient $\bar{B}_0$ from Eq. (150), we modify the particular solutions in S_2 as,

$$v_2^* = \sum_{i=1}^{M} \bar{B}_i \phi_i(\xi,\eta) \qquad \text{in } S_2\,, \tag{162}$$

Table 5. The leading coefficients $\bar{A}_i$ and $\bar{B}_i$ by the collation TM for Motz's problem at $L = 34$ and $M = N = 26$ by the Gaussian rule of six nodes with $\bar{M} = 36$ along $\overline{AB}$

i	$\bar{A}_i$	$\bar{B}_i$
0	−.324796539532479756(3)	.913597470506873464(2)
1	.232607975243312204(2)	.106454402511134788(3)
2	.749635805883399087(1)	.114358406001622779(2)
3	−.105838719294348760(1)	−.475017155047903383(1)
4	−.797801480899109294	−.712640969353020681
5	.133844127561235676	.522088231291717109
6	.115978763171984939	.889809563419886873(−1)
7	−.206752186506053218(−1)	−.773689919897273398(−1)
8	−.194544291456736733(−1)	−.139185098079888867(−1)
9	.358689091375572920(−2)	.131391643034147309(−1)
10	.354875859185069991(−2)	.243632356556054957(−2)
11	−.668408209585087893(−3)	−.241627546046592912(−2)
12	−.683773705768343765(−3)	−.456865396633695327(−3)
13	.130697557610920205(−3)	.468227837388299315(−3)
14	.136926484810349767(−3)	.897479231523365764(−4)
15	−.264552436403060604(−4)	−.941587491224513969(−4)
16	−.282148351496040590(−4)	−.182308972521741703(−4)
17	.549510252114114052(−5)	.194624182351141110(−4)
18	.594196246642652178(−5)	.379789786965780228(−5)
19	−.116159250071523603(−5)	−.410078867091007969(−5)
20	−.126951842989415352(−5)	−.805726533744082841(−6)
21	.244038929588239169(−6)	.861952521981058757(−6)
22	.268711101149843939(−6)	.170880067660462860(−6)
23	−.462051393846600188(−7)	−.164639262402756537(−6)
24	−.510670724598834558(−7)	−.331977632058908638(−7)
25	.564846976060578345(−8)	.205003005242903636(−7)
26	.624656884317372375(−8)	.424996984523008565(−8)

where the particular solutions are

$$\phi_i(\xi, \eta) = (-1)^{i+1} + \xi^{2i} \cos(2i\eta), \quad i = 1, 2, \dots \tag{163}$$

For the simplified hybrid TM, we choose the particular solutions

$$v_0 = \sum_{i=0}^{L} \bar{D}_i r^{i+\frac{1}{2}} \cos(i + \frac{1}{2})\theta \quad \text{in } S_0 \,, \tag{164}$$

$$v_1 = 500 + \sum_{i=0}^{M} \bar{A}_i \rho^{2i+1} \cos(2i+1)\phi \quad \text{in } S_1 \,, \tag{165}$$

$$v_2^* = \sum_{i=1}^{M} \bar{B}_i \phi_i(\xi, \eta) \quad \text{in } S_2 \,. \tag{166}$$

Since there exists a constant solution in v_1 and v_2^*, only the simplified hybrid TM in Eq. (62) can be applied for $v^+ = S_0$ and $S^- = S_1 \cup S_2$, based on Remark 1. Otherwise, the remedy in Remark 2 is needed. The computed solutions $u_{L,M,N}$ are obtained, and the error and the condition number are listed in Tables 6 and 7. When

$$L = 34, \quad M = N = 15, \quad \bar{M} = 3840, \tag{167}$$

Table 6. The condition number and errors of leading coefficients from the simplified hybrid TM for Motz's problem by the Gaussian rule of six nodes rule as $L = 34$, $M = N$ and $\bar{M} = 30$

M	$\|\varepsilon\|_{\infty,\Gamma_0}$	$\|\varepsilon_n\|_{\infty,\Gamma_0}$	$\|\varepsilon\|_1$	$\|\frac{\Delta D_0}{D_0}\|$	$\|\frac{\Delta D_1}{D_1}\|$	$\|\frac{\Delta D_2}{D_2}\|$	$\|\frac{\Delta D_3}{D_3}\|$	Cond
21	13.9	0.155(4)	0.726(3)	0.627(−4)	0.114(−3)	0.815(−3)	0.192(−3)	0.502(9)
19	0.612	64.3	38.2	0.212(−5)	0.411(−5)	0.120(−4)	0.235(−4)	0.392(8)
17	0.516(−1)	5.37	4.34	0.985(−7)	0.202(−6)	0.237(−6)	0.191(−5)	0.353(7)
15	0.516(−1)	5.26	2.45	0.306(−7)	0.317(−6)	0.954(−6)	0.744(−7)	0.199(7)
13	0.444(−1)	4.66	2.48	0.386(−7)	0.333(−6)	0.984(−6)	0.111(−6)	0.200(7)
11	0.425(−1)	3.74	2.50	0.396(−7)	0.335(−6)	0.110(−5)	0.351(−6)	0.199(7)
9	0.418(−1)	3.64	2.51	0.400(−7)	0.345(−6)	0.637(−6)	0.485(−7)	0.195(7)
7	0.414(−1)	3.58	2.52	0.389(−7)	0.362(−6)	0.225(−5)	0.133(−4)	0.189(7)

Table 7. The condition number and errors of leading coefficients from the simplified hybrid TM for Motz's problem by the Gaussian rule of six nodes rule as $L = 34$ and $M = N = 15$

$\bar{M}$	$\|\varepsilon\|_{\infty,\Gamma_0}$	$\|\varepsilon_n\|_{\infty,\Gamma_0}$	$\|\varepsilon\|_1$	$\|\frac{\Delta D_0}{D_0}\|$	$\|\frac{\Delta D_1}{D_1}\|$	$\|\frac{\Delta D_2}{D_2}\|$	$\|\frac{\Delta D_3}{D_3}\|$	Cond
30	0.516(−1)	5.26	2.45	0.306(−7)	0.317(−6)	0.954(−6)	0.744(−7)	0.199(7)
60	0.180(−4)	0.212(−2)	0.643(−3)	0.162(−12)	0.194(−10)	0.218(−9)	0.593(−9)	0.192(7)
120	0.402(−5)	0.249(−3)	0.499(−4)	0.417(−13)	0.485(−11)	0.561(−10)	0.158(−9)	0.192(7)
240	0.119(−5)	0.822(−4)	0.112(−4)	0.106(−13)	0.121(−11)	0.140(−10)	0.402(−10)	0.191(7)
480	0.357(−6)	0.295(−4)	0.298(−5)	0.227(−14)	0.303(−12)	0.352(−11)	0.105(−10)	0.191(7)
960	0.147(−6)	0.161(−4)	0.140(−5)	0.425(−15)	0.757(−13)	0.911(−12)	0.314(−11)	0.191(7)
1920	0.899(−7)	0.128(−4)	0.902(−6)	0.113(−14)	0.199(−13)	0.220(−12)	0.129(−11)	0.191(7)
3840	0.767(−7)	0.119(−4)	0.625(−6)	0*	0.340(−14)	0.680(−13)	0.106(−11)	0.191(7)
7680	0.741(−7)	0.118(−4)	0.439(−5)	0.893(−14)	0.190(−14)	0.381(−13)	0.947(−12)	0.191(7)

the leading coefficients are listed in Tables 8 and 9, where the coefficient $\bar{D}_0$ also has 17 significant digits. Note that the large $\bar{M}$ should be used to balance the exponential convergence rates, based on Theorem 4. This is distinct from the CTM where $\bar{M} = 30$ is small.

Let us compare the errors from the CTM and the hybrid TM. The best results are given from Table 1 with Eq. (159) and from Table 8 with Eq. (167), respectively,

$$|\varepsilon|_{\infty,\Gamma_0} = 0.401(-10), \qquad |\varepsilon_n|_{\infty,\Gamma_0} = 0.454(-9), \qquad |\varepsilon|_1 = 0.498(-10), \tag{168}$$

$$|\varepsilon|_{\infty,\Gamma_0} = 0.767(-7), \qquad |\varepsilon_n|_{\infty,\Gamma_0} = 0.119(-4), \qquad |\varepsilon|_1 = 0.625(-6). \tag{169}$$

Obviously, the global errors of the solutions by the CTM are more accurate than the hybrid TM. Moreover, less CPU is needed because that $\bar{M} = 30$ is much smaller than $\bar{M} = 3840$. Note that D_0 has 17 significant digits by both TMs. Hence, the accuracy of $\bar{D}_0$ is just one criteria to evaluate the numerical methods for singularity problems.

Last, we consider the Debye–Huckel equation $-\Delta u + u = 0$ with Motz's boundary conditions in Eq. (146). The piecewise particular solutions can be found as

$$v = \begin{cases} v_0 = \sum_{i=0}^{L} \bar{d}_i \dfrac{I_{i+\frac{1}{2}}(r)}{I_{i+\frac{1}{2}}(\frac{1}{2})} \cos(i + \dfrac{1}{2})\theta \qquad \text{in } S_0\,, \\ v_1 = 500\exp(-\rho\cos\phi) + \sum_{i=1}^{M} \bar{a}_i \dfrac{I_{2i+1}(\rho)}{I_{2i+1}(\frac{1}{2})} \cos(2i+1)\phi \qquad \text{in } S_1\,, \\ v_2 = \sum_{i=0}^{N} \bar{b}_i \dfrac{I_{2i}(\xi)}{I_{2i}(\frac{1}{2})} \cos 2i\eta \qquad \text{in } S_2\,, \end{cases} \tag{170}$$

Table 8. The leading coefficients $\bar{D}_i$ by the simplified hybrid TM for Motz's problem at $L = 34$ and $M = N = 15$ by the Gaussian rule of six nodes with $\bar{M} = 3840$ along $\overline{AE}$

i	$\bar{D}_i$	i	$\bar{D}_i$
0	.401162453745234416(3)	18	.119539425331155125(−4)
1	.876559201950882141(2)	19	−.997377878424553307(−5)
2	.172379150794479834(2)	20	−.218799988043284786(−5)
3	−.807121525970671527(1)	21	.123113699808094708(−4)
4	.144027271700217074(1)	22	−.861590585674063637(−7)
5	.331054886077918553	23	.112519712035018431(−4)
6	.275437344367627068	24	.101379541374730469(−4)
7	−.869329929651373179(−1)	25	−.213038668080027716(−4)
8	.336048812445294623(−1)	26	−.232568778053804656(−6)
9	.153843598338551015(−1)	27	−.128254949790181612(−4)
10	.730230881089648492(−2)	28	−.104808433332291741(−4)
11	−.318417994710996129(−2)	29	.192089636657504856(−4)
12	.122055808683248419(−2)	30	.122621067574900189(−5)
13	.531301275801784813(−3)	31	.522797705053408068(−5)
14	.271419809942469279(−3)	32	.411209726337550316(−5)
15	−.119177132893171268(−3)	33	−.653035080167190711(−5)
16	.514919906538434415(−4)	34	−.679587260047955378(−6)
17	.202800107498274560(−4)		

Table 9. The leading coefficients $\bar{A}_i$ and $\bar{B}_i$ by the simplified hybrid TM for Motz's problem at $L = 34$ and $M = N = 15$ by the Gaussian rule of six nodes with $\bar{M} = 3840$ along $\overline{AE}$

i	$\bar{A}_i$	$\bar{B}_i$
0	−.324796539532481859(3)	.913597470506882559(2)
1	.232607975243296679(2)	.106454402511139278(3)
2	.749635805872579919(1)	.114358406001376700(2)
3	−.105838719285406513(1)	−.475017155034118410(1)
4	−.797801467080026638	−.712640963620977508
5	.133844120961793767	.522088218619536693
6	.115978358085103983	.889807533637062276(−1)
7	−.206750825527241000(−1)	−.773686693148138288(−1)
8	−.194501444783602567(−1)	−.139161110779844714(−1)
9	.358572990720075610(−2)	.131359927538752877(−1)
10	.352852142710858522(−2)	.242420048076527270(−2)
11	−.663646884204361471(−3)	−.240186646894774295(−2)
12	−.636922005089684043(−3)	−.427600140136292529(−3)
13	.120671171672690993(−3)	.435524978752034240(−3)
14	.832019792254850188(−4)	.553634704333649501(−4)
15	−.156455892524474174(−4)	−.569941742981822615(−4)

where $\bar{d}_i$, $\bar{a}_i$ and $\bar{b}_i$ are the coefficients to be sought, and $I_\mu(r)$ are the Bessel function for a purely imaginary arguments defined by

$$I_\mu(r) = \sum_{i=0}^{\infty} \frac{1}{\Gamma(k+1)\,\Gamma(k+\mu+1)} \left(\frac{r}{2}\right)^{2k+\mu}. \tag{171}$$

The hybrid TM with $\alpha = \beta = \frac{1}{2}$ and the direct TM can be used for the Debye–Huckel equation with the piecewise particular solutions (170).

7. Final remarks

To close this paper, let us make some remarks.

1. New analysis is made for the simplified hybrid TM, the hybrid plus penalty TM and the Lagrange multiplier (i.e. the direct) TM by using piecewise particular solutions, and the exponential convergence rates may be achieved. When the TMs involve integration approximation, only polynomial convergence rates can be obtained. By means of piecewise particular solutions, not only may the stability be improved significantly, but also the solution errors can decline, because the local particular solutions may better approach the true solution. More importantly, the hybrid and other TMs can be applied to more complicated PDEs involving multiple singularities, see Li *et al.* [48].

2. The symmetric simplified hybrid TM is explored in this paper. Some limitations as in Remark 1 exist for the Laplace equation. In order to remove such limitations, the remedy in Remark 2 is needed, or the hybrid plus penalty TM may be suggested.

3. The numerical solutions for Motz's problem given in this paper are better than those given in Li, *et al.* [37, 44, 52]. The reasons are twofold. (i) The partition by the straight interior boundary as in Fig. 3 is the best, based on the stability analysis in Li and Mathon [51], while the partition by the arc was reported in [52]. (ii) The Gaussian rule with high order is used to raise the accuracy of the leading coefficient D_0, while the central or the Simpson rules were chosen in [37, 44, 52].

4. From the numerical experiments for Motz's problem, the CTM is also superior to the simplified hybrid TM, the detailed comparisons of the TMs in analysis and computation are explored in [50].

5. Let us describe the hybrid techniques applied to the Dirichlet and Neumann conditions, by removing the Lagrange multiplier in IV in Section 1. For III to remove the extra-multiplier variables, the simplified hybrid techniques are applied to the Dirichlet condition on $\overline{AB}$ as in [37, 49]. The simplified hybrid Trefftz method (SHTM) reads: To seek $u_N^* \in H_M^1(S)$ such that

$$D_d(u_N^*, v) = f(v), \quad \forall\, (v, \mu) \in H_M^1(S), \tag{172}$$

where

$$D_d(u, v) = \iint_S \nabla v \cdot \nabla u + \int_{\overline{AB}} u_\nu v - \int_{\overline{AB}} v_\nu u, \tag{173}$$

and $f(v)$ is given in (176). For IV to remove the extra- multiplier variables, the simplified hybrid techniques are applied to the Neumann condition on $\overline{AB}$ as in [37, 49]. The simplified hybrid Trefftz method reads: To seek $u_N \in H_M^1(S)$ such that

$$D(u_N, v) = f(v), \quad \forall\, v \in H_M^1(S), \tag{174}$$

where

$$D(u,v) = \iint_S \nabla v \cdot \nabla u + \int_{\overline{BCD}} u_\nu v - \int_{\overline{BCD}} v_\nu u, \tag{175}$$

$$f(v) \quad = 500 \int_{\overline{AB}} v_\nu \,. \tag{176}$$

The optimal error bounds are also achieved in [39]. Since $\overline{BCD}$ and $\overline{AB}$ consist of two and one piecewise lines, respectively, the SHTM (172) to couple the Dirichlet condition is more efficient.

6. In elasticity problem, the hybrid-Trefftz method is often used by using Lagrange multiplier to couple the traction (i.e. Neumann) condition, and reported in many papers [4, 12–15, 23, 25–29, 58, 59, 62, 63, 69, 70, 72]. This algorithm is similar to the Lagrange multiplier Trefftz method in Eq. (12), to couple the displacement (i.e. Dirichlet) condition. The Lagrange multipliers are, indeed, extra-variables, to cause more CPU time and computer storage. When such extra Lagrange multipliers are removed, the following three Trefftz methods are recommended:

   (a) The collocation Trefftz method (CTM) in Eq. (3).

   (b) The simplified hybrid Trefftz method (SHTM) in Eq. (5).

   (c) The penalty plus hybrid method in Eq. (8).

   The error bounds of Eqs. (3), (5) and (8) are given in [44, 49] only for uniform particular solutions, and in this paper for piecewise particular solutions. From our computational experiments, the CTM is the best in accuracy, stability and computer complexity.

References

[1] M.S. Abou-Dina. Implementation of Trefftz method for the solution of some elliptic boundary value problems. *Appl. Math. Comp.*, **127**: 125–147, 2002.

[2] I. Babuška. The finite element method with Lagrangian multipliers. *Numer. Math.*, **20**: 179–192, 1973.

[3] I. Babuška, A.K. Aziz. Survey lectures on the mathematical foundations of the finite element method. In: A.K. Aziz, ed., *The Mathematical foundations of the Finite Element with Applications to Partial Differential Equations.* Academic Press, Inc., pp. 3–358, 1971.

[4] I. Babuška, J.T. Oden, J.K. Lee. Mixed-hybrid finite element approximations of second-order elliptic boundary value problems. Part 2 – Weak-hybrid methods. *Comput. Meth. Appl. Mech. Engrg.*, **14**: 1–22, 1978.

[5] H.J.C. Barbosa, T.J.R. Hughes. Boundary Lagrange multipliers in finite element methods: error analysis in natural norms. *Numer. Math.*, **62**: 1–15, 1992.

[6] J.H. Bramble. The Lagrange multiplier methods for Dirichlet's problem. *Math. Comp.*, **37**(155): 1–12, 1981.

[7] F. Brezzi, M. Fortin. *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, 1991.

[8] Y.K. Cheung, W.G. Jin, O.C. Zienkiewicz. Solution of Helmholtz equation by Trefftz methods. *Inter. J. Numer. Meth. Engrg.*, **32**(1): 63–78, 1991.

[9] C.Y. Dong, S.H. Lo, Y.K. Cheung, K.Y. Lee. Anisotropic thin plate bending problem by Trefftz boundary collocation method. *Engrg. Anal. Bound. Elem.*, **28**: 1017–1024, 2004.

[10] J.S. Domingues, A. Portela, P.M.S.T. de Castro. Trefftz boundary element method applied to fracture mechanics. *Engrg. Fract. Mech.*, **64**: 67–85, 1999.

[11] G.M. Fix. Hybrid finite element methods. *SIAM Review*, **18**(3): 460–485, 1976.

[12] J.A.T. de Freitas. Hybrid-Trefftz displacement and stress elements for elastodynamic analysis in the frequency domain. *Comput. Assisted Mech. Engrg. Sci.*, **4**: 345–368, 1997.

[13] J.A.T. de Freitas. Formulation of elastostatic hybrid-Trefftz stress elements. *Comput. Meth. Appl. Mech. Engrg.*, **153**: 127–151, 1998.

[14] J.A.T. de Freitas. Formulation and implementation of hybrid-Trefftz stress elements for cohesive fracture. *Proceeding of Trefftz.08, 5th International Workshop on Trefftz methods*, pp. 105–120. Leuven, March 31–April 2, 2008.

[15] J.A.T. de Freitas, Z.-Y. Ji. Hybrid-Trefftz equilibrium model for crack problems. *Inter. J. Numer. Meth. Engrg.*, **39**: 569–584, 1996.

[16] J.A.T. de Freitas, V.M.A. Leitao. A boundary integral Trefftz formulation with symmetric collocation. *Comp. Mech.*, **25**(6): 515–523, 2003.

[17] I. Herrera. *Boundary Method: An Algebraic Theory.* Pitman, Boston, 1984.

[18] I. Herrera. Trefftz–Herrera domain decomposition. *Adv. Engrg. Softw.*, **24**: 43–56, 1995.

[19] I. Herrera. Trefftz method: A general theory. *Numer. Meth. PDEs*, **16**(6): 561–580, 2000.

[20] I. Herrera, M. Diáz. Indirect methods of collocation: Trefftz–Herrera collocation. *Numer. Meth. PDEs*, **15**: 709–738, 1999.

[21] I. Herrera, J. Solano. A non-overlapping TH-domain decomposition. *Adv. Engrg. Softw.*, **28**(4): 223–229, 1997.

[22] I. Herrera, M. Diáz. General theory of domain decomposition: Indirect methods. *Numer. Meth. PDEs*, **18**: 296–322, 2002.

[23] J. Jirousek. Basis for development of large finite elements locally satisfying all field equations. *Comput. Meth. Appl. Mech. Engrg.*, **14**: 65–92, 1978.

[24] J. Jirousek, N. Leon. A powerful finite element for plate blending. *Comput. Meth. Appl. Mech. Engrg.*, **12**: 77–96, 1977.

[25] J. Jirousek, Q.H. Qin. Application of hybrid-Trefftz element approach to transient heat conduction analysis. *Comp. Struct.*, **58**(1): 195–201, 1996.

[26] J. Jirousek, A. Venkatesh. A simple stress error estimator for hybrid-Trefftz p-version elements. *Inter. J. Numer. Meth. Engrg.*, **28**: 211–236, 1989.

[27] J. Jirousek, A. Venkatesh. Hybrid Trefftz plane elasticity elements with p-method capabilities. *Inter. J. Numer. Meth. Engrg.*, **35**: 1443–1472, 1992.

[28] J. Jirousek, A. Wróblewski. T-elements: a finite element approach with advantages of boundary solution methods. *Adv. Engrg. Softw.*, **24**: 71–88, 1995.

[29] J. Jirousek, A. Wróblewski. T-elements: State of the art and future trends. *Arch. Comput. Meth. Engrg.*, **3**: 323–434, 1996.

[30] W.G. Jin, Y.K. Cheung. Trefftz direct method. *Adv. Engrg. Softw.*, **24**: 65–69, 1995.

[31] N. Kamiya, E. Kita. Trefftz method 70 years. *Adv. Engrg. Softw.*, **24**(1–3): 1, 1995.

[32] E. Kita, N. Kamiya. Trefftz method: An overview. *Adv. Engrg. Softw.*, **24**: 3–12, 1995.

[33] E. Kita, N. Kamiya, T. Iio. Application of a direct Trefftz method with domain decomposition to 2D potential problems. *Engrg. Anal. Bound. Elem.*, **23**(7): 539–548, 1999.

[34] P. Lee. A Lagrange multiplier method for the interface equations from electromagnetic applications. *SIAM J. Numer. Anal.*, **30**(2): 478–506, 1993.

[35] V.M.A. Leitao. Application of multi-region Trefftz-collocation to fracture mechanics. *Engrg. Anal. Bound. Elem.*, **22**(3): 251–256, 1998.

[36] Z.C. Li. Lagrange multipliers and other coupling techniques for combined methods and other coupling techniques for combined methods for elliptic equations. *Inter. J. Information*, **1**(2): 5–21, 1998.

[37] Z.C. Li. *Combined Methods for Elliptic Equations with Singularities, Interfaces and Infinities.* Kluwer Academic Publishers, Dordrecht, Boston, 1998.

[38] Z.C. Li. Global superconvergence of simplified hybrid combinations for elliptic equations with singularities, I. Basic theory. *Computing*, **65**: 27–44, 2000.

[39] Z.C. Li. *Error Analysis for Hybrid Trefftz Methods Coupling Neumann Conditions.* Technical report. Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, Taiwan, 2008.

[40] Z.C. Li, T.D. Bui. Generalized hybrid-combined methods for singularity problems of homogeneous equations. *Inter. J. Numer. Meth. Engrg.*, **26**: 785–803, 1988.

[41] Z.C. Li, T.D. Bui. Six combinations of the Ritz–Galerkin and finite element methods for elliptic boundary value problems. *Numer. Meth. PDEs*, **4**: 197–218, 1988.

[42] Z.C. Li, T.D. Bui. The simplified hybrid-combined methods for Laplace equation with singularities. *J. Comp. Appl. Math.*, **29**: 171–193, 1990.

[43] Z.C. Li, T.D. Bui. Coupling techniques in boundary-combined methods. *Engrg. Anal. Bound. Elem.*, **10**: 75–85, 1992.

[44] Z.C. Li, Y.L. Chen, G.C. Georgiou, C. Xenophontos. Special boundary approximation methods for Laplace equation problems with boundary singularities – Applications to the Motz problem. *Inter. Comp. Math. Appl.*, **51**: 115–142, 2006.

[45] Z.C. Li, H.T. Huang. Global superconvergence of simplified hybrid combinations of the Ritz–Galerkin and FEMs for elliptic equations with singularities, II. Lagrange Elements and Adini's Elements. *Appl. Numer. Math.*, **43**: 253–273, 2002.

[46] Z.C. Li, C.S. Huang, R.C.D. Cheng. Interior boundary conditions in the Schwarz alternating method for the Trefftz method. *Engrg. Anal. Bound. Elem.*, **29**: 477–493, 2005.

[47] Z.C. Li, G.P. Liang. On the simplified hybrid-combined method. *Math. Comp.*, **41**: 1–25, 1983.

[48] Z.C. Li, T.T. Lu, H.Y. Hu, A.H.-D. Cheng. Particular solutions of Laplace's equations on polygons and new models involving mild Singularities. *Engrg. Anal. Bound. Elem.*, **29**: 59–75, 2005.

[49] Z.C. Li, T.T. Lu, H.Y. Hu, A.H.-D. Cheng. *Trefftz and Collocation Methods.* WITpress, Southampton, Boston, January 2008.

[50] Z.C. Li, T.T. Lu, H.T. Huang, A.H.-D. Cheng. Trefftz, collocation and other boundary methods – A comparison. *Numer. Meth. PDEs*, **23**: 93–144, 2007.

[51] Z.C. Li, R. Mathon. Error and stability analysis of boundary methods for elliptic problems with interfaces. *Math. Comp.*, **54**: 1–61, 1990.

[52] Z.C. Li, R. Mathon, P. Sermer. Boundary methods for solving elliptic problem with singularities and interfaces. *SIAM J. Numer. Anal.*, **24**: 487–498, 1987.

[53] Z.C. Li, T.T. Lu. Singularities and treatments of elliptic boundary value problems. *Math. Comp. Model.*, **31**: 79–145, 2000.

[54] G.P. Liang, P. Liang. Non-conforming domain decomposition with hybrid method. *J. Comp. Math.*, **8**(4): 363–370, 1990.

[55] G.P. Liang, J.H. He. The non-conforming domain decomposition method for elliptic problems with Lagrangian multipliers (in Chinese). *Math. Numer. Sinica*, **14**(2): 207–215, 1992.

[56] T.T. Lu, H.Y. Hu, Z.C. Li. Highly accurate solutions of Motz's and the cracked beam problems. *Engrg. Anal. Bound. Elem.*, **28**: 1387–1403, 2004.

[57] J. Mandel, R. Tezaur. Convergence of a substructuring method with Lagrange multipliers. *Numer. Math.*, **73**: 473–487, 1996.

[58] I.D. Moldovan, J.A.T. de Freitas. Hybrid-Trefftz stress and displacement elements for dynamic analysis of bounded and unbounded saturated porous media. *Proceeding of Trefftz.08, 5th International Workshop on Trefftz methods*, pp. 307–321. Leuven, March 31–April 2, 2008.

[59] J. Pitkaranta. Boundary subspaces for the finite element method with Lagrange multipliers. *Numer. Math.*, **33**: 273–289, 1979.

[60] J. Pitkaranta. The finite element method with Lagrange multipliers for domains with corners. *Math. Comp.*, **37**(155): 13–30, 1981.

[61] A. Potela, A. Charafi. Trefftz boundary element-multi-region formulation. *Inter. J. Numer. Meth. Engrg.*, **45**: 821–840, 1999.

[62] Q.H. Qin. *The Trefftz Finite and Boundary Element Methods*. WITpress, Southampton, Boston, 2000.

[63] Q.H. Qin, K.Y. Wang. Application of hybrid-Trefftz finite element method to frictional contact problems. *Proceeding of Trefftz.08, 5th International Workshop on Trefftz methods*, pp. 65–87. Leuven, March 31–April 2, 2008.

[64] P.A. Raviart, J.M. Thomas. Primal hybrid finite element methods for 2nd order elliptic equations. *Math. Comp.*, **31**(138): 391–413, 1977.

[65] S. Reutskiy. A boundary method of Trefftz type with approximate trial functions. *Engrg. Anal. Bound. Elem.*, **26**(4): 341–353, 2002.

[66] P. Ruge. The complete Trefftz method. *Acta Mechanica*, **78**(3–4): 234–242, 1989.

[67] J. Sladek, V. Sladek, R. van Keer. Global and local Trefftz boundary integral formulation for sound vibration. *Adv. Engrg. Softw.*, **33**: 469–476, 2002.

[68] R.P. Shaw, S.C. Huang, C.X. Zhao. The embedding integral and the Trefftz method for potential problems with partitioning. *Engrg. Anal. Bound. Elem.*, **9**(1): 83–90, 1992.

[69] K.Y. Sze, G.H. Liu. Hybrid-Trefftz finite element models for plane Helmholtz problems. *Proceeding of Trefftz.08, 5th International Workshop on Trefftz methods*, pp. 401–415. Leuven, March 31–April 2, 2008.

[70] M. Toma, J.A.T. de Freitas. Hybrid-Trefftz stress and displacement elements for transient analysis of incompressible saturated porous media. *Proceeding of Trefftz.08, 5th International Workshop on Trefftz methods*, pp. 401–415. Leuven, March 31–April 2, 2008.

[71] E. Trefftz. Ein Gegenstuck zum Ritz'schen Verfahren. *Proceeding of the 2nd Inter. Cong. Appl. Mech.*, pp. 131–137. Zurich, 1926.

[72] A.P. Zieliński. Special Trefftz elements and improvement of their conditioning. *Commun. Numer. Meth. Engrg.*, **13**(10): 765–775, 1997.

[73] A.P. Zieliński. On trial functions applied in the generalized Trefftz method. *Adv. Engrg. Softw.*, **24**: 147–155, 1995.

[74] A.P. Zieliński, I. Herrera. Trefftz method-fitting boundary conditions. *Inter. J. Numer. Meth. Engrg.*, **24**(5): 871–891, 1987.

[75] O.C. Zienkiewicz, D.W. Kelley, P. Bettess. The coupling of the finite element method and boundary solution procedures. *Inter. J. Numer. Meth. Engrg.*, **11**(2): 355–375, 1977.