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Application of genetic algorithms for optimal positions of source points in the method of fundamental solutions

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This paper describes the application of the method of fundamental solutions for 2-D harmonic and biharmonic problems. Also, genetic algorithm is presented as a numerical procedure used for the determination of source points positions. Choosing good locations of source points is crucial in the MFS as it has a great impact on the quality of the solution. Genetic algorithm is applied in order to find such an arrangement of source points, which provides the solution of sufficient accuracy.

Keywords: method of fundamental solutions, genetic algorithm, multicriteria optimization, Motz problem, biharmonic problem

1. Introduction

In the very new paper [1] the equivalence between Trefftz method and Method of Fundamental Solutions (MFS) has been shown for harmonic and biharmonic problems. In the MFS for a given boundary value problem the solution is represented as a linear combination of fundamental solutions of the governing equation [11]. The unknown coefficients, which occur in the assumed form of solution, are determined by satisfying approximately the appropriate boundary conditions. Those boundary conditions are only satisfied exactly in selected boundary points in which the conditions are collocated.

Apart from points of collocations there are also points located outside the considered domain in which singularities occur — these are called source points. Although the position and the number of source points are important and the final solution depends on it, there is no proven procedure how to arrange the source points.

Generally there are two possible approaches to the problem of source points arrangement. First method is based on the assumption that we know the position of the sources. It means that the coordinates of the points are freely chosen and are given to the numerical procedure as already known parameters. Another possibility is to take the position of source points as unknowns as well, so the final locations are determined during the calculation, but in this case the problem becomes nonlinear. For that reason, publications presenting MFS with unknown source position are rare. Fairweather and Karageorghis [2–4] proposed the adaptive scheme, in which the coefficients of the linear representation of the solution as well as the position of the sources, which are given as a fixed number, are chosen by a non-linear least-squares algorithm.

In this work Genetic Algorithm [6] is used in order to determine the optimal (or suboptimal) position of source points for boundary value problems with 2-D biharmonic equation and 2-D Laplace equation. In the process of optimization the best solution is chosen according to the evaluation function as a minimum value of squares of errors on the boundaries. For the 2-D problem $2N$ coordinates represent N number of source points. Each coordinate is related with one dimension in the search space, so for the large number of source points the problem gets difficult and time-consuming. Genetic Algorithms, however, have been designed for such multidimensional problems, so in this work GA have been used as an effective method for source points arrangement. Similar approach has been also applied by Nisimura [8–10] to a problem of potential distribution around electrodes in the charge simulation method.

Moreover, in this paper the Motz problem is studied. For such a problem with singularity which occurs on the boundary many different approaches are used. Li [5] proposed Trefftz method which uses both MFS and particular solution of a singularity. Similarly, in this paper the solution is proposed as a sum of two parts. The first part is a combination of fundamental solutions whereas the latter describes the singularity. Due to the fact that several different boundary conditions are applied in the Motz problem, multicriteria optimization is performed as some boundary conditions are easily satisfied whereas other are difficult to satisfy.

2. Source points arrangement

The Method of Fundamental Solution is an instance of Trefftz Method, which means that differential equation is satisfied exactly in the considered domain, whereas the boundary conditions are satisfied approximately. In this work, the boundary conditions are collocated in points (collocation points) on the boundary of the domain. The source points are the points in which the singularities of fundamental solution occur and so they should be located outside the considered region. Usually, one of the arrangements presented in Fig. 1 is assumed. The optimization of source points is possible in each case, if it is a geometry-based contour, the distance between geometry and contour is optimized [7, 12], in the circular arrangement: the radius of the circle might be optimized. The last approach, however, requires optimization of the position of each point, which implies high computational costs. In addition to this, in each case the number of source points might be also one of the unknown coefficients, which are optimized.

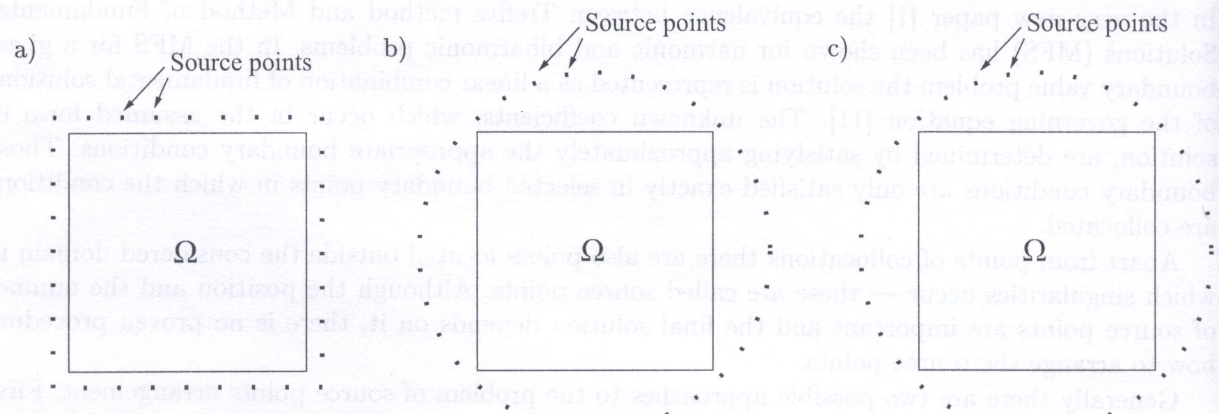

Fig. 1. Methods of locating source points: a) on a contour of scaled geometry, b) on the circle, c) randomly

3. Genetic algorithm

3.1. General concept

Genetic algorithm is a numerical technique used in order to find exact or approximate solution to optimization and search problems. The method is derived from the biological mechanism of evolution; hence some terms and procedures are inspired by biology as well. The optimization process is based on the following scheme presented in Fig. 2.

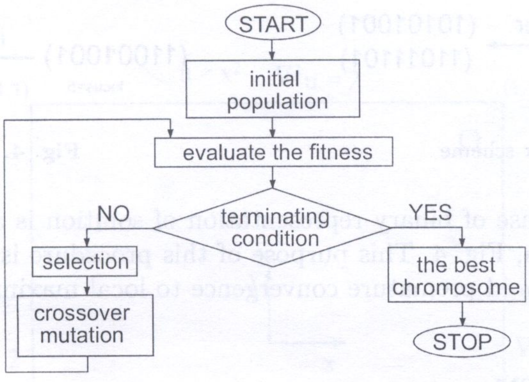

Fig. 2. Genetic algorithm scheme

In this paper the objective of optimization process is an optimal or suboptimal source points arrangement. Every such arrangement of source points in which source points are located in assumed region is called a feasible solution and is called a chromosome. In genetic algorithms, in each step several chromosomes are considered and such set is a population. At the beginning the initial population is created in such a way that the chromosomes are chosen randomly and they are represented by binary strings, though real-value encoding is also possible. Next, the iterative process takes place, in which every individual chromosome in population is ranked by the fitness function and based on these ranks (fitness), best chromosomes are selected and modified with genetic operators to create a new generation. Iteration can be terminated. There are several terminating conditions possible for the iterative process, such as: maximum number of iterations (generations) reached, calculation time exceeded, a solution found with assumed accuracy.

3.2. Binary representation

Consider that the domain is 2-D and there are N source points distributed outside, the position of each source point is defined by its coordinates, which means there are $2N$ parameters to be optimized. Furthermore, it is assumed that each coordinate is determined with finite precision and limited to a certain range. Consequently, for the i-th parameter, $x_i \in \langle a, b \rangle$, where $i \in \langle 1, 2N \rangle$ and a, b denoting the lower and upper limit of the range respectively, there are $d = (b - a) * 10^q$ feasible solutions, where q stands for assumed accuracy. In such situation, K bits are required in order to encode all feasible solutions, where $2^K > d$.

3.3. Genetic procedures

The iterative process in which new generations are created involves such procedures as selection, mutation and cross-over. Selection is the procedure used in order to chose the best chromosomes from each population to create the new one. Mutation and cross-over are used to modify the chromosomes, and so to find new solutions.

In selection, parents for the new generation are chosen using the fitness function, in this case, using fitness proportionate selection also called roulette-wheel selection. Based on values assigned to each solution by fitness function, the probability of being selected is calculated for every individual chromosome. Consequently, the candidate solution whose fitness is low will be less likely to be selected as a parent whereas it is more probable for candidates with higher fitness to become a parent.

Cross-over operation requires two chromosomes (parents) which are cut in one, randomly chosen point (locus) and since this point the binary code is swapped between the chromosomes creating two, new chromosomes, as it is shown in Fig. 3.

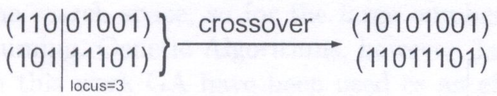

Fig. 3. Cross-over scheme

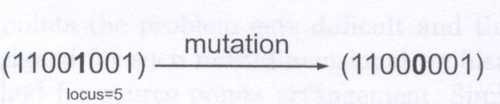

Fig. 4. Mutation scheme

Mutation procedure in case of binary representation of solution is an operation of bit inversion at randomly chosen position, Fig. 4. This purpose of this procedure is to introduce some diversity into population and so to avoid premature convergence to local maximum.

4. Biharmonic problem

Biharmonic equation is used in applied mechanics for modelling such problems as flexure of thin plates, slow viscous flow of Newtonian fluids and plane problems of elasticity theory. Generally, the problem is defined as follows,

$$\nabla^{r} u = 0 \qquad \text{in the domain } \Omega, \tag{1}$$

$$u = g_1, \quad \frac{\partial u}{\partial n} = h_1 \qquad \text{on the boundary } \partial\Omega_1, \tag{2}$$

$$u = g_2, \quad \frac{\partial^2 u}{\partial n^2} = h_2 \qquad \text{on the boundary } \partial\Omega_2, \tag{3}$$

where $\frac{\partial u}{\partial n}$ denotes the outward normal derivative, and g_1, g_2, h_1 and h_2 are prescribed functions.

The geometry in which the biharmonic problem is solved is a square with boundary conditions presented in Fig. 5. The solution of biharmonic problem, by means of MFS is represented by the linear combination of fundamental solution and it takes the following form,

$$\hat{u}(\mathbf{x}) = \sum_{j=1}^{N} c_j \varphi_1(\mathbf{x}) + \sum_{j=N+1}^{2N} d_j \varphi_2(\mathbf{x}), \tag{4}$$

$$\varphi_1(\mathbf{x}) = \log r_j, \tag{5}$$

$$\varphi_2(\mathbf{x}) = r_j^2 \log r_j, \tag{6}$$

$$r_j = \sqrt{(x - x_j)^2 + (y - y_j)^2}, \tag{7}$$

where φ_1, φ_2 are the fundamental solutions of harmonic and biharmonic equations respectively, c_j, d_j are unknown coefficients, x_j, y_j are source points coordinates. The unknown coefficients are determined by satisfying appropriate boundary conditions by means of boundary collocation method. During the calculation the total number of collocation points $NC = 120$ was assumed (30 points on each boundary) whereas the total number $NS = 12$ of source points was used. Due to the fact that the number of collocation points is greater than the number of source points, the linear system is over-determined and so collocated boundary conditions are satisfied in least-square sense.

Table 1 presents some of the numerical results i.e. maximum error and mean square error which are related to exact solution which is known for this problem. There is a comparison between solution for the calculations performed with source points located on the contour and source points distributed by GA: The difference is remarkable, especially in case of maximum error.

The arrangement of source points distributed on the contour similar to the geometry of the considered problem is presented in Fig. 6. The offset of the source points contour from the boundary of the domain is 0.2. The plots of the solution and solution error, which were calculated using source points distributed on the contour are presented below in Fig. 7.

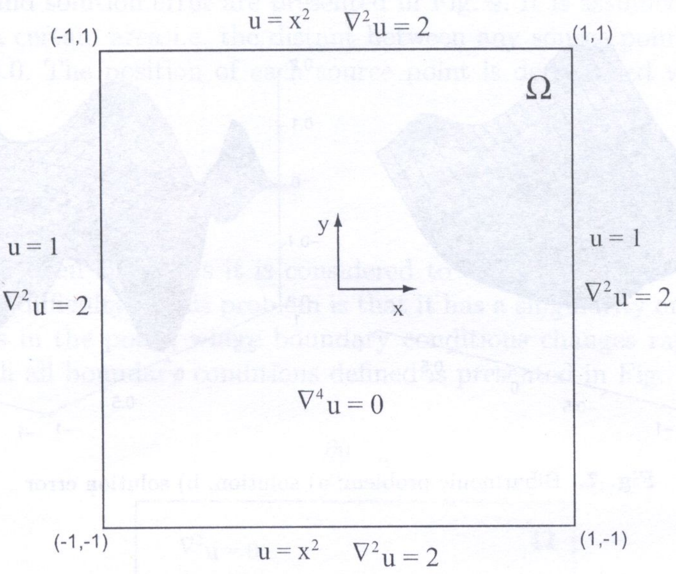

Fig. 5. Geometry of the biharmonic problem

Table 1. Biharmonic problem: Errors on the boundaries

	GA	Contour
Mean Sqr. Err. $\nabla^2 u$	0.0002	0.001
Mean Sqr. Err. u	0.0001	0.0004
Max Err. $\nabla^2 u$	0.035	0.18
Max Err. u	0.012	0.06

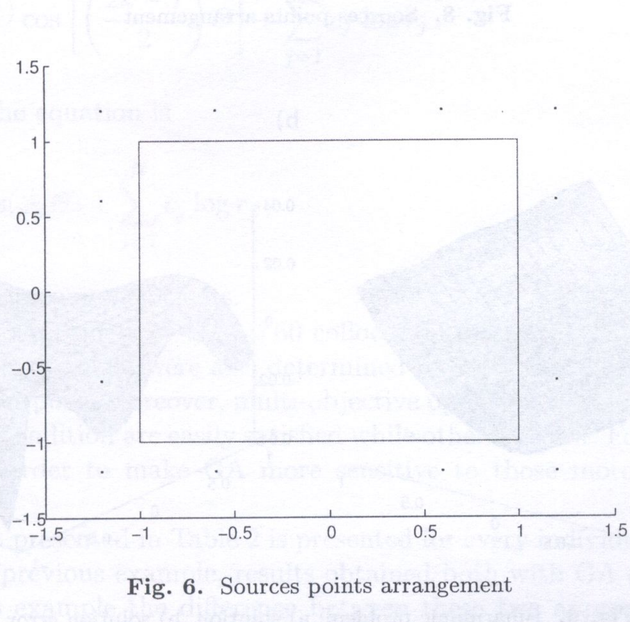

Fig. 6. Sources points arrangement

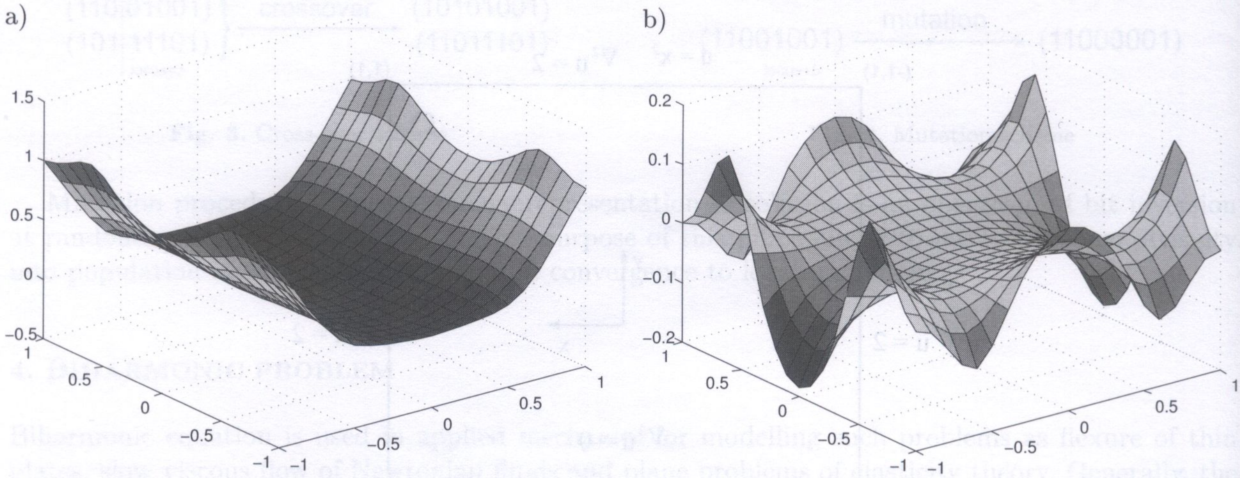

Fig. 7. Biharmonic problem: a) solution, b) solution error

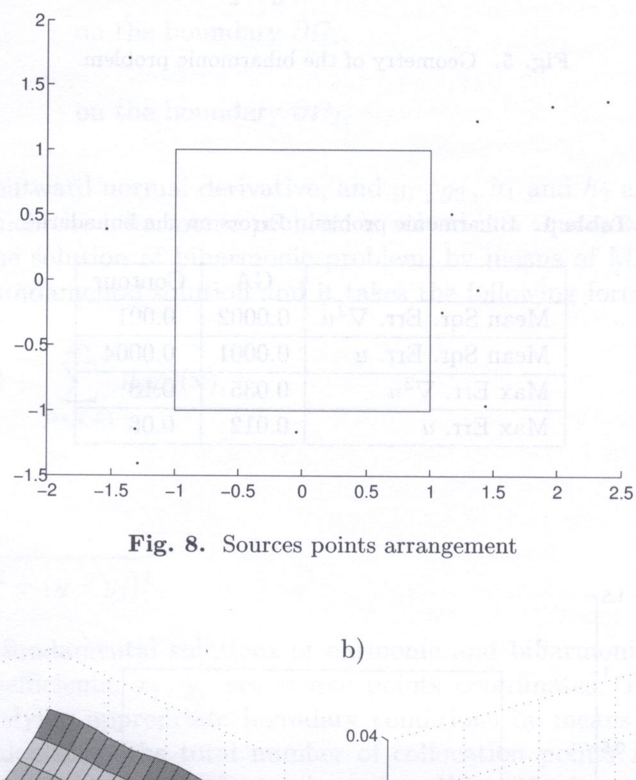

Fig. 8. Sources points arrangement

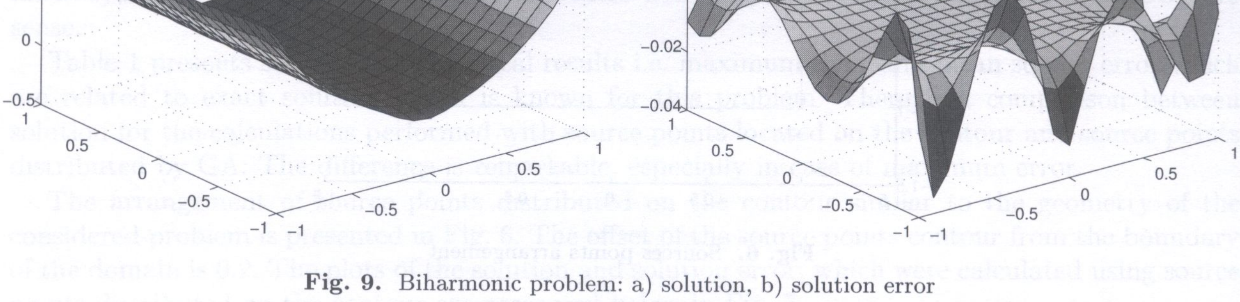

Fig. 9. Biharmonic problem: a) solution, b) solution error

The plot of source points arrangement determined by genetic algorithm is presented in Fig. 8 whereas the solution and solution error are presented in Fig. 9. It is assumed that all source points should be located in a certain area i.e. the distant between any source point and geometry should not be greater than 2.0. The position of each source point is determined with assumed accuracy $e = 10^{-3}$.

5. The Motz problem

The Motz problem has been chosen as it is considered to be a benchmark problem for Boundary Element Methods. The difficulty of this problem is that it has a singularity on one of its boundaries. The singularity occurs in the point, where boundary conditions changes rapidly The geometry of the Motz problem with all boundary conditions defined is presented in Fig. 10.

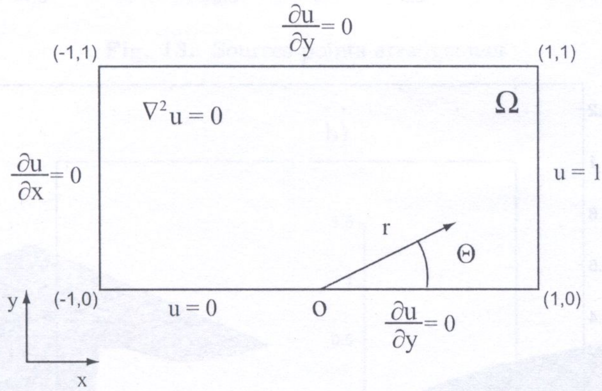

Fig. 10: Geometry of the Motz problem

The solution of the Motz problem, due to the singularity, which occurs at one of the boundaries, consists of two parts. The first part is the linear combination of fundamental solutions whereas the second part describes the singularity. Hence, the solution takes the following form,

$$u_n(\mathbf{x}) = \sum_{k=1}^{N} \alpha_k \, r^{\left(\frac{2k-1}{2}\right)} \cos\left[\left(\frac{2k-1}{2}\right)\Theta\right] + \sum_{j=1}^{N} c_j \log r_j \,. \tag{8}$$

The simplified form of the equation is

$$u_n(\mathbf{x}) = \sum_{k=1}^{N} \alpha_k r^{\beta_k} \cos(\beta_k \Theta) + \sum_{j=1}^{N} c_j \log r_j \,, \tag{9}$$

where: α_k , β_k , c_j are unknown coefficients.

The calculation were performed for $NC = 60$ collocation points and $NS = 12$ source points. In this case, positions of source points were also determined by GA or located on the scaled geometry contour for comparison purpose. Moreover, multi-objective optimization was performed, as it turned out that some boundary condition are easily satisfied while others are not. For that reason, weighting method was applied in order to make GA more sensitive to those more problematic boundary conditions.

The calculation result presented in Table 2 is presented for every individual boundary, which was considered. Similarly to previous example, results obtained both with GA application and without it are compared. In this example the difference between these two approaches is also significant. The numbers in the table are the maximum errors of each boundary condition fulfillment.

Table 2. Motz problem: maximum error at boundaries

Boundary		GA	Contour
$-1 < x < 0$	$y = 0$	0.064	0.006
$0 < x < 1$	$y = 0$	0.016	0.014
$-1 < x < 0$	$y = 1$	0.053	0.248
$0 < x < 1$	$y = 1$	0.093	0.348
$x = -1$	$0 < y < 1$	0.065	0.032
$x = 1$	$0 < y < 1$	0.033	0.051

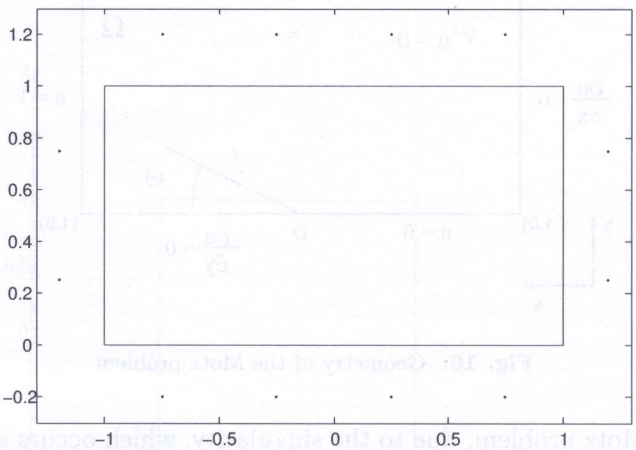

Fig. 11. Sources points arrangement

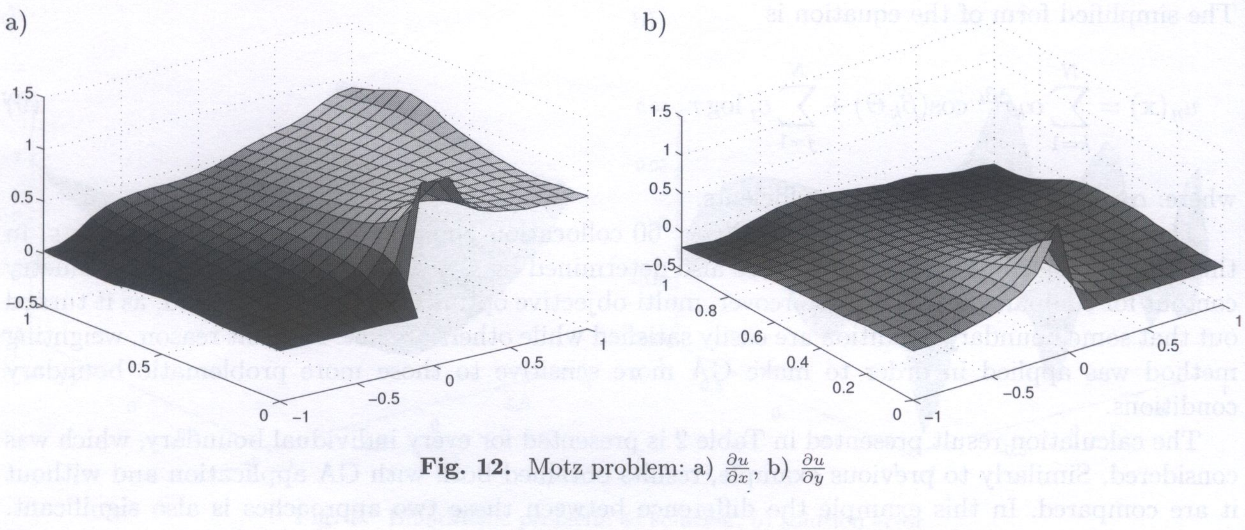

Fig. 12. Motz problem: a) $\frac{\partial u}{\partial x}$, b) $\frac{\partial u}{\partial y}$

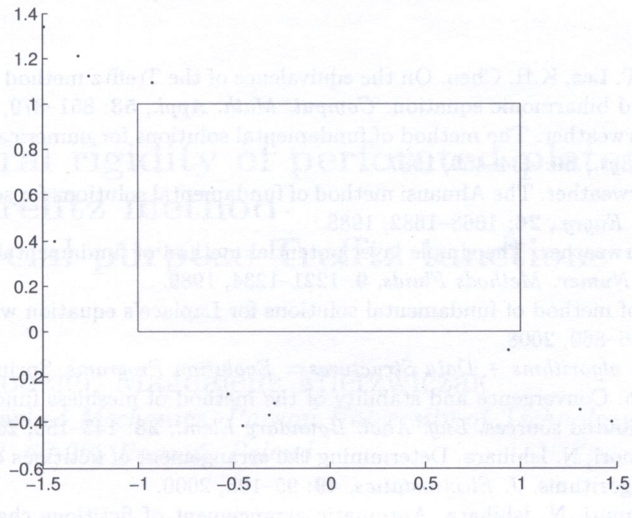

Fig. 13. Sources points arrangement

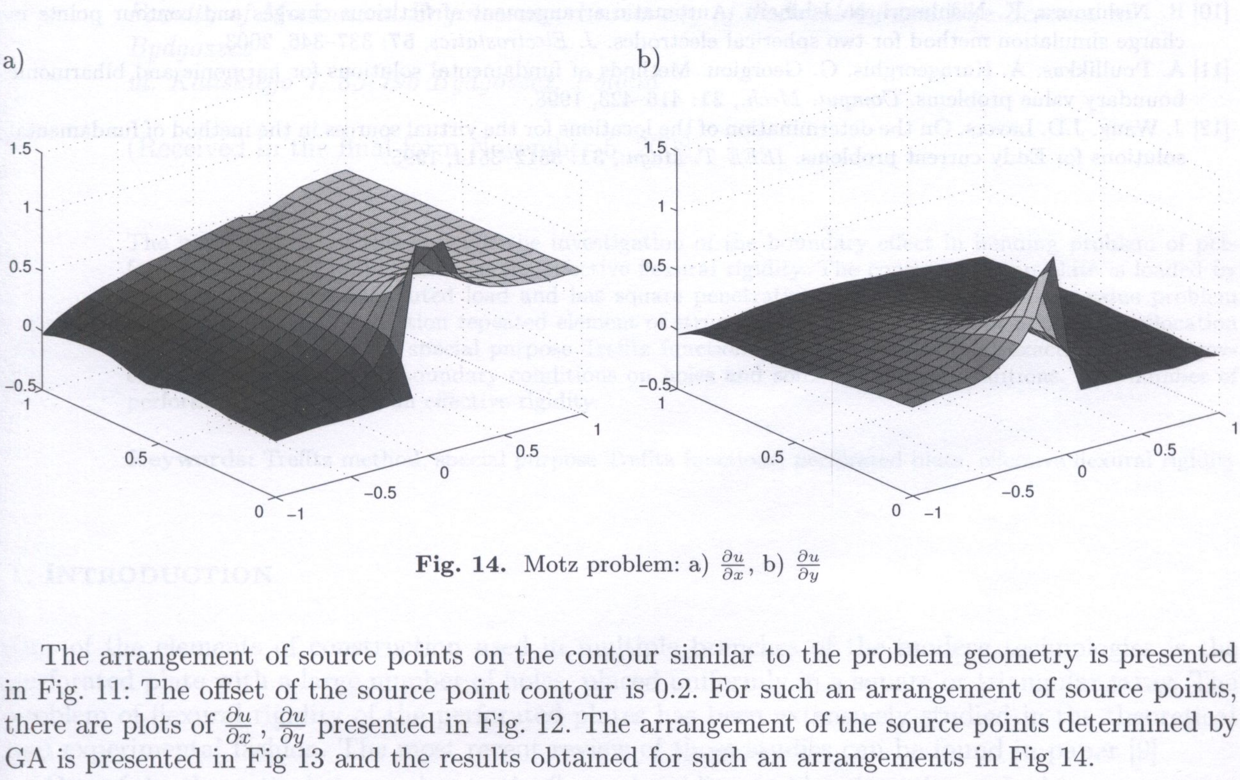

Fig. 14. Motz problem: a) $\frac{\partial u}{\partial x}$, b) $\frac{\partial u}{\partial y}$

The arrangement of source points on the contour similar to the problem geometry is presented in Fig. 11: The offset of the source point contour is 0.2. For such an arrangement of source points, there are plots of $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ presented in Fig. 12. The arrangement of the source points determined by GA is presented in Fig 13 and the results obtained for such an arrangements in Fig 14.

6. Conclusion

Application of genetic algorithm to find optimal or suboptimal arrangement of source point outside the considered domain improves the results as the boundary conditions are satisfied more accurate. The weakness of this approach is computational cost as finding the optimal solution large number of parameters is difficult and for that reason, practically genetic algorithm is used to find suboptimal solution. In this work, multi-objective optimization was successfully applied for the Motz problem, where some boundary conditions are easily satisfied while the others are not. Furthermore, the results for the Motz problem are more accurate when the solution is assumed as a sum of linear combination of fundamental solution and truncated series of singular function.

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