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Effective flexural rigidity of perforated plates by means of Trefftz method with using special purpose Trefftz functions

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The main purpose of this paper is the investigation of the boundary effect in bending problem of perforated plates and its influence on the effective flexural rigidity. The considered strip plate is loaded by constant uniformly distributed load and has square penetration pattern. The boundary value problem for determination of deflection repeated element of structure is solved by means of boundary collocation method with a use of the special purpose Trefftz functions. These functions fulfil exactly not only governing equation but also boundary conditions on holes and some symmetry conditions. The number of perforations is discussed on effective rigidity.

Keywords: Trefftz method, special purpose Trefftz functions, perforated plate, effective flexural rigidity

1. Introduction

One of the elements of construction used in multiple branches of the modern technologies is the perforated plate with a large number of holes, placed uniformly in a square or triangular array. The problem of flexural rigidity of the perforated plates has been extensively studied in the theoretical and experimental fashion. The most recent review of those studies can be found in paper [9].

One of the theoretical approaches to the flexural rigidity problem introduces the homogenization of the medium and postulates the existence of the effective elasticity constants or the effective bending stiffness of the plate. The authors supporting this particular approach accept that the uniformly perforated plate is infinite, so as the consequence the boundary effects are not taken into consideration. The supporters of this concept in unison use the St. Venant's principle. For example, the authors of [5] use the following arguments to support their solution of the deflection of an infinite perforated plate with square array: *"Although the above condition associated with on infinity plate may not occur always in practice, it follows from st. Venant's principle that when the number of perforations is large the above simplifying assumption is justified. That is, the solution for on infinite plate is applicable to a plate of finite size except in on area near the boundary of the plate corresponding to a lateral distance equal to, say, two or three limes the pitch of the holes."* Similar justification is found in paper [1]: *"While technically the solution is valid only for plates of infinite extent in the orthogonal plane directions having an infinite number of perforations uniformly distributed in a square array, it is well known that such restrictions are not necessary. It follows* *from St. Venant's principle that the solution can be applied to a large number of holes expect in a strip near the boundary the perforated portion of the plate."*

Assuming the St. Venant's principle one must agree that the boundary effect, although small, depends on many factors, such as:

- surface ratio of perforation,
- mesh type,
- the way the plate is attached,
- type of load.

Furthermore, in a particular case, it is not known how big the boundary effect is — how many rows of the edge perforations are affected. The answer to such a question is quite important if the concept of homogenization is to be used for a plate with a small number of holes. In other words, one has to determine how many holes are needed in order for the homogenization to make sense.

The main goal of this work is to present a method which allows in a fairly easy manner to determine the boundary effect for a perforated plate. To explain the problem, a strip of plate perforated in a square pattern, with a finite number of holes, deflected by a constant load of a fixed value is taken into consideration. The edges of the strip are simply supported. In order to determine the value of the strip deflection, the equation of thin plates with proper boundary conditions is used. The boundary value problem is formulated for deflection and in the next is solved by means of boundary collocation method with a use of the special purpose Trefftz functions [12]. By the way two new special purpose test functions are proposed. These functions fulfil exactly not only governing equation, what is essence of Trefftz method, but also boundary conditions on holes and some symmetry conditions.

2. Formulation of boundary value problem for perforated strip and its solution

2.1. Governing equation

Let's consider a simply supported strip plate perforated in a square pattern with a finite number M of rows of holes (Fig. 1).

The strip can have an odd number of holes (Fig. 1a) and in such a case the axis of symmetry passes through a middle row or an even number of rows when the axis of symmetry passes in the mid-distance between the two middle rows (Fig. 1b). The differentiation between the two cases is important for the proposed analytical-numerical algorithm. In both cases we introduce the following values to characterize the perforation: diameter of the holes $2a$, distance between the centers of the neighbouring holes $2b$ and the distance between the edge rows and a wall b. The ratio between the diameters of the holes to the distance between the centres of the neighbouring holes is marked as $\eta = \frac{a}{b}$ and its dimensionless radius of holes. This value is connected to surface ratio of penetration of strip φ by the formula $\varphi = \frac{\pi\eta^2}{4}$. It is assumed that the strip is loaded by the constant load q, which influences the plate without influencing the holes. We accept as a known value the flexural rigidity of the material of strip

$$D = \frac{Eh^3}{12(1-\nu)}, \tag{1}$$

where E is the Young modulus, ν is the Poisson number, and h is the plate thickness.

The governing equation for deflection of plate has the form

$$\nabla^2\nabla^2 w = \frac{q}{D}, \tag{2}$$

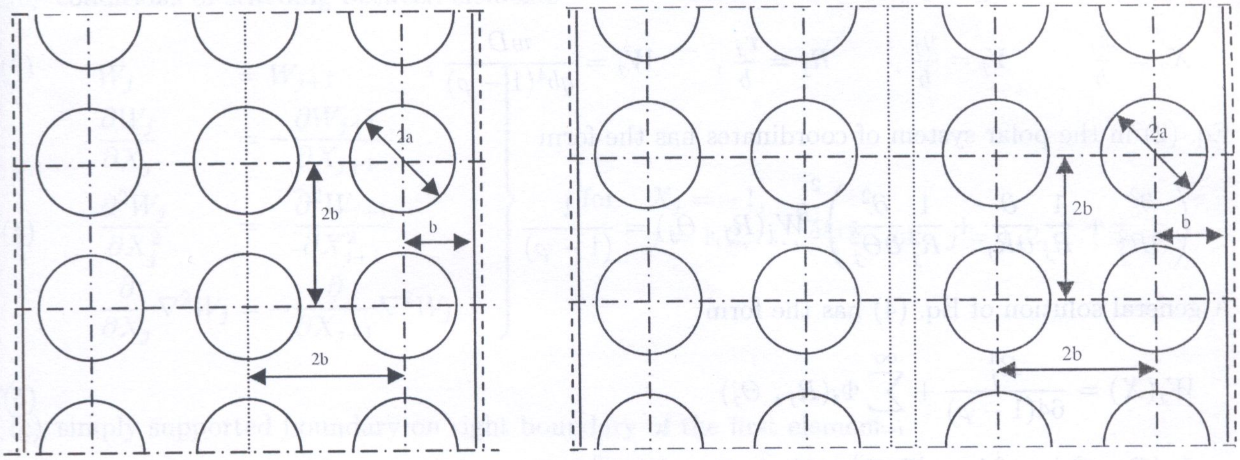

Fig. 1. Perforated plate strips

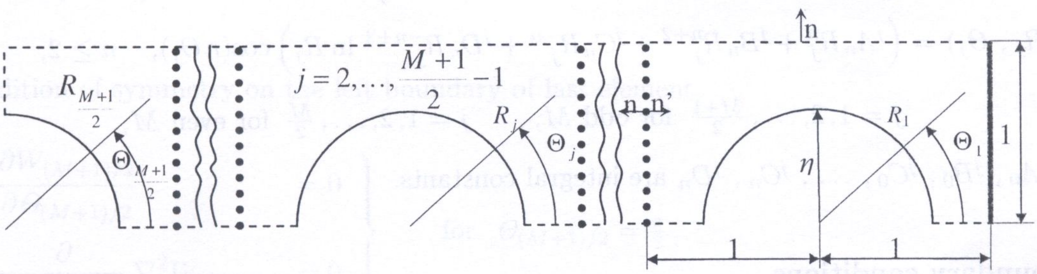

Fig. 2. Repeated element of perforated plate strip with even number of holes

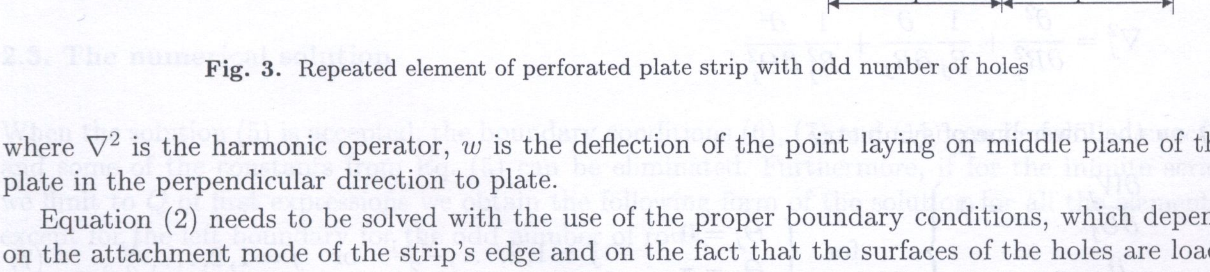

Fig. 3. Repeated element of perforated plate strip with odd number of holes

where ∇^2 is the harmonic operator, w is the deflection of the point laying on middle plane of the plate in the perpendicular direction to plate.

Equation (2) needs to be solved with the use of the proper boundary conditions, which depend on the attachment mode of the strip's edge and on the fact that the surfaces of the holes are load-free. From an infinite strip we isolate the smallest, repeating area, which is limited by the axes of symmetry of the strip and by the one of the edges. Such an area for the even and odd number of rows is presented in Figs. 2 and 3, respectively. We will attempt to find the solution for Eq. (2) within this repeating area. In order to do that we divide the repeating area into the large finite elements. In the case of odd number of rows the number of elements equals $\frac{M+1}{2}$ and for the even number of rows the number of elements equals $\frac{M}{2}$. For each of the elements we assume a local polar system of coordinates (R_j, Θ_j), so that the origin of the system lines up with the centres of the holes, where $j = 1, 2, \ldots, \frac{M+1}{2}$ for an odd number of rows and $j = 1, 2, \ldots, \frac{M}{2}$ for an even number of rows.

After introducing dimensionless variables

$$X_j = \frac{x_j}{b}, \qquad Y_j = \frac{y_j}{b}, \qquad R_j = \frac{r_j}{b}, \qquad W_j = \frac{wD}{qb^4(1-\varphi)}, \tag{3}$$

Eq. (2) in the polar system of coordinates has the form

$$\left(\frac{\partial^2}{\partial R_j^2} + \frac{1}{R_j}\frac{\partial}{\partial R_j} + \frac{1}{R_j^2}\frac{\partial^2}{\partial \Theta_j^2}\right)^2 W_j(R_j\,,\,\Theta_j) = \frac{1}{(1-\varphi)}. \tag{4}$$

A general solution of Eq. (4) has the form

$$W_j(X) = \frac{R_j^4}{64(1-\varphi)} + \sum_{i=0}^{\infty}\Phi_i(R_j\,,\,\Theta_j) \tag{5}$$

$$\Phi_0(R_j\,,\,\Theta_j) = {}^jA_0 + {}^jB_0R_j^2 + {}^jC_0\ln R_j + {}^jD_0R_j^2\ln R_j\,,$$

$$\Phi_1(R_j\,,\,\Theta_j) = \left({}^jA_1R_j + {}^jB_1R_j^3 + {}^jC_1R_j^{-1} + {}^jD_1R_j\ln R_j\right)\cos\Theta_j$$

$$\Phi_n(R_j\,,\,\Theta_j) = \left({}^jA_nR_j^n + {}^jB_nR_j^{n+2} + {}^jC_nR_j^{-n} + {}^jD_nR_j^{-n+2}\ln R_j\right)\cos(n\Theta_j), \qquad n \geq 2,$$

$$j = 1, 2, \ldots, \tfrac{M+1}{2} \text{ for odd } M, \qquad j = 1, 2, \ldots, \tfrac{M}{2} \text{ for even } M,$$

where ${}^jA_0\,, {}^jB_0\,, {}^jC_0\,, \ldots\,, {}^jC_n\,, {}^jD_n$ are integral constants.

2.2. Boundary conditions

For determination of integral constants we have the following boundary conditions:

(a) on the free boundary (boundary of holes)

$$\left.\begin{aligned} &\frac{\partial^2 W_j}{\partial R_j^2} + \nu\left(\frac{1}{R_j}\frac{\partial W_j}{\partial R_j} + \frac{1}{R_j^2}\frac{\partial^2 W_j}{\partial \Theta_j^2}\right) = 0 \\ &\frac{\partial}{\partial R_j}\nabla^2 W_j + \frac{1-\nu}{R_j}\frac{\partial^2}{\partial R_j \partial \Theta_j}\left(\frac{1}{R_j}\frac{\partial W_j}{\partial \Theta_j}\right) = 0 \end{aligned}\right\} \text{ for } \begin{aligned} &R_j = \eta \\ &j = 1, 2, \ldots, \tfrac{M+1}{2} \quad \text{or} \\ &j = 1, 2, \ldots, \tfrac{M}{2}, \end{aligned} \tag{6}$$

$$\nabla_j^2 = \frac{\partial^2}{\partial R_j^2} + \frac{1}{R_j}\frac{\partial}{\partial R_j} + \frac{1}{R_j^2}\frac{\partial^2}{\partial \Theta_j^2},$$

(b) on the lower line of symmetry

$$\left.\begin{aligned} &\frac{\partial W_j}{\partial \Theta_j} = 0 \\ &\frac{\partial}{\partial \Theta_j}\nabla^2 W_j = 0 \end{aligned}\right\} \text{ for } \left\{\begin{aligned} &\Theta_j = 0 \\ &\Theta_j = \pi \end{aligned}\right. \quad j = 1, 2, \ldots, \tfrac{M+1}{2} \text{ or } j = 1, 2, \ldots, \tfrac{M}{2}, \tag{7}$$

(c) on the upper line of symmetry

$$\left.\begin{aligned} &\frac{\partial W_j}{\partial Y_j} = 0 \\ &\frac{\partial}{\partial Y_j}\nabla^2 W_j = 0 \end{aligned}\right\} \text{ for } Y_j = 1, \quad j = 1, 2, \ldots, \tfrac{M+1}{2} \text{ or } j = 1, 2, \ldots, \tfrac{M}{2}, \tag{8}$$

(d) conditions of stitching between elements

$$\left.\begin{aligned} W_j &= W_{j+1} \\ \frac{\partial W_j}{\partial X_j} &= -\frac{\partial W_{j+1}}{\partial X_{j+1}} \\ \frac{\partial^2 W_j}{\partial X_j^2} &= -\frac{\partial^2 W_{j+1}}{\partial X_{j+1}^2} \\ \frac{\partial}{\partial X_j}\nabla^2 W_j &= -\frac{\partial}{\partial X_{j+1}}\nabla^2 W_{j+1} \end{aligned}\right\} \qquad \text{for} \quad \begin{aligned} &X_j = -1, \\ &j = 1,2,\dots,\tfrac{M+1}{2} \quad \text{or} \quad j = 1,2,\dots,\tfrac{M}{2}\,, \end{aligned} \tag{9}$$

(e) simply supported boundary on right boundary of the first element

$$\left.\begin{aligned} W_1 &= 0 \\ \nabla^2 W_1 &= 0 \end{aligned}\right\} \qquad \text{for} \quad X_1 = 1, \tag{10}$$

(f) condition of symmetry on the left boundary of last element

$$\left.\begin{aligned} \frac{\partial W_{(M+1)/2}}{\partial \Theta_{(M+1)/2}} &= 0 \\ \frac{\partial}{\partial \Theta_{(M+1)/2}}\nabla^2 W_{(M+1)/2} &= 0 \end{aligned}\right\} \qquad \text{for} \quad \Theta_{(M+1)/2} = \tfrac{\pi}{2}\,, \tag{11}$$

or

$$\left.\begin{aligned} \frac{\partial W_{M/2}}{\partial X_{M/2}} &= 0 \\ \frac{\partial}{\partial X_{M/2}}\nabla^2 W_{M/2} &= 0 \end{aligned}\right\} \qquad \text{for} \quad X_{M/2} = -1. \tag{12}$$

2.3. The numerical solution

When the solution (5) is accepted, the boundary conditions (6), (7) and (11) can be fulfilled exactly and some of the constants from Eq. (5) can be eliminated. Furthermore, if for the infinite series we limit to Q of first expressions we obtain the following form of the solution for all the elements, except for the left boundary for the odd number of rows,

$$W(X) = W^* + \sum_{i=0}^{Q} {}^jA_i F_i(R_j\,,\,\Theta_j\,,\eta,\nu) + \sum_{i=0}^{Q} {}^jB_i G_i(R_j\,,\,\Theta_j\,,\eta,\nu) \tag{13}$$

where

$$W^* = \frac{R^4}{64(1-\varphi)} - \frac{\eta^2}{8}R^2\ln R + \frac{\eta^2\ln R}{12(\nu-1)}\,(4(\nu+1)\ln\eta + 3 + \nu)\,,$$

$$
\begin{aligned}
F_0(R_j\,,\,\Theta_j\,,\eta,\nu) &= 1,\\
F_1(R_j\,,\,\Theta_j\,,\eta,\nu) &= R_j\cos\Theta_j\,,\\
F_n(R_j\,,\,\Theta_j\,,\eta,\nu) &= \left[R_j^n+\eta^{2(n-1)}R_j^{-n}\frac{(1-\nu)}{(\nu+3)}\left(\eta^2(n-1)+nR_j^2\right)\right]\cos(n\Theta_j),\\
G_0(R_j\,,\,\Theta_j\,,\eta,\nu) &= R_j^2+2\eta^2\frac{(\nu+1)}{(1-\nu)}\ln R_j,\\
G_1(R_j\,,\,\Theta_j\,,\eta,\nu) &= \left(R_j^3-\frac{\eta^4(\nu+3)}{R_j(\nu-1)}\right)\cos\Theta_j\,,\\
G_n(R_j\,,\,\Theta_j\,,\eta,\nu) &= \left(R_j^{2+n}+\frac{\eta^{2n}R_j^{-n}\left(\eta^2\left(n^2(\nu-1)^2+8(\nu+1)\right)-nR_j^2(\nu+1)(\nu-1)^2\right)}{n(\nu-1)(\nu+3)}\right)\times\\
&\qquad\times\cos(n\Theta_j).
\end{aligned}
$$

For the last element in the case of the odd number of rows, the condition (11) is fulfilled exactly thanks to the introduction of the particular form of the solution (13):

$$
W(X)=\;W^*+\sum_{i=0}^{Q}{}^jA_iF_i(R_j\,,\,\Theta_j\,,\eta,\nu)+\sum_{i=0}^{Q}{}^jB_iG_i(R_j\,,\,\Theta_j\,,\eta,\nu),\tag{14}
$$

$$
\begin{aligned}
F_0(R_j\,,\,\Theta_j\,,\eta,\nu) &= 1,\\
F_n(R_j\,,\,\Theta_j\,,\eta,\nu) &= \left[R_j^{2n}+\eta^{2(2n-1)}R_j^{-2n}\frac{(1-\nu)}{(\nu+3)}\left(\eta^2(2n-1)+2nR_j^2\right)\right]\cos(2n\Theta_j),\\
G_0(R_j\,,\,\Theta_j\,,\eta,\nu) &= R_j^2+2\eta^2\frac{(\nu+1)}{(1-\nu)}\ln R_j\,,\\
G_n(R_j\,,\,\Theta_j\,,\eta,\nu) &= \left(R_j^{2(1+n)}+\frac{\eta^{4n}R_j^{-2n}\left(2\eta^2\left(n^2(\nu-1)^2+2(\nu+1)\right)-nR_j^2(2\nu+1)(\nu-1)^2\right)}{n(\nu-1)(\nu+3)}\right)\times\\
&\qquad\times\cos(2n\Theta_j),
\end{aligned}
$$

Functions in Eqs. (13) and (14) $F_n, G_n, n=0,1,2,..$ are derived here special purpose Trefftz functions. Similar functions for plate with hole was derived in paper [7], but without symmetry of elements.

In the solution (13) and (14), ${}^jA_i\,,\,{}^jB_i$ are constants which should be determined with the help of the boundary conditions (8), (9), (10) and (12). Because the exact fulfilment of these conditions seems to be somewhat problematic, we will use the boundary collocation method [8]. In order to do that on the each segment of the boundary of unit length, we choose np equidistant collocation (Figs. 4,5) points and assume that the appropriate boundary conditions are exactly fulfilled in those points. Each large element has $4np$ of collocation points with the exception of the plate with the odd number of perforation rods where the last strip (half of the strip) has $2np$ of collocation points. In each collocation point two boundary conditions should be satisfied then $8np$ equations we have for each element with exception last element for odd number of perforation rods where we have $4np$ equations.

This leads us to the algebraic system of equations, where the unknowns are the undetermined coefficients ${}^jA_i\,,\,{}^jB_i$ in the solution (13) or (14). The dimension of the linear system depends on the number of the collocation points and the number of the large elements that are taken into consideration and equals $N=4np\cdot M$ for even and odd case of rods. The coefficients obtained with the Gauss elimination method allow determining the value of the deflection of the plate in any of its points.

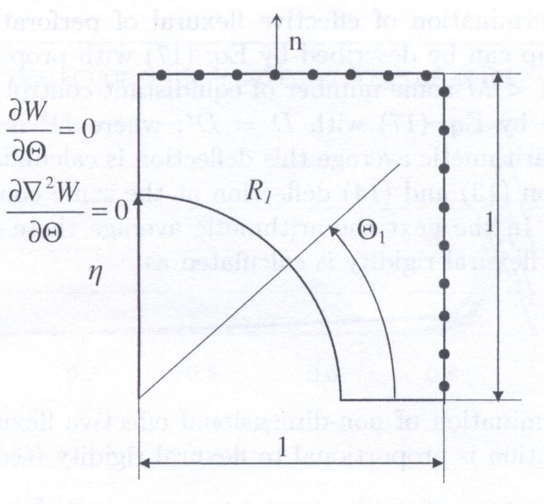

Fig. 4. Distribution of the collocation points in element

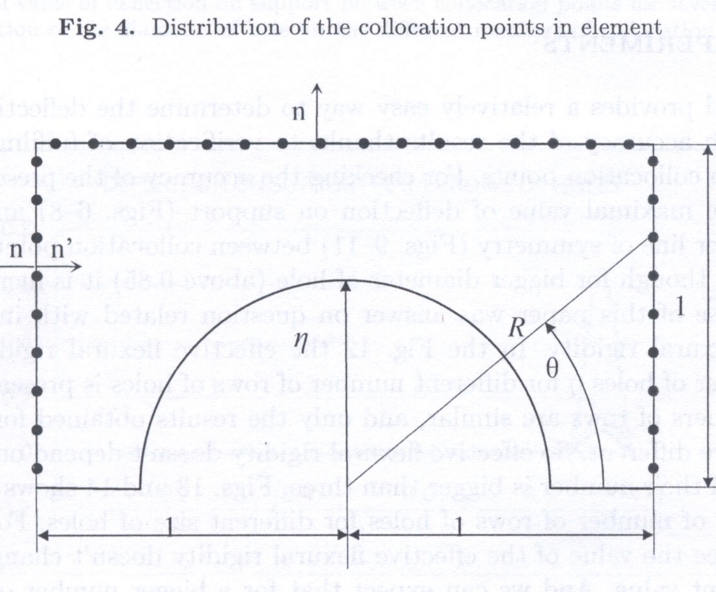

Fig. 5. Distribution of the collocation points in element

3. Manner of determination of effective flexural rigidity

For a non-perforated plate with infinite length and width L, loaded with a constant load q, we determine the deflection as follows [10, p. 212],

$$w(x) = \frac{4q}{L \cdot D} \sum_{n=1}^{\infty} \frac{\sin(\alpha_n \cdot x)}{\alpha_n^5}, \qquad \alpha_n = \frac{n\pi}{L}, \tag{15}$$

or dimensionless form

$$W(X) = \frac{wD \cdot L}{q} = 4 \sum_{n=1}^{\infty} \frac{\sin(\alpha_n \cdot X)}{\alpha_n^5} \tag{16}$$

In the case of a plate with M strips, where b is width of the strip, we can assume that $L = 2Mb$, thus deflection has no value,

$$W(X) = \frac{wD}{qL^4} = 64 \frac{M^4}{\pi^5} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2M} \cdot X\right)}{n^5} \tag{17}$$

For the purpose of determination of effective flexural of perforated plate it is assumed that deflection of perforated strip can be described by Eq. (17) with proper rigidity – effective flexural rigidity. On segment $0 < X < M$ some number of equidistant control points are chosen. For these points the deflection given by Eq. (17) with $D = D^*$, where D^* is effective flexural rigidity, is calculated and in the next arithmetic average this deflection is calculated as $W_h^{average}$. On the other hand on the base of solution (13) and (14) deflection at the same control points for X coordinate and $Y = 1$ are calculated. In the next the arithmetic average these deflections are calculated as $W_{Trefftz}^{average}$. Non-dimensional flexural rigidity is calculated as

$$D_Z = \frac{D^*}{D} = \frac{W_h^{average}}{W_{Trefftz}^{average}}. \tag{18}$$

Such a manner of determination of non-dimensional effective flexural rigidity is based on fact that non-dimensional deflection is proportional to flexural rigidity (see Eqs. (3) and (17)).

4. Numerical experiments

The proposed method provides a relatively easy way to determine the deflection of the perforated plate, with fairly high accuracy of the results thanks to verification of fulfilment of the boundary effects on the between collocation points. For checking the accuracy of the presented Trefftz method the calculation of the maximal value of deflection on support (Figs. 6–8) and normal derivative deflection on the upper line of symmetry (Figs. 9–11) between collocation points is made. The error can be accepted even though for bigger diameter of hole (above 0.85) it is increasing.

One of the purpose of this paper was answer on question related with influence of boundary effect on effective flexural rigidity. In the Fig. 12 the effective flexural rigidity as a function of dimensionless diameter of holes η for different number of rows of holes is presented. We see that the results for 4–11 numbers of rows are similar, and only the results obtained for one, two and three rows of perforation are different. So effective flexural rigidity doesn't depend on the number of rows of holes in the strip, if their number is bigger than three. Figs. 13 and 14 shows the effective flexural rigidity as a function of number of rows of holes for different size of holes. For number of rows of holes bigger than three the value of the effective flexural rigidity doesn't change too much and it's going towards constant value. And we can expect that for a bigger number of rows it will be the same.

Problem of effective flexural rigidity is not new in literature (see review [11]) and was considered by many authors. Figure 15 shows the comparison of the effective flexural rigidity of perforated plates obtained by of various author with results obtained by proposed method. One can observe

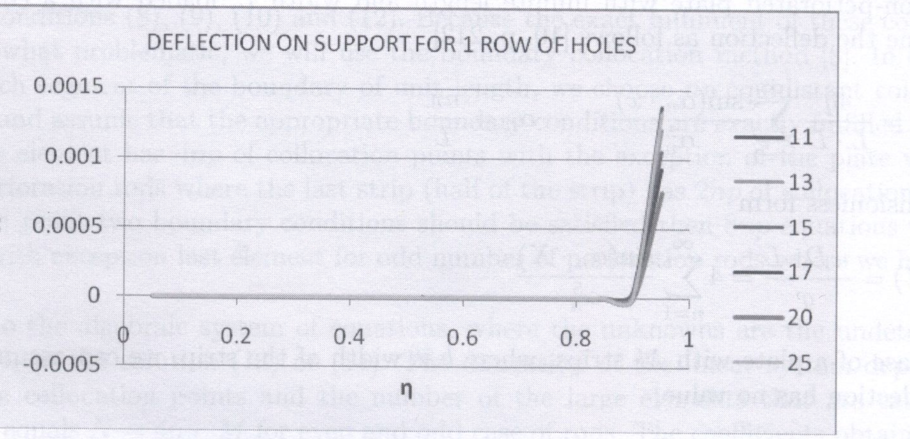

Fig. 6. Maximal value of deflection on support between collocation points for one row of holes as a function of the diameter of hole for the different numbers of collocation points (11, 13, 15, 17, 20, and 25)

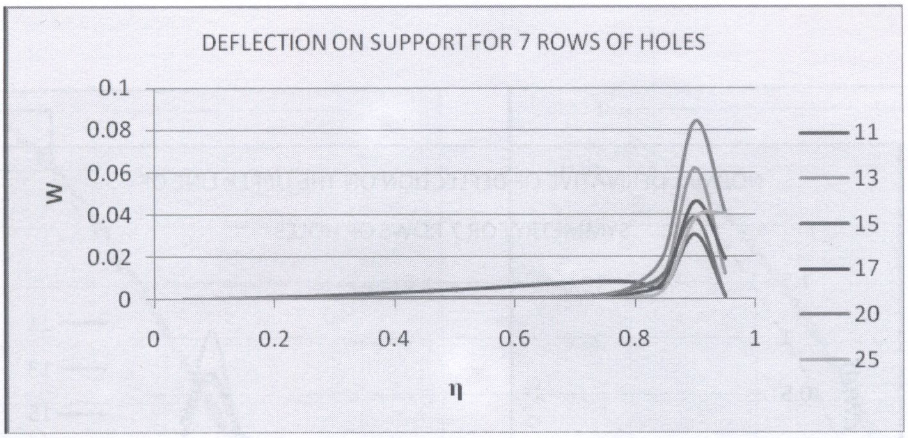

Fig. 7. Maximal value of deflection on support between collocation points for seven rows of holes as a function of the diameter of hole for the different numbers of collocation points

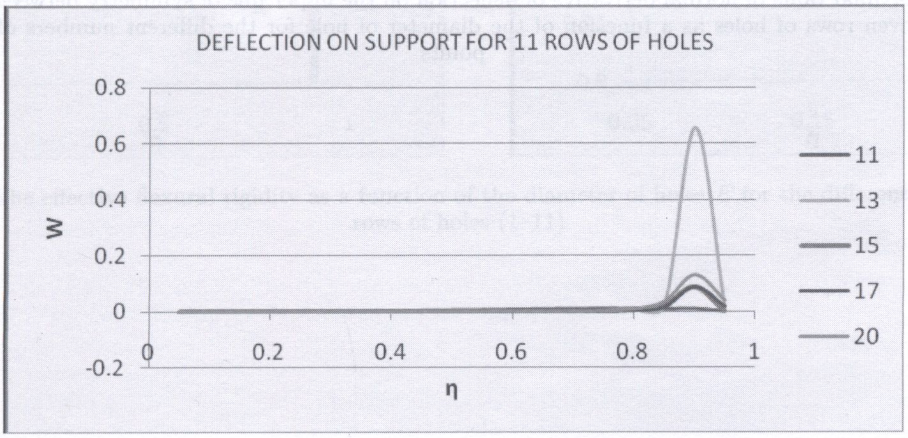

Fig. 8. Maximal value of deflection on support between collocation points for eleven rows of holes as a function of the diameter of hole for the different numbers of collocation points

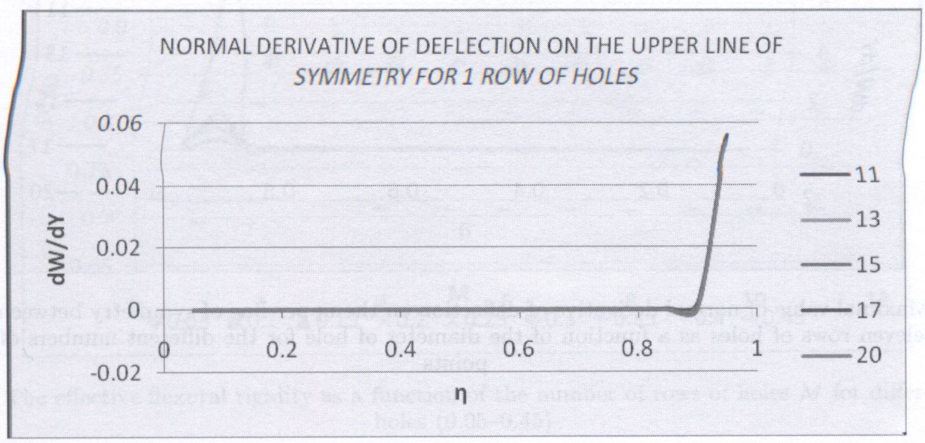

Fig. 9. Maximal value of normal derivative of deflection on the upper line of symmetry between collocation points for one row of holes as a function of the diameter of hole for the different numbers of collocation points

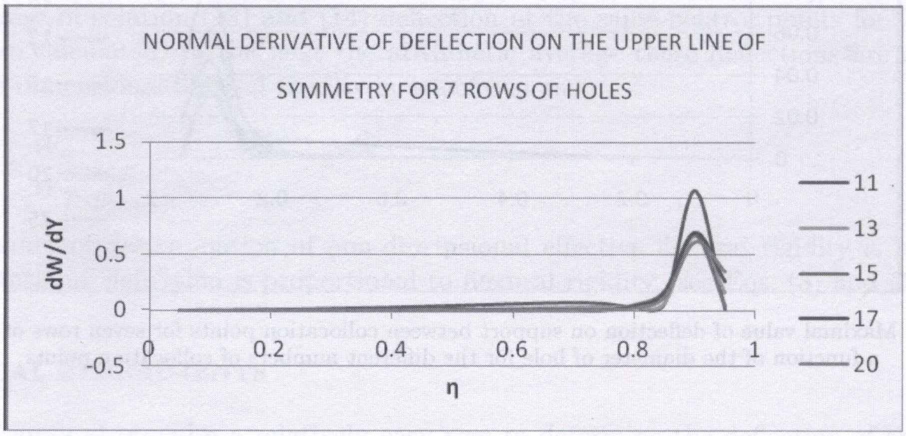

Fig. 10. Maximal value of normal derivative of deflection on the upper line of symmetry between collocation points for seven rows of holes as a function of the diameter of hole for the different numbers of collocation points

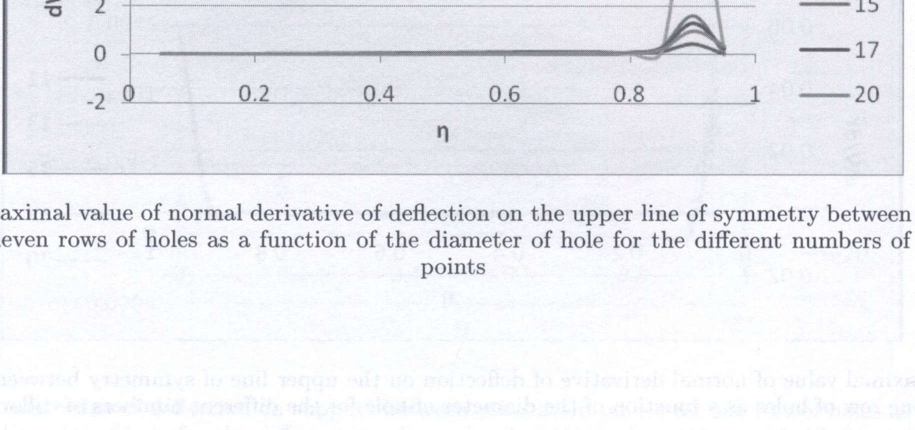

Fig. 11. Maximal value of normal derivative of deflection on the upper line of symmetry between collocation points for eleven rows of holes as a function of the diameter of hole for the different numbers of collocation points

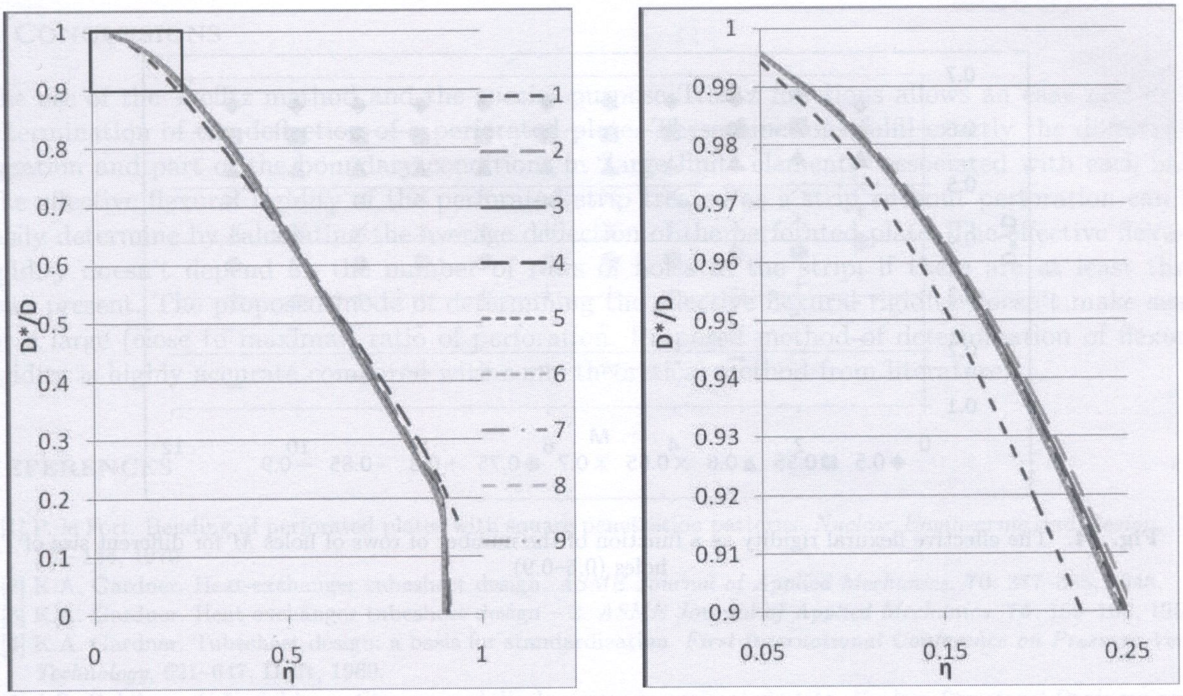

Fig. 12. The effective flexural rigidity as a function of the diameter of holes E for the different numbers of rows of holes (1–11)

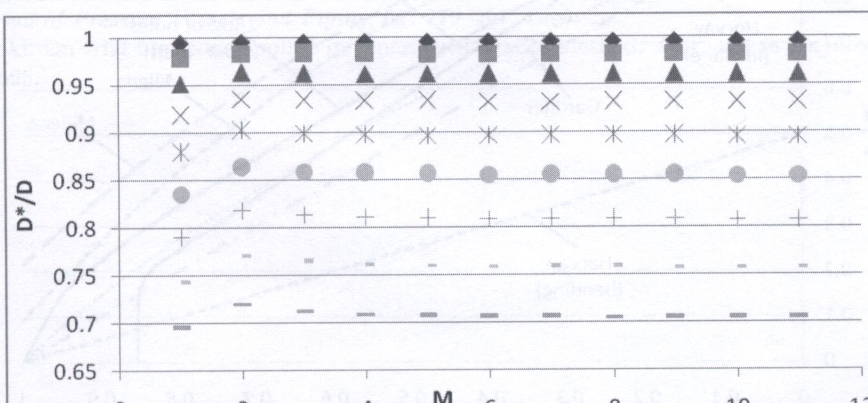

Fig. 13. The effective flexural rigidity as a function of the number of rows of holes M for different size of holes (0.05–0.45)

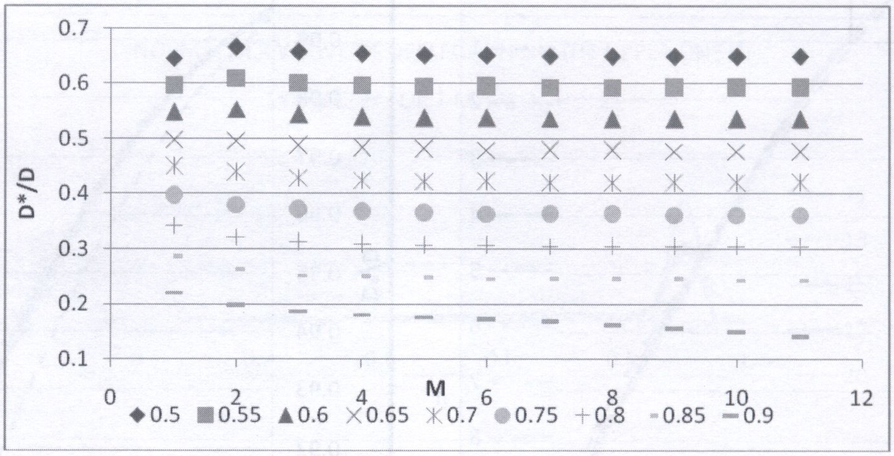

Fig. 14. The effective flexural rigidity as a function of the number of rows of holes M for different size of holes (0.5–0.9)

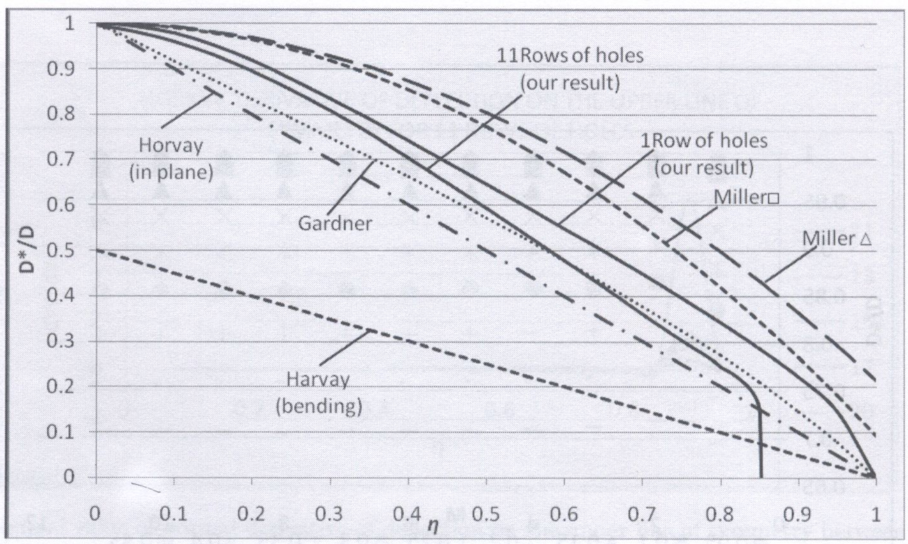

Fig. 15. Comparison non-dimensional effective flexural rigidity of the obtained results for one and eleven rods of perforation with results from literature [2–4, 6, 9].

that the difference between our results for strips with one and eleven rows of perforation is smaller then differences between results another authors on the base different methods.

5. Conclusions

The use of the Trefftz method and the special purpose Trefftz functions allows an easy and exact determination of the deflection of a perforated plate. These functions fulfil exactly the differential equation and part of the boundary conditions in "large finite elements" associated with each hole. The effective flexural rigidity of the perforated strip treated as a strip without perforation can be easily determine by calculating the average deflection of the perforated plate. The effective flexural rigidity doesn't depend on the number of rows of holes in the strip, if there are at least three rows present. The proposed mode of determining the effective flexural rigidity doesn't make sense for a large (close to maximal) ratio of perforation. Proposed method of determination of flexural rigidity is highly accurate compared with some theoretical method from literature.

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