

Trefftz functions and application to 3D elasticity

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When solving complex boundary value problems, the primary advantage of the Trefftz method is that Trefftz functions *a priori* satisfy the governing differential equations. For the treatment of three-dimensional isotropic elasticity problems, it is proposed that the bi-harmonic solutions in Boussinesq's method can be expressed as half-space Fourier series to bypass the difficulties of integration. A total of 29 Trefftz terms for each component of the displacement vectors are derived from the general solutions of the elasticity system. Numerical assessments on the proposed formulations are performed through two examples (a cubic and a cylindrical body). Results are compared with those from the method of fundamental solutions (MFS) and the commercial finite element method (FEM) software *STRAND 7*, suggesting that Trefftz functions can provide pseudo-stability, faster convergence and reduced error margins.

1. INTRODUCTION

The solutions for three-dimensional isotropic elasticity problems are of great importance when more precise stress analysis is required in three-dimensional bodies where two-dimensional or axisymmetric analyses are not feasible [8]. In addition, it is suggested that the three-dimensional elasticity solutions could have useful applications in fracture mechanics such as solving problems involving voids, inclusions and cracks in three-dimensional spaces [4].

Engineering approaches to the elasticity problems are based on the classical continuum theory, in which a material with a continuous geometric volume contains infinitesimal segments, and represents their average behaviour [3]. The outcomes of this are the governing equilibrium equations in differential forms together with the elastic properties, which can be used to seek the displacements in Lamé's equations or the stresses solutions in Beltrami–Michell equations. Boussinesq showed that Lamé's equations could be reduced to bi-harmonic equations which present the three components of displacement u_1 , u_2 and u_3 as three bi-harmonic functions. Later, Papkovitch showed that Boussinesq's solution could be simplified and presented in four harmonic functions [12]. Piltner [10] developed the complex-valued functions method, using a set of displacement trial functions as an alternative to the bi-harmonic functions approach for solving three-dimensional elasticity problems. Wang and Huang [14] developed the classical potential functions method to solve three-dimensional transversely isotropic piezoelectric problems, while many other researchers employed polynomials as an alternative solution. For example, Barber [1] used polynomials to approximate the analytical solution for the prismatic bar.

However, analytical solutions for these differential equations in three-dimensional spaces are always difficult to obtain and are available only for a few problems with simple geometries and boundary conditions such as axisymmetric bodies, half-spaces and layers. In addition to the development of general solutions for three-dimensional elasticity, numerical approaches such as MFS and FEM

were developed in the 19th and 20th centuries. These two methods provided alternative methods approaches to approximate the solutions for three-dimensional elasticity problems [8]. Both methods analyse continuous bodies as a series of finite elements. The MFS employed Green's function on the element surface to satisfy the governing differential equations. Although this method provides accurate results, the introduction of singularity properties makes it problematic for solving complex boundary value problems. Besides the MFS, the core idea of the FEM was originated by Ritz [12]. His approach was to approximate the true solutions for the governing differential equations via a series of functions satisfying the boundary conditions.

Alternatively, Trefftz methods, first developed by Trefftz in 1926, have been considered the most effective numerical method for solving three-dimensional elasticity problems [13]. As a counterpart to Ritz's method, Trefftz proposed a new concept of using trial functions satisfying *a priori* the governing differential equations but violating the boundary conditions so as to provide upper and lower bounds to the exact solutions of boundary value problems [9]. In contrast to the other analytical solutions, the Trefftz functions are usually derived from the complete set of general solutions without restrictive boundary conditions [5]. This concept of the Trefftz functions not only removes the singularity problem occurred in the MFS, it also avoids any approximation of the boundary integrals [15]. All of these features allow the Trefftz method to yield very accurate and rapidly convergent results when used in parallel with the analytical treatment of complex boundary value problems.

From the point view of basic unknown variables used, Trefftz formulations can be classified as direct or indirect formulations [6]. In the direct formulation, a relatively new formulation presented by Cheung *et al.* [2], the weighted residual expression of the governing equation is derived by taking the regular T-complete solutions satisfying the governing equation as the weighting function, and then the boundary integral equation is obtained by applying twice the Gauss' divergence formula to it. The resulting boundary integral equation, as in the boundary element method, is discretised and solved for the boundary unknowns. In the indirect formulation [7], which is thought to be the original one presented by Trefftz, the solution of the problem is approximated by the superposition of the functions satisfying the governing differential equation, and then the unknown parameters are determined so that the approximate solution satisfies the boundary condition by means of the collocation, least square or Galerkin method.

In this paper, particular emphasis is placed on deriving the general solutions following Boussinesq's method. Once the general solutions for the three-dimensional elasticity problem have been obtained Trefftz functions can be easily established. In addition, the indirect Trefftz method is presented based on the established Trefftz functions to solve three-dimensional elasticity problems. The classical MFS is also evaluated through two numerical examples and the results are compared with those from the indirect Trefftz method.

2. PROBLEM FORMULATIONS

2.1. Basic equations

The equations of the three-dimensional isotropic elasticity system are governed by equilibrium, strain-displacement and constitutive equations as given in Eqs. (1), (2) and (3).

(a) The equilibrium equations are

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} &= -F_x, \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} &= -F_y, \\ \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} &= -F_z, \end{aligned} \quad (1)$$

where $\sigma_{ij} = \sigma_{ji}$, ($i, j = x, y, z$) is the stress component and F_i is the component i of body force vector.

(b) The strain-displacement equations are

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u_1}{\partial x}, & \varepsilon_{yy} &= \frac{\partial u_2}{\partial y}, & \varepsilon_{zz} &= \frac{\partial u_3}{\partial z}, \\ \varepsilon_{xy} &= \frac{1}{2} \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right), & \varepsilon_{xz} &= \frac{1}{2} \left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} \right), & \varepsilon_{yz} &= \frac{1}{2} \left(\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} \right), \end{aligned} \quad (2)$$

where $\varepsilon_{ij} = \varepsilon_{ji}$ is the strain component and u_1, u_2, u_3 are the displacement vectors corresponding to x, y and z directions, respectively.

(c) The constitutive equations are

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2C_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{Bmatrix}, \quad (3)$$

where $C_{11} = C_{12} + 2C_{44}$ and C_{12}, C_{44} are elastic constants.

(d) By combining Eqs. (1), (2) and (3), Lamé's equations are obtained:

$$0 = (C_{12} + C_{44}) \frac{\partial}{\partial \xi_i} \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} \right) + C_{44} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \zeta_i, \quad (4)$$

for $(\xi_1, \xi_2, \xi_3) = (x, y, z)$ and $(\zeta_1, \zeta_2, \zeta_3) = (u_1, u_2, u_3)$

2.2. Boussinesq's method

The trial solutions proposed by Boussinesq [3] are given by

$$u_1 = \varphi_1 + z \frac{\partial \psi}{\partial x}, \quad u_2 = \varphi_2 + z \frac{\partial \psi}{\partial y}, \quad u_3 = \varphi_3 + z \frac{\partial \psi}{\partial z}, \quad (5)$$

where $\varphi_1, \varphi_2, \varphi_3, \psi$ are harmonic functions so that the trial solutions satisfy the elasticity system (4).

Substituting the Boussinesq's solution into the Lamé's equation (4), we obtain

$$\frac{\partial \psi_1}{\partial z} = -\frac{k+1}{k+3} \left(\frac{\partial \varphi_1}{\partial x} + \frac{\partial \varphi_2}{\partial y} + \frac{\partial \varphi_3}{\partial z} \right) \quad (6)$$

where $k = \frac{C_{12}}{C_{44}}$, $\psi_1 = \psi - Ez$, and E is a constant value.

It is noted that by using the Boussinesq's method, the problem is reduced to finding the explicit solution of $\psi_1(x, y, z)$.

3. BASIC FORMULATION FOR TREFFTZ APPROACH

Consider a partial differential equation which governs the elasticity problem [11]:

$$Lu(x) + F = 0 \quad (7)$$

where L , $u(x)$ and F are the partial differential operator, true solution and known function.

Since Eq. (7) is linear, its corresponding solution u can be divided into two parts, a homogeneous solution $u_h(x)$ and a particular solution $u_p(x)$, that is

$$u(x) = u_h(x) + u_p(x). \quad (8)$$

Accordingly, they should, respectively, satisfy

$$Lu_h(x) = 0 \quad \text{and} \quad Lu_p(x) + F = 0. \quad (9)$$

To obtain Trefftz functions to equation (9)₁, we approximate the homogeneous solution $u_h(x)$ in the form

$$u_h(x) = \sum_{i=1}^n c_i N_i(x), \quad (10)$$

such that

$$LN_i(x) = 0 \quad (11)$$

where N_i are known as Trefftz functions and c_i are the unknown coefficients which can be determined using boundary conditions.

4. DERIVATION OF TREFFTZ FUNCTIONS

4.1. Method of variable separation

We first consider the general solutions for harmonic functions, φ_i , for $i = 1, 2, 3$ in Boussinesq's method, Eqs. (5) and (6),

$$\frac{\partial^2 \varphi_i}{\partial x^2} + \frac{\partial^2 \varphi_i}{\partial y^2} + \frac{\partial^2 \varphi_i}{\partial z^2} = 0. \quad (12)$$

Using the variable separation technique, we obtain

$$\varphi_i = (a_{1i}e^{\alpha x} + a_{2i}e^{-\alpha x})(b_{1i}e^{\beta y} + b_{2i}e^{-\beta y})(c_{1i} \sin \gamma z + c_{2i} \cos \gamma z) \quad (13)$$

where a, b, c are the coefficients to be determined and α, β, γ are any arbitrary constants such that $\alpha^2 + \beta^2 = \gamma^2$.

4.2. Fourier series

Expressing the harmonic solutions, φ_i , as half-space Fourier series using Eq. (13) and substituting them into Eq. (6), we obtain

$$\begin{aligned} \frac{\partial \psi_1}{\partial z} = & -\frac{k+1}{k+3} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(A_{mn}^{(1)} m + B_{mn}^{(1)} n - C_{mn}^{(5)} M \right) \exp(mx + ny) \sin Mz \\ & + \left(A_{mn}^{(2)} m - B_{mn}^{(2)} n - C_{mn}^{(6)} M \right) \exp(mx - ny) \sin Mz \end{aligned}$$

$$\begin{aligned}
 &+ \left(-A_{mn}^{(3)}m + B_{mn}^{(3)}n - C_{mn}^{(7)}M\right) \exp(-mx + ny) \sin Mz \\
 &+ \left(-A_{mn}^{(4)}m - B_{mn}^{(4)}n - C_{mn}^{(8)}M\right) \exp(-mx - ny) \sin Mz \\
 &+ \left(A_{mn}^{(5)}m + B_{mn}^{(5)}n + C_{mn}^{(1)}M\right) \exp(mx + ny) \cos Mz \\
 &+ \left(A_{mn}^{(6)}m - B_{mn}^{(6)}n + C_{mn}^{(2)}M\right) \exp(mx - ny) \cos Mz \\
 &+ \left(-A_{mn}^{(7)}m + B_{mn}^{(7)}n + C_{mn}^{(3)}M\right) \exp(-mx + ny) \cos Mz \\
 &+ \left(-A_{mn}^{(8)}m - B_{mn}^{(8)}n + C_{mn}^{(4)}M\right) \exp(-mx - ny) \cos Mz
 \end{aligned} \tag{14}$$

where $M^2 = m^2 + n^2$ and A, B, C are the coefficients to be determined.

Integrating Eq. (14) and observing that $f(x, y)$ must be a two dimensional harmonic function because all of the remaining terms after integration are harmonic functions, we obtain

$$\begin{aligned}
 \psi_1 = &-\frac{k+1}{k+3} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -\frac{A_{mn}^{(1)}m + B_{mn}^{(1)}n - C_{mn}^{(5)}M}{M} \exp(mx + ny) \cos Mz \\
 &- \frac{A_{mn}^{(2)}m - B_{mn}^{(2)}n - C_{mn}^{(6)}M}{M} \exp(mx - ny) \cos Mz \\
 &- \frac{-A_{mn}^{(3)}m + B_{mn}^{(3)}n - C_{mn}^{(7)}M}{M} \exp(-mx + ny) \cos Mz \\
 &- \frac{-A_{mn}^{(4)}m - B_{mn}^{(4)}n - C_{mn}^{(8)}M}{M} \exp(-mx - ny) \cos Mz \\
 &+ \frac{A_{mn}^{(5)}m + B_{mn}^{(5)}n + C_{mn}^{(1)}M}{M} \exp(mx + ny) \sin Mz \\
 &+ \frac{A_{mn}^{(6)}m - B_{mn}^{(6)}n + C_{mn}^{(2)}M}{M} \exp(mx - ny) \sin Mz \\
 &+ \frac{-A_{mn}^{(7)}m + B_{mn}^{(7)}n + C_{mn}^{(3)}M}{M} \exp(-mx + ny) \sin Mz \\
 &+ \frac{-A_{mn}^{(8)}m - B_{mn}^{(8)}n + C_{mn}^{(4)}M}{M} \exp(-mx - ny) \sin Mz \\
 &+ D_m^{(1)} \exp(mx) \cos my + D_m^{(2)} \exp(mx) \sin my \\
 &+ D_m^{(3)} \exp(-mx) \cos my + D_m^{(4)} \exp(-mx) \sin my.
 \end{aligned} \tag{15}$$

Note that by following the same procedure from Eqs. (12) and (13), the two-dimensional harmonic function $f(x, y)$ is chosen as

$$f(x, y) = (a_1 e^{\alpha x} + a_2 e^{-\alpha x}) (b_1 \sin \beta y + b_2 \cos \beta y) \tag{16}$$

where α and β are arbitrary constants such that $\alpha = \beta$ and D is the coefficient to be determined so as to satisfy the boundary conditions for the corresponding potential function.

4.3. T-complete functions

Substituting Eq. (15) into Eqs. (5) and (6), we can obtain the displacement functions u_1 , u_2 and u_3 with coefficients A , B , C and D . The Trefftz function, or so called T-complete function is then extracted from Eq. (10). For instance, the Trefftz functions for u_1 is

$$\begin{aligned}
 N_{u1} &= \exp(mx + ny) \left(\sin Mz - l \frac{m^2}{M} z \cos Mz \right) & N_{u17} &= lmz \exp(mx + ny) \sin Mz \\
 N_{u2} &= \exp(mx - ny) \left(\sin Mz - l \frac{m^2}{M} z \cos Mz \right) & N_{u18} &= lmz \exp(mx - ny) \sin Mz \\
 N_{u3} &= \exp(-mx + ny) \left(\sin Mz - l \frac{m^2}{M} z \cos Mz \right) & N_{u19} &= -lmz \exp(-mx + ny) \sin Mz \\
 N_{u4} &= \exp(-mx - ny) \left(\sin Mz - l \frac{m^2}{M} z \cos Mz \right) & N_{u20} &= -lmz \exp(-mx - ny) \sin Mz \\
 N_{u5} &= \exp(mx + ny) \left(\cos Mz + l \frac{m^2}{M} z \sin Mz \right) & N_{u21} &= lmz \exp(mx + ny) \cos Mz \\
 N_{u6} &= \exp(mx - ny) \left(\cos Mz + l \frac{m^2}{M} z \sin Mz \right) & N_{u22} &= lmz \exp(mx - ny) \cos Mz \\
 N_{u7} &= \exp(-mx + ny) \left(\cos Mz + l \frac{m^2}{M} z \sin Mz \right) & N_{u23} &= -lmz \exp(-mx + ny) \cos Mz \\
 N_{u8} &= \exp(-mx - ny) \left(\cos Mz + l \frac{m^2}{M} z \sin Mz \right) & N_{u24} &= -lmz \exp(-mx - ny) \cos Mz \\
 N_{u9} &= -l \frac{mn}{M} z \exp(mx + ny) \cos Mz & N_{u25} &= mz \exp(mx) \cos my \\
 N_{u10} &= l \frac{mn}{M} z \exp(mx - ny) \cos Mz & N_{u26} &= mz \exp(mx) \sin my \\
 N_{u11} &= l \frac{mn}{M} z \exp(-mx + ny) \cos Mz & N_{u27} &= -mz \exp(-mx) \cos my \\
 N_{u12} &= -l \frac{mn}{M} z \exp(-mx - ny) \cos Mz & N_{u28} &= -mz \exp(-mx) \sin my \\
 N_{u13} &= l \frac{mn}{M} z \exp(mx + ny) \sin Mz & N_{u29} &= 0 \\
 N_{u14} &= -l \frac{mn}{M} z \exp(mx - ny) \sin Mz \\
 N_{u15} &= -l \frac{mn}{M} z \exp(-mx + ny) \sin Mz \\
 N_{u16} &= l \frac{mn}{M} z \exp(-mx - ny) \sin Mz
 \end{aligned} \tag{17}$$

where $l = \left(-\frac{k+1}{k+3} \right)$. The Trefftz functions for displacement u_2 and u_3 can be derived in a way similar to that described above.

5. NUMERICAL IMPLEMENTATION

5.1. Collocation method

The main concept of the collocation method for the Trefftz method is its use of weighted residual [7],

$$R_1 = \bar{u} - u \quad \text{and} \quad R_2 = \bar{t} - t \tag{18}$$

where \bar{u}, \bar{t} are the prescribed solution for displacements and surface traction and

$$t = \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} \cos(n, x) \\ \cos(n, y) \\ \cos(n, z) \end{Bmatrix} \tag{19}$$

As the approximated tractions t are derived from the approximated displacement obtained from Eq. (10) through the strain–displacement equation (2) and the constitutive equations (3), we therefore have the opportunity to unify the Trefftz coefficients for traction and displacement

$$t(x) = \sum_{i=1}^n c_i T_i(x) \tag{20}$$

where T and c represent the Trefftz function for traction as derived from displacement and the arbitrary coefficients after super-positioning.

After forcing the weighted residual in Eq. (18) to zero, we can obtain one equation in each direction for each collocation point corresponding to the prescribed boundary conditions. For M_1 prescribed displacement and M_2 prescribed stresses, we have $Kc = f$ where

$$K = \begin{bmatrix} N_{11} & N_{12} & \dots & N_{1n} \\ N_{21} & N_{22} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \\ N_{M_1 1} & \dots & \dots & N_{M_1 n} \\ T_{11} & T_{12} & \dots & T_{1n} \\ T_{21} & T_{22} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \\ T_{M_2 1} & \dots & \dots & T_{M_2 n} \end{bmatrix}, \quad c = \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{Bmatrix}, \quad f = \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_{M_1} \\ \bar{t}_1 \\ \bar{t}_2 \\ \vdots \\ \bar{t}_{M_2} \end{Bmatrix} \tag{21}$$

The coefficient a can be easily determined by taking $c = K^{-1}f$.

5.2. Galerkin virtual work method

Starting from the Galerkin formulation as shown in Eq. (22),

$$\int_{\Gamma_1} \alpha_1 R_1 \partial \Gamma_1 + \int_{\Gamma_2} \alpha_2 R_2 \partial \Gamma_2 = 0, \tag{22}$$

where we let $\alpha_1 = T, \alpha_2 = -N$ and Γ_1, Γ_2 are the surface domains for prescribed displacements and prescribed tractions.

We can then obtain the following explicit formulation for three-dimensional elasticity,

$$\left[\int_{\Gamma_u} [T]^T [N^T] \partial \Gamma_u - \int_{\Gamma_\sigma} [N]^T [T^T] \partial \Gamma_\sigma \right] [c] = \int_{\Gamma_u} [T]^T [\bar{u}] \partial \Gamma_u - \int_{\Gamma_\sigma} [N]^T [\bar{t}] \partial \Gamma_\sigma \tag{23}$$

or

$$[k][c] = [f],$$

where

$$[k] = \left[\int_{\Gamma_u} [T]^T [N^T] \partial \Gamma_u - \int_{\Gamma_\sigma} [N]^T [T^T] \partial \Gamma_\sigma \right],$$

$$[f] = \int_{\Gamma_u} [T]^T [\bar{u}] \partial \Gamma_u - \int_{\Gamma_\sigma} [N]^T [\bar{t}] \partial \Gamma_\sigma,$$

$$[\bar{u}] = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{bmatrix},$$

$$[\bar{t}] = \begin{bmatrix} \bar{t}_1 \\ \bar{t}_2 \\ \bar{t}_3 \end{bmatrix},$$

$$[N]^T = [[N_1] \quad [N_2] \quad [N_3]]_{M_T \times 3},$$

$$[T]^T = [[T_1] \quad [T_2] \quad [T_3]]_{M_T \times 3}.$$

M_T is the number of Trefftz terms being used.

6. NUMERICAL ASSESSMENTS

6.1. Modeling

Numerical assessment has been performed on a cubic body (2 m × 2 m × 2 m) and a cylindrical (π m² × 2 m) body by using the indirect Trefftz method including the collocation and Galerkin methods. For the elastic constants, we choose the value of steel, $C_{12} = 1.725$ and $C_{44} = 1.15$. The imposed boundary conditions for cubic and cylindrical bodies are listed below.

- Imposed boundary condition for cubic body:

$$\begin{aligned} \bar{u}_{1,2,3} &= 0 && \text{at } z = -1, \\ \bar{t}_x = \bar{t}_y &= 0 && \text{at } x = \pm 1 \text{ or } y = \pm 1 \text{ or } z = 1, \\ \bar{t}_z &= 0 && \text{at } x = \pm 1 \text{ or } y = \pm 1, \\ \bar{t}_z &= -1000 \text{ Pa} && \text{at } z = 1. \end{aligned}$$

- Imposed boundary condition for cylindrical body:

$$\begin{aligned} \bar{u}_{x,y,z} &= 0 && \text{at } z = -1, \\ \bar{t}_x = \bar{t}_y &= 0 && \text{at } x^2 + y^2 = 1 \text{ or } z = 1, \\ \bar{t}_z &= 0 && \text{at } x^2 + y^2 = 1, \\ \bar{t}_z &= -1000 \text{ Pa} && \text{at } z = 1. \end{aligned}$$

The singular values of matrix $[k]$ referring to Eq. (23), known as the ‘norms’ are first computed for different number of Trefftz functions. Next, the u_3 displacements are also computed with respect to the increase of the number of Trefftz functions. All the computing values are on the specified test point [0, 0, 1].

6.2. Discussion

6.2.1. Cubic model in Galerkin method

Figure 1 shows the reference axes and cubic model used for the Galerkin method. Using this method, Fig. 2 shows that although the matrix norm initially grows rapidly with the number of Trefftz functions, the norm stabilizes when there are around 16 Trefftz terms. This improved stability with

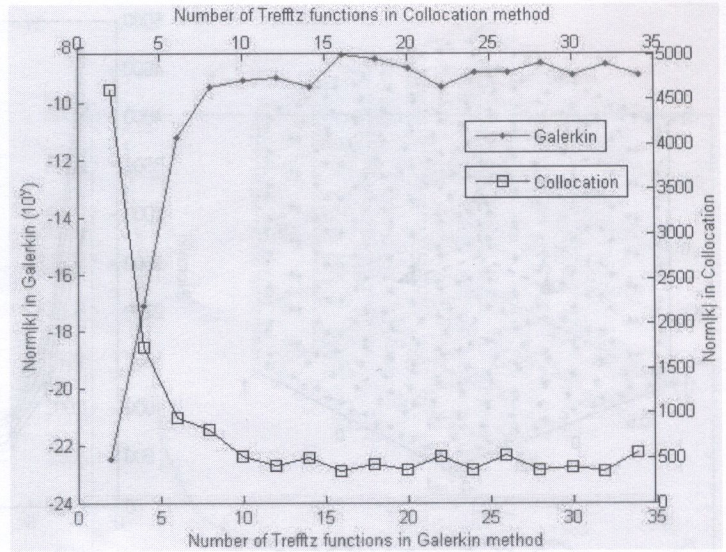
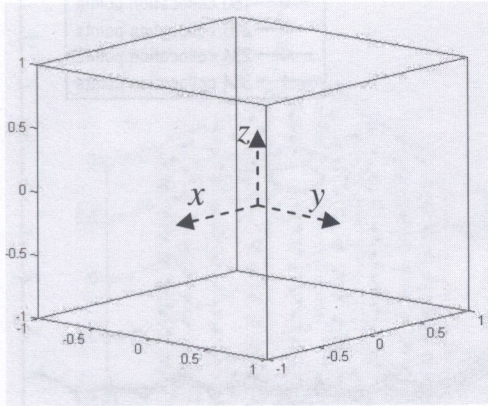


Fig. 1. Cubic model and axes used in the Galerkin method

Fig. 2. Norm versus number of Trefftz functions in Galerkin and collocation cubic models

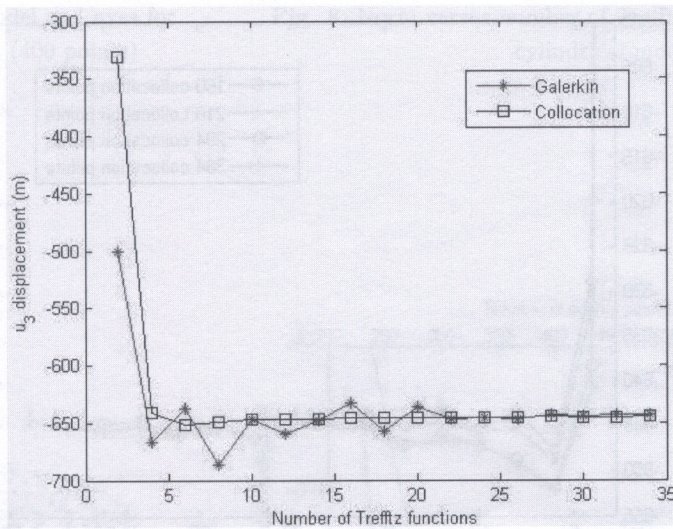


Fig. 3. u_3 displacement versus number of Trefftz functions in Galerkin and collocation cubic models showing upper and lower bound error estimations in the Galerkin method

the number of Trefftz functions is consistent with the improvement in the stability of displacement u_3 shown in Fig. 3. Displacement enters a pseudo-stable region beyond 22 Trefftz functions, a number for which the norm also shows a high level of stability.

6.2.2. Cubic model in collocation method

Numerical assessments using the cubic model in collocation method show the norm stabilize after around 12 Trefftz terms compared to 16 terms for the cubic Galerkin method, as shown in Fig. 5. Similarly, Fig. 6 shows that the displacement u_3 converges quickly and becomes pseudo-stable at the use of only 6 Trefftz terms in the collocation method compared to the 22 terms in the Galerkin method. Overall, the cubic model in collocation method shows better stability and convergence than the cubic Galerkin model.

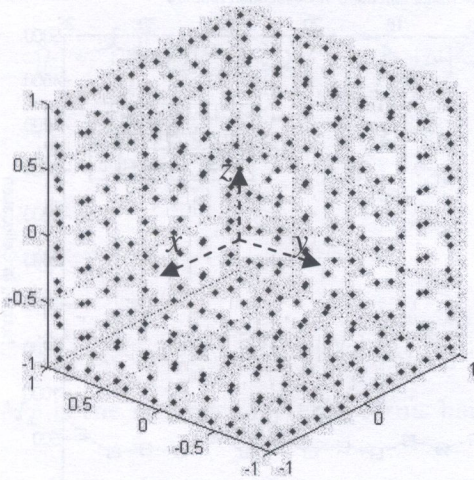


Fig. 4. Overview model of collocation points (64×4 points) and axes on cubic surfaces

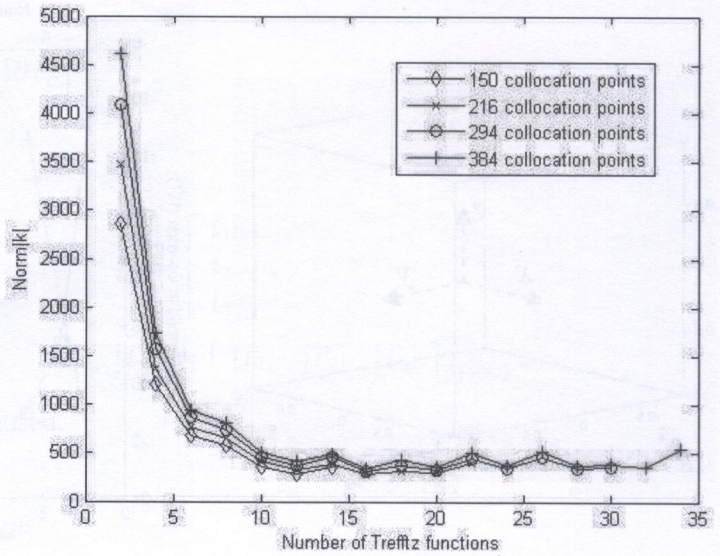


Fig. 5. Norm versus number of Trefftz functions in collocation cubic model with different choices of collocation points

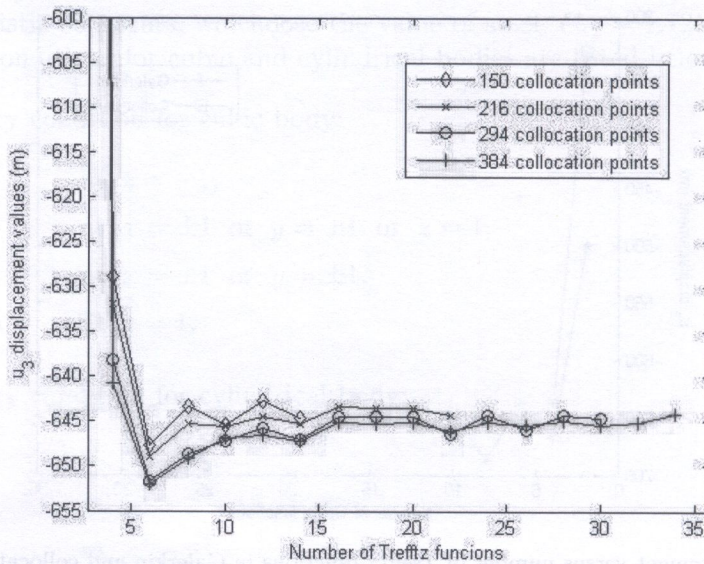


Fig. 6. u_3 displacement versus number of Trefftz functions with different choices of collocation points showing pseudo-stability with the use of few Trefftz functions

6.2.3. Cylindrical model in collocation method

Using the collocation cylindrical model as shown in Fig. 7, Fig. 8 shows that the norm becomes reasonably stable after 12 Trefftz terms, which is similar to the cubic model in collocation. As shown in Fig. 10, the displacement u_3 converges quickly and became pseudo-stable with around 8 Trefftz terms in the cylindrical collocation method. Overall, this model yielded similar results to the cubic model in collocation method.

6.2.4. Conventional MFS and FEM

Figure 9 shows the field point and source point distribution in the cylindrical model for MFS. The results obtained from the proposed formulation are compared with those from MFS. It is evident

1. SUMMARY

1.1. Conclusion

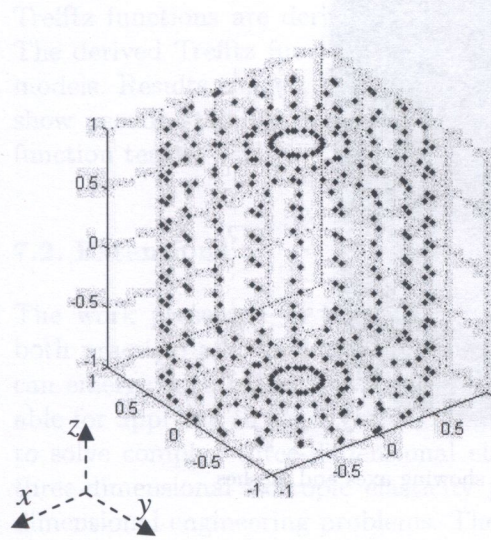


Fig. 7. Cylindrical model and axes for collocation method (400 points)

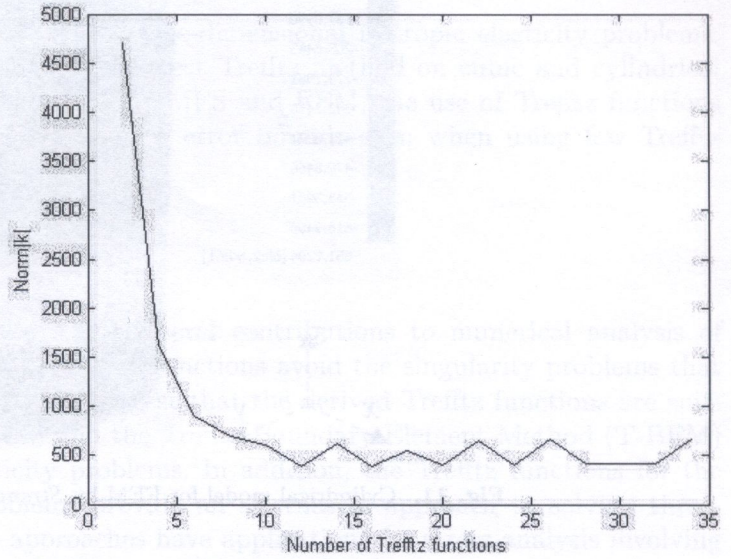


Fig. 8. Norm versus number of Trefftz functions in collocation cylindrical model

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REFERENCES

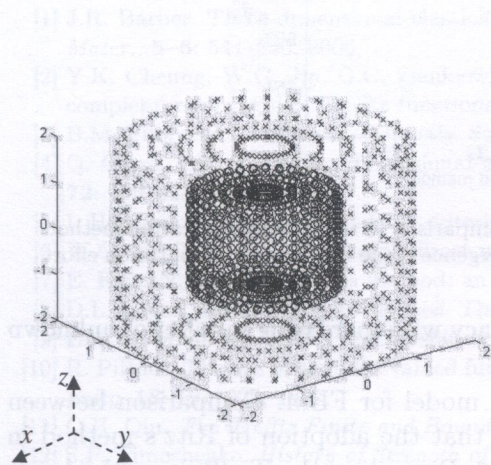


Fig. 9. Cylindrical model for MFS showing 400 field points and 400 source points

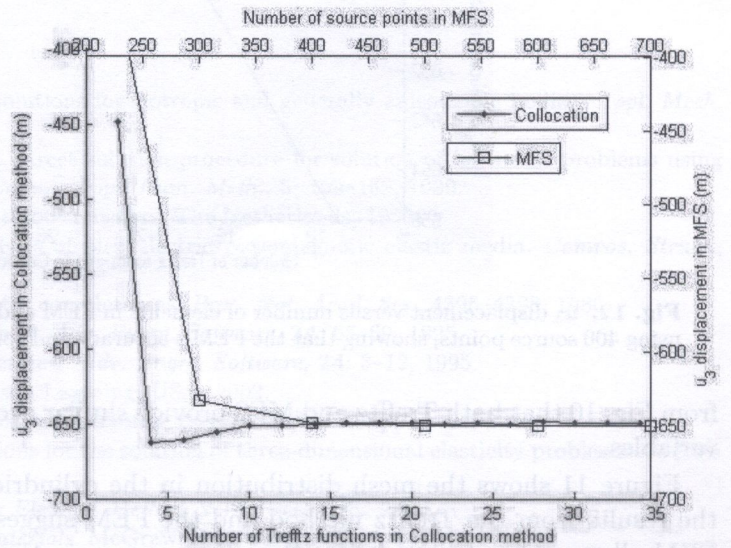


Fig. 10. u_3 displacement versus number of source points in MFS and comparison to the Trefftz collocation method (400 source points) showing that the MFS provides reasonable accuracy but poor robustness due to singularity problems

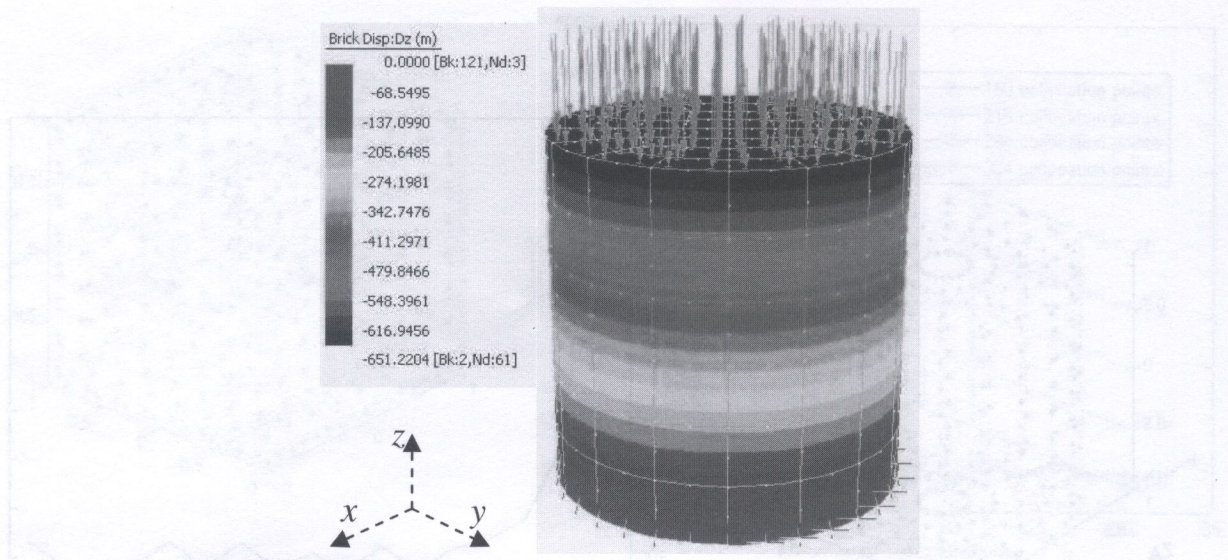


Fig. 11. Cylindrical model for FEM by Strand 7 showing axes and meshes

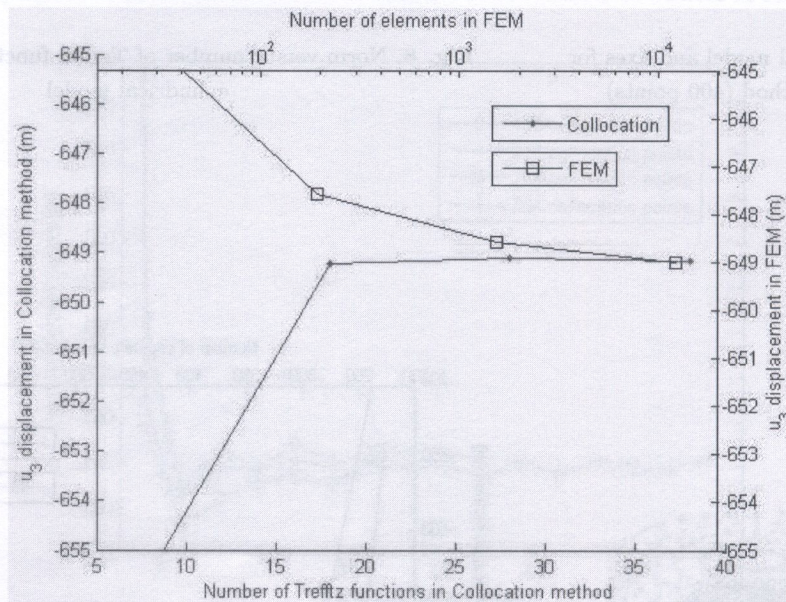


Fig. 12. u_3 displacement versus number of elements in FEM and comparison to the Trefftz collocation method using 400 source points, showing that the FEM's accuracy and convergence trade-off to huge computation effort

from Fig. 10 that both Trefftz and MFS provide similar accuracy with increasing number of unknown variables.

Figure 11 shows the mesh distribution in the cylindrical model for FEM. Comparison between the results from the Trefftz method and the FEM suggests that the adoption of Ritz's method in FEM allows upper bound error estimation as seen in Fig. 12. Regarding the Trefftz's method, not only the upper bound but also the lower bound error estimation can be achieved.

It is concluded that the derived Trefftz function for three-dimensional elasticity shows its fast convergence and stability with accuracy when comparing to the MFS and FEM. In addition, the use of collocation method will give higher stability and faster convergence with a few number of T-complete functions used as seen in Fig. 3 while the employment of Galerkin method in processing Trefftz solution requires less computation effort as the norm size is dramatically reduced as shown in Fig. 2, the logarithm plot.

7. SUMMARY

7.1. Conclusion

Trefftz functions are derived to numerically solve three-dimensional isotropic elasticity problems. The derived Trefftz functions are verified by the indirect Trefftz method on cubic and cylindrical models. Results suggest that when comparing to the MFS and FEM, the use of Trefftz functions show pseudo-stability, faster convergence and smaller error bounds even when using few Trefftz function terms.

7.2. Extensions

The work presented in this paper has made the several contributions to numerical analysis of both practical and theoretical values. As the Trefftz functions avoid the singularity problems that can emerge with fundamental solutions, results suggest that the derived Trefftz functions are suitable for applying to the Hybrid-Trefftz FEM and the Trefftz Boundary Element Method (T-BEM) to solve complex three-dimensional elasticity problems. In addition, the Trefftz functions for the three-dimensional isotropic elasticity problems provide an alternative approach to solving three-dimensional engineering problems. These approaches have applications for stress analysis involving voids, inclusions and three-dimensional cracks, potentially giving with more precise results with less computation effort.

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REFERENCES

- [1] J.R. Barber. Three-dimensional elasticity solutions for isotropic and generally anisotropic bodies. *Appl. Mech. Mater.*, **5–6**: 541–550, 2006.
- [2] Y.K. Cheung, W.G. Jin, O.C. Zienkiewicz. Direct solution procedure for solution of harmonic problems using complete, non-singular, Trefftz functions. *Comm. Appl. Num. Meth.*, **5**: 159–169, 1989.
- [3] B.M. Filonenko. *Theory of Elasticity*. Scientific Publishers, The Netherlands, 1959.
- [4] Q. Guan, S.R. He. Three-dimensional analysis of piezoelectric/piezomagnetic elastic media. *Compos. Struct.*, **72**: 419–428, 2006.
- [5] I. Herrera. Boundary methods: A criterion for completeness. *Proc. Nat. Acad. Sci.*, 4395–4398, 1980.
- [6] W.G. Jin, Y.K. Cheung. Trefftz direct method. *Adv. Engrg. Software*, **24**: 65–69, 1995.
- [7] E. Kita, N. Kamiya. Trefftz method: an overview. *Adv. Engrg. Software*, **24**: 3–12, 1995.
- [8] D.L. Logan. *Finite Element Method*. Thomson Learning, USA, 2002.
- [9] E.A.W. Maunder. Trefftz in Translation. *3rd International Trefftz Conference*, pp. 1–24, 2002.
- [10] R. Piltner. The use of complex valued functions for the solution of three-dimensional elasticity problems. *J. Elasticity*, **18**: 191–225, 1987.
- [11] Q.H. Qin. *The Trefftz Finite and Boundary Element Method*. WIT Press, UK, 2000.
- [12] S.P. Timoshenko. *History of Strength of Materials*. McGraw-Hill, USA, 1953.
- [13] E. Trefftz. A counterpart to Ritz's method. In: *Proceedings of the 2nd International Congress of Applied Mechanics*: 131–137, 1926.
- [14] Z.K. Wang, S.H. Huang. The general solution of three dimensional problems in piezoelectric media. *Int. J. Solids Struct.*, **32**: 105–115, 1995.
- [15] A.P. Zieliński. On trial functions applied in the generalized Trefftz method. *Adv. Engrg. Software*, **24**: 147–155, 1995.