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2D wave polynomials as base functions in modified FEM

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The paper presents solutions of a two-dimensional wave equation by using Trefftz functions. Two ways of obtaining different forms of these functions are shown. The first one is based on a generating function for the wave equation and leads to recurrent formulas for functions and their derivatives. The second one is based on a Taylor series expansion and additionally uses the inverse Laplace operator. Obtained wave functions can be used to solve the wave equation in the whole considered domain or can be used as base functions in FEM. For solving the problem three kinds of modified FEM are used: nodeless, continuous and discontinuous FEM. In order to compare the results obtained with the use of the aforementioned methods, a problem of membrane vibrations has been considered.

Keywords: Trefftz functions, wave functions, inverse operations, FEM

1. Introduction

The essence of the method of solving functions is to find a complete system of functions which satisfy identically the considered equation. Therefore this is a variant of Trefftz method [15]. An approximate solution is represented in the form of a linear combination of solving functions. Unknown coefficients of this approximation are sought from the minimization of the proper functional of fitting the approximate solution to the given conditions according to the accepted criterion. This method, for the first time described in [14], was dealing with a one dimensional heat conduction equation. It was extended to more dimensions not only in Cartesian coordinates in works [2, 5, 6]. In these works the derivation of Trefftz functions for the heat conduction equation was connected with the expansion into a power series of a so called generating function which solves this equation. The new technique of generating Trefftz functions for certain type of partial differential equations which uses the expansion of a function into a Taylor series and inverse operations was presented in [1, 2, 8, 11–13] for different coordinate systems. The obtained functions were applied as the base functions in FEM in the works [2, 7, 9, 11] to solve both direct as well as inverse problems of heat conduction. The use of this method gave satisfactory results and stable solutions. Since the technique of generating Trefftz functions for different type of equations is to a certain extent the same it seems reasonable to apply the function which satisfy the wave equation as a base function in FEM.

In the finite element method base functions in general do not satisfy the given differential equation. The use of Trefftz functions to the given equation implies that the functional for the finite element method will have a different form than the one presented in [15]. The application of a classical finite element method to solutions of partial differential equations assures the continuity of a function between the elements. If one omits this postulate it leads to the discontinuous Galerkin method. Three approaches to the modification of the classical FEM will be presented in this work.

The first approach, similarly as in [7], is based on an introduction of a partition of the domain in which the solution of a given problem is sought, into such subdomains where no nodes are introduced. In each of the elements the approximate solution is sought in terms of a linear combination of Trefftz functions for the wave equation.

The second modification of the classical FEM consists of an introduction of space-time elements in which the solution is sought as a liner combination of base functions with the unknown coefficients which are the values of the sought solution in nodes. The base functions depend on all variables and they satisfy governing equation. The unknown coefficients of the linear combination are determined by the minimization, in the mean-square sense, approximate solution to given conditions (initial, boundary). Additionally one should take into account the adjustment of solutions and their normal derivatives at the common boundaries between the neighbouring elements. Two variants of the presented method are considered in this work. One is based on the assumption of continuity of the function in the common for the neighbouring elements nodes and in the other this assumption is neglected.

2. The Trefftz functions for the 2D wave equation

Trefftz functions for the wave equation called wave functions or wave polynomials can be determined by different methods. The first method, described in [8] uses the so called generating function and leads to the recurrent formulas for wave functions (wave polynomials) and theirs derivatives. The same wave functions but in terms of explicit formulas can be obtained by first using Taylor expansion and next eliminating in this expansion the derivative with respect to time variable with the use of the governing equation. Other Trefftz functions (which are also polynomials) are described in [12, 13]. To derive these functions, apart from the Taylor expansion of the function satisfying the wave equation, additionally one uses the inverse Laplace operator. Both techniques of derivation of wave functions will be shortly discussed.

2.1. Derivation of wave function with the use of a generating function

Let us consider the 2D non-dimensional wave equation, as in [8],

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial t^2}. \tag{1}$$

The generating function

$$e^{i(ax+by+ct)} \tag{2}$$

which satisfies Eq. (1) is extended into a power series with respect to the unknown parameters. After the substitution $c^2 = a^2 + b^2$ one gets the following form of the expansion

$$e^{i(ax+by+ct)} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{\substack{l=0 \\ l<2}}^{n-k} R_{(n-k-l)kl}(x,y,t)\, a^{n-k-l} b^k c^l \tag{3}$$

in which both the real part (4) as well as the imaginary part (5) of the function $R(x,y,t)$ satisfy the wave equation

$$P_{(n-k-l)kl}(x,y,t) = \operatorname{Re}\left(R_{(n-k-l)kl}(x,y,t)\right), \tag{4}$$

$$Q_{(n-k-l)kl}(x,y,t) = \operatorname{Im}\left(R_{(n-k-l)kl}(x,y,t)\right). \tag{5}$$

In this way one gets two linearly independent sequences of functions called wave polynomials. Differentiation of the generating function (2) with respect to the variable. x and the comparison of the appropriate coefficients at the same powers leads to the recurrent formulas for wave polynomials [8],

$$\begin{aligned}
P_{000}(x,y,t) &= 1,\\
Q_{000}(x,y,t) &= 0,\\
P_{(n-k)k0}(x,y,t) &= \frac{1}{n}\big(-xQ_{(n-k-1)k0}(x,y,t) - yQ_{(n-k)(k-1)0}(x,y,t)\\
&\qquad - tQ_{(n-k-2)k1}(x,y,t) - tQ_{(n-k)(k-2)1}(x,y,t)\big),\\
P_{(n-k-1)k1}(x,y,t) &= \frac{1}{n}\big(-xQ_{(n-k-2)k1}(x,y,t) - yQ_{(n-k-1)(k-1)1}(x,y,t)\\
&\qquad - tQ_{(n-k-1)k0}(x,y,t)\big),\\
Q_{(n-k)k0}(x,y,t) &= \frac{1}{n}\big(xP_{(n-k-1)k0}(x,y,t) + yP_{(n-k)(k-1)0}(x,y,t)\\
&\qquad + tP_{(n-k-2)k1}(x,y,t) + tP_{(n-k)(k-2)1}(x,y,t)\big),\\
Q_{(n-k-1)k1}(x,y,t) &= \frac{1}{n}\big(xP_{(n-k-2)k1}(x,y,t) + yP_{(n-k-1)(k-1)1}(x,y,t)\\
&\qquad + tP_{(n-k-1)k0}(x,y,t)\big),
\end{aligned}\tag{6}$$

and on their derivatives [9],

$$\begin{aligned}
\frac{\partial P_{(n-k-l)kl}(x,y,t)}{\partial x} &= -Q_{(n-k-l-1)kl}(x,y,t),\\
\frac{\partial P_{(n-k-l)kl}(x,y,t)}{\partial y} &= -Q_{(n-k-l)(k-1)l}(x,y,t),\\
\frac{\partial Q_{(n-k-l)kl}(x,y,t)}{\partial x} &= P_{(n-k-l-1)kl}(x,y,t),\\
\frac{\partial Q_{(n-k-l)kl}(x,y,t)}{\partial y} &= P_{(n-k-l)(k-1)l}(x,y,t),\\
\frac{\partial P_{(n-k)k0}(x,y,t)}{\partial t} &= -Q_{(n-k-2)k1}(x,y,t) - Q_{(n-k)(k-2)1}(x,y,t),\\
\frac{\partial P_{(n-k-1)k1}(x,y,t)}{\partial t} &= -Q_{(n-k-1)k0}(x,y,t),\\
\frac{\partial Q_{(n-k)k0}(x,y,t)}{\partial t} &= P_{(n-k-2)k1}(x,y,t) + P_{(n-k)(k-2)1}(x,y,t),\\
\frac{\partial Q_{(n-k-1)k1}(x,y,t)}{\partial t} &= P_{(n-k-1)k0}(x,y,t).
\end{aligned}\tag{7}$$

2.2. Derivation of wave functions with the use of inverse operations

2.2.1. Taylor series expansion

To obtain a different form of a Trefftz function [12, 13], one uses an expansion into a Taylor series of the solution $w = w(x,y,t)$ of Eq. (1) in the vicinity of a given point $(x_0\,,\, y_0\,,\, t_0)$,

$$w(x,y,t) = w(x_0\,,y_0\,,t_0) + \sum_{n=1}^{\infty}\frac{d^n w(x_0\,,y_0\,,t_0)}{n!}(x-x_0\,,\,y-y_0\,,\,t-t_0),\tag{8}$$

where

$$d^n w(x_0\,, y_0\,, t_0)(x-x_0\,, y-y_0\,, t-t_0) = \left(\frac{\partial}{\partial x}(x-x_0) + \frac{\partial}{\partial y}(y-y_0) + \frac{\partial}{\partial t}(t-t_0)\right)^n w\Big|_{(x_0,y_0,t_0)}. \tag{9}$$

In this expansion one eliminates the derivative $\frac{\partial^2 w}{\partial t^2}$ using the Eq. (1). The obtained form of the expansion [8] includes as coefficients the functions which identically satisfy the wave equation. These coefficients are the same as those obtained with the help of a generating function. However, if apart from the elimination of the time derivative, we perform transformations leading to the extraction of Δw in the expansion (8), then we obtain the following form of the expansion

$$\begin{aligned} w(x,y,t) &= w_0 + \frac{\partial w}{\partial x}\bar{x} + \frac{\partial w}{\partial y}\bar{y} + \frac{\partial w}{\partial t}\bar{t} \\ &+ \frac{\partial^2 w}{\partial x^2}\left(\frac{\bar{x}^2}{2!} - \frac{\bar{y}^2}{2!}\right) + \frac{\partial^2 w}{\partial x \partial y}\bar{x}\bar{y} + \frac{\partial^2 w}{\partial x \partial t}\bar{x}\bar{t} + \frac{\partial^2 w}{\partial y \partial t}\bar{y}\bar{t} + \Delta w\left(\frac{\bar{t}^2}{2!} + \frac{\bar{y}^2}{2!}\right) \\ &+ \frac{\partial^3 w}{\partial x^3}\left(\frac{\bar{x}^3}{3!} - \frac{\bar{x}\bar{y}^2}{2!}\right) + \frac{\partial^3 w}{\partial x^2 \partial y}\left(\frac{\bar{x}^2\bar{y}}{2!} - \frac{\bar{y}^3}{3!}\right) + \frac{\partial^3 w}{\partial x^2 \partial t}\bar{t}\left(\frac{\bar{x}^2}{2!} - \frac{\bar{y}^2}{2!}\right) + \frac{\partial^3 w}{\partial x \partial y \partial t}\bar{t}\bar{x}\bar{y} \\ &+ \frac{\partial}{\partial t}\Delta w\left(\frac{\bar{t}^3}{3!} + \bar{t}\frac{\bar{y}^2}{2!}\right) + \frac{\partial}{\partial x}\Delta w\left(\frac{\bar{t}^2}{2!}\bar{x} + \frac{\bar{x}\bar{y}^2}{2!}\right) + \frac{\partial}{\partial y}\Delta w\left(\frac{\bar{t}^2}{2!}\bar{y} + \frac{\bar{y}^3}{3!}\right) + \cdots \end{aligned} \tag{10}$$

where $\bar{x} = x - x_0\,, \bar{y} = y - y_0\,, \bar{t} = t - t_0$.

Now, in this expansion, the coefficients at partial derivatives evaluated at point $(x_0\,, y_0\,, t_0)$ are functions which satisfy identically Eq. (1), hence these are Trefftz functions. They are different from functions obtained by the previous method. Moreover, the derivatives $\frac{\partial^n w}{\partial x^n}\,, \frac{\partial^{n-1}}{\partial x^{n-1}}\left(\frac{\partial w}{\partial y}\right)$ are multiplied by the known harmonic functions (11) and (12), respectively,

$$F_n(x,y) = \mathrm{Re}\left[\frac{(x+iy)^n}{n!}\right] = \sum_{k=0,2,\ldots}^{n \geq k} (-1)^{\frac{k}{2}} \frac{x^{n-k}}{(n-k)!}\frac{y^k}{k!}\,, \qquad n \geq 0, \tag{11}$$

$$G_n(x,y) = \mathrm{Im}\left[\frac{(x+iy)^n}{n!}\right] = \sum_{k=1,3,\ldots}^{n \geq k} (-1)^{\frac{k-1}{2}} \frac{x^{n-k}}{(n-k)!}\frac{y^k}{k!}\,, \qquad n \geq 1. \tag{12}$$

The functions derived in this way are different from those obtained with the help of the generating function and can be expressed by the inverse operations [12, 13].

2.2.2. An inverse Laplace operator for harmonic functions

Since each of the harmonic functions $F_n(x,y)$, $G_n(x,y)$ can be represented as the sum of monomials of the form $\frac{x^m}{m!}\frac{y^k}{k!}$ hence to calculate inverse operations of harmonic functions it is enough to know $\Delta^{-n}\left(\frac{x^m}{m!}\frac{y^k}{k!}\right)$.

An inverse Laplace operator of monomial $\frac{x^m}{m!}\frac{y^k}{k!}$ can be defined in the following manner [2],

$$\Delta^{-1}\left(\frac{x^m}{m!}\frac{y^k}{k!}\right) = \begin{cases} \dfrac{x^m}{m!}\dfrac{y^{k+2}}{(k+2)!} & m=0,\, m=1,\, k \geq 0 \\ \dfrac{x^m}{m!}\dfrac{y^{k+2}}{(k+2)!} - \Delta^{-1}\left(\dfrac{x^{m-2}}{(m-2)!}\dfrac{y^{k+2}}{(k+2)!}\right) & m \geq 2,\, k \geq 0. \end{cases} \tag{13}$$

Harmonic functions $F_n(x,y)$, $G_n(x,y)$ are symmetrical with respect to variables x and y, whereas calculations of successive inverse operations in accordance with the formula (13) distinguish the variable y (observe that $\Delta^{-1}(1) = \frac{y^2}{2!}$). By symmetry of Δ with respect to the variables x, y it is possible to define Δ^{-1} which distinguishes the variable x. Both possibilities are shown in [11–13].

2.2.3. Representation of wave functions in term of inverse Laplace operations

The Taylor series expansion (10) can be represented with the help of inverse operations in the following way,

$$
\begin{aligned}
w(x,y,t) = {} & w_0 1 + \frac{\partial w}{\partial x}\bar{x} + \frac{\partial w}{\partial t}\bar{t} \\
& + \frac{\partial^2 w}{\partial x^2}F_2 + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial t}\right) tF_1 + \Delta w\left(\frac{t^2}{2!}F_0 + \Delta^{-1}F_0\right) \\
& + \frac{\partial^3 w}{\partial x^3}F_3 + \frac{\partial^2}{\partial x^2}\left(\frac{\partial w}{\partial t}\right) tF_2 + \frac{\partial}{\partial x}\Delta w\left(\frac{t^2}{2!}F_1 + \Delta^{-1}F_1\right) \\
& + \Delta w\left(\frac{\partial w}{\partial t}\right)\left(\frac{t^3}{3!}F_0 + t\,\Delta^{-1}F_0\right) \\
& + \frac{\partial^4 w}{\partial x^4}F_4 + \frac{\partial^3}{\partial x^3}\left(\frac{\partial w}{\partial t}\right) tF_3 + \frac{\partial^2}{\partial x^2}\Delta w\left(\frac{t^2}{2!}F_2 + \Delta^{-1}F_2\right) \\
& + \frac{\partial}{\partial x}\left(\Delta\frac{\partial w}{\partial t}\right)\left(\frac{t^3}{3!}F_1 + t\Delta^{-1}F_1\right) + \Delta\Delta w\left(\frac{t^4}{4!}F_0 + \frac{t^2}{2!}\Delta^{-1}F_0 + \Delta^{-2}F_0\right) + \cdots \\
& + \frac{\partial w}{\partial y}G_1 + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)G_2 + \frac{\partial^2 w}{\partial t\partial y}tG_1 \\
& + \frac{\partial^2}{\partial x^2}\left(\frac{\partial w}{\partial y}\right)G_3 + \frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial t\partial y}\right)tG_2 + \Delta\left(\frac{\partial w}{\partial y}\right)\left(\frac{t^2}{2!}G_1 + \Delta^{-1}G_1\right) \\
& + \frac{\partial^3}{\partial x^3}\left(\frac{\partial w}{\partial y}\right)G_4 + \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 w}{\partial t\partial y}\right)tG_3 + \frac{\partial}{\partial x}\Delta\left(\frac{\partial w}{\partial y}\right)\left(\frac{t^2}{2!}G_2 + \Delta^{-1}G_2\right) \\
& + \Delta u\left(\frac{\partial^2 w}{\partial t\partial y}\right)\left(\frac{t^3}{3!}G_1 + t\Delta^{-1}G_1\right) \\
& + \frac{\partial^4}{\partial x^4}\left(\frac{\partial w}{\partial y}\right)G_5 + \frac{\partial^3}{\partial x^3}\left(\frac{\partial^2 w}{\partial t\partial y}\right)tG_4 + \frac{\partial^2}{\partial x^2}\Delta\left(\frac{\partial w}{\partial y}\right)\left(\frac{t^2}{2!}G_3 + \Delta^{-1}G_3\right) \\
& + \frac{\partial}{\partial x}\Delta\left(\frac{\partial^2 w}{\partial t\partial y}\right)\left(\frac{t^3}{3!}G_2 + t\Delta^{-1}G_2\right) + \cdots \\
= {} & \sum_{n=0}^{N}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{\partial^{n-2k}}{\partial x^{n-2k}}(\Delta^k w)\left(\sum_{j=0}^{k}\frac{t^{2j}}{(2j)!}\Delta^{-k+j}F_{n-2k}(x,y)\right) \\
& + \sum_{n=1}^{N}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\frac{\partial^{n-1-2k}}{\partial x^{n-1-2k}}\left(\Delta^k\frac{\partial w}{\partial t}\right)\left(\sum_{j=0}^{k}\frac{t^{2j+1}}{(2j+1)!}\Delta^{-k+j}F_{n-1-2k}(x,y)\right) \\
& + \sum_{n=1}^{N}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\frac{\partial^{n-1-2k}}{\partial x^{n-1-2k}}\left(\Delta^k\frac{\partial w}{\partial y}\right)\left(\sum_{j=0}^{k}\frac{t^{2j}}{(2j)!}\Delta^{-k+j}G_{n-2k}(x,y)\right) \\
& + \sum_{n=2}^{N}\sum_{k=0}^{\left[\frac{n-2}{2}\right]}\frac{\partial^{n-2-2k}}{\partial x^{n-2-2k}}\left(\Delta^k\frac{\partial w}{\partial t\partial y}\right)\left(\sum_{j=0}^{k}\frac{t^{2j+1}}{(2j+1)!}\Delta^{-k+j}G_{n-1-2k}(x,y)\right) \\
& + R_{N+1}\,.
\end{aligned}
\tag{14}
$$

The form of the expansion obtained in this way consists of two independent sequences of wave polynomials, expressed through the inverse operation of harmonic functions. Obviously, one can eliminate the other derivatives in the Taylor expansion (8) and for each of such eliminations there exists a possibility of calculating the inverse operations from harmonic functions which distinguish one of the variables x or y. All these possibilities were discussed in the work [12, 13].

3. Wave functions as base functions in FEM

Trefftz functions for the wave equation can serve in the construction of base functions in FEM. The general idea of such an approach is the same regardless of the FEM type.

Let us consider the non-dimensional wave equation:

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}, \tag{15}$$

in the bounded space-time domain $D \times \langle 0, \tau \rangle$. Additionally the following conditions are given,

$$w(x, y, 0) = w_0(x, y), \tag{16}$$

$$\frac{\partial w(x, y, 0)}{\partial t} = w_0(x, y), \tag{17}$$

$$w(x, y, t)|_{(x,y)\in\Gamma} = w_\Gamma(x, y, t), \tag{18}$$

where Γ is the boundary of D. We introduce the mesh division of the domain. We divide the domain into subdomains (rectangles, triangles, etc.) Ω_j, $j = 1, 2, 3, \ldots J$, and the time interval into intervals $\langle k\Delta t,\ (k+1)\Delta t\rangle$, $k = 0, 1, \ldots, K$.

A solution of the Eq. (15) is sought in successive time intervals $\langle k\Delta t,\ (k+1)\Delta t\rangle$, $k = 0, 1, \ldots, K$. We introduce a local system of coordinates in the space-time domains $\Omega_j \times \langle k\Delta t,\ (k+1)\Delta t\rangle$. Because of the property of wave polynomials (the Runge phenomenon of waving of polynomials at the boundary) the division is in such a way so that the relatively high degree of approximation in the element was kept , and so that the matrices which form the system of equations were not badly conditioned. In a local system of coordinate we perform the rescaling of this system in such a way so that the space — time coordinates in the element do not exceed one. The division of the domain into elements is the same regardless of the type of FEM. In the element with the number j the approximate solution of Eq. (15) is represented as a linear combination of base functions satisfying Eq. (15) identically. Determination of the unknown coefficients of such linear combinations and the use of base functions differ from each over depending on the type of FEM.

3.1. Nodeless FEM

In the nodeless method after partition of a domain into the space-time elements, in each element with number j one seeks an approximate solution in the form of a linear combination of base functions which are wave polynomials

$$u_j(x, y, t) = \sum_{n=1}^{N} A_{jn} v_n(x, y, t). \tag{19}$$

The unknown coefficients in this combination A_{jn} are determined through the minimization of the functional which takes into account mean-squared fitting of the solution approximated to:

- the initial condition (16), (17),
- the boundary conditions (18),
- the values of the approximate solution and its derivative normal between the elements [2].

The undoubted advantage of this approach is the fact that the number of base functions taken to approximate the approximated solution in the element is not bounded by the number of element's nodes. Therefore it is possible to divide the domain into a few "large" elements. In each of these elements the approximation of the solution is done by the polynomial of possible highest degree (up to the given differential). The drawback of this approach is the lack of continuity of the approximate solution between the elements. Moreover, increasing the number of functions which are used to in the approximation, creates a numerical problem.

3.2. Continuity FEM

In the continuity FEM after partition of a domain into the space-time elements, in each of these elements we introduce the mesh of nodes location — for each element the same. The approximate solution in each element is expressed by the relation

$$w_j(x,y,t) \approx u_j(x,y,t) = \sum_{n=1}^{N} A_{jn} v_n(\bar{x},\bar{y},\bar{t}). \tag{20}$$

Moreover, it is assumed that in the nodes of neighbouring elements the approximate solution is continuous between the elements. The lack of the full continuity between the elements results from a finite number of base functions in each finite element.

The procedure of determining the constants A_{jn} and base functions is the same as the one presented in [2] for the equation of heat condition. It is assumed that for the fixed element (with number j) one knows the values of sought function $u_{j1}, u_{j2}, \ldots, u_{jN}$ in nodes of the element $P_1 = (x_1, y_1, t_1), \ldots, P_N = (x_N, y_N, t_N)$. The coefficients A_{jn} are determined with the solution of the system of equations

$$\begin{bmatrix} v_1(\bar{x}_1,\bar{y}_1,\bar{t}_1) & v_2(\bar{x}_1,\bar{y}_1,\bar{t}_1) & \cdots & v_N(\bar{x}_1,\bar{y}_1,\bar{t}_1) \\ v_1(\bar{x}_2,\bar{y}_2,\bar{t}_2) & v_2(\bar{x}_2,\bar{y}_2,\bar{t}_2) & \cdots & v_N(\bar{x}_2,\bar{y}_2,\bar{t}_2) \\ \vdots & \vdots & \ddots & \vdots \\ v_1(\bar{x}_N,\bar{y}_N,\bar{t}_N) & v_2(\bar{x}_N,\bar{y}_N,\bar{t}_N) & \cdots & v_N(\bar{x}_N,\bar{y}_N,\bar{t}_N) \end{bmatrix} \begin{bmatrix} A_{j1} \\ A_{j2} \\ \vdots \\ A_{jN} \end{bmatrix} = \begin{bmatrix} u_{j1} \\ u_{j2} \\ \vdots \\ u_{jN} \end{bmatrix} \tag{21}$$

or in the matrix form

$$[v][A] = [U]. \tag{22}$$

Therefore, if the matrix $[v]$ is non-singular, then to $[A] = [v]^{-1}[U] = [V][U]$,

$$A_n = \sum_{i=1}^{N} V_{ni} u_i \,. \tag{23}$$

After substitution of Eq. (23) into Eq. (20) we get for the element

$$w_j(x,y,t) \approx \sum_{n=1}^{N}\left(\sum_{i=1}^{N} V_{ni}u_{ji}\right) v_{jn}(x,y,t) = \sum_{i=1}^{N}\left(\sum_{n=1}^{N} V_{ni}v_n(x,y,t)\right) u_j = \sum_{i=1}^{N} \varphi_{ji}(x,y,t) u_{ji} \tag{24}$$

where

$$\varphi_{jk}(\bar{x},\bar{y},\bar{t}) = \sum_{n=1}^{N} V_{nk} v_n(\bar{x},\bar{y},\bar{t}). \tag{25}$$

Base functions φ_{jk} have the following properties connected with the properties of wave functions,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{\partial}{\partial t}\right)\varphi_{jk}(\bar{x},\bar{y},\bar{t})=0,$$

$$\varphi_{jk}(\bar{x}_m\,,\,\bar{y}_m\,,\,\bar{t}_m)=\begin{cases}1 & k=m,\\ 0 & k\neq m,\end{cases}\qquad (\bar{x}_m\,,\,\bar{y}_m\,,\,\bar{t}_m)\ \text{– nodes of element},\quad k=1,\ldots,N,\quad m=1,\ldots,M.$$

We assume the approximation (24) of the exact solution in each element. To determine the values of function u_{jn} in the element's nodes one constructs the functional which fits the approximate solution

- to the initial condition (16), (17),
- to the boundary conditions (18),
- to the fitting a boundary condition at the common interfaces (fitting of solutions and their normal derivatives between the elements).

It is worthwhile to notice that part of the value u_{jn} is known from the given condition of a problem (the initial-boundary conditions).

3.3. Discontinuity FEM

A general idea of using Trefftz functions as base functions in the discontinuous FEM is similar to the previous case. The essence of this approach relays on resignation from the continuity of the approximate solution in the element's nodes. Analogously to the previous case, the space-time domain in which one seeks a solution, is divided into elements. In each element the solution is sought in the form of the following linear combination,

$$w_j(x,y,t)\approx\sum_{i=1}^{N}\varphi_{ji}(x,y,t)u_{ji}\,. \tag{26}$$

Similarly to the previous case, one constructs base functions in an arbitrary element by solving the system of equations (21). The unknown coefficients are in this case the values of sought functions in the element's nodes. They are determined in the analogous way as in nodeless FEM by the minimization of the appropriate functional. This functional adjusts approximation to given initial and boundary conditions (in mean square sense). Moreover, it fits solutions (in elements) and their derivatives in the common edge of elements. Because of the lack of the continuity in nodes between neighbouring elements, the search for the approximate solution requires finding a double number of the unknowns than in the case of a continuous FEM. This causes the increase of the calculation time.

4. Numerical example

The effectiveness of presented methods is tested on the example describing vibrations of a squared membrane which are described by the following relations. The two-dimensional wave equation

$$\frac{\partial^2 w}{\partial t^2}=\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\,,\qquad (x,y)\in(0,1)\times(0,1),\quad t\geq 0, \tag{27}$$

together with conditions

$$w(0,y,t)=w(1,y,t)=w(x,0,t)=w(x,1,t)=0, \tag{28}$$

$$w(x,y,0)=xy(x-1)(y-1),\qquad \frac{\partial w}{\partial t}(x,y,0)=0, \tag{29}$$

has the exact solution given by [10]

$$w(x,y,t) = \frac{64}{\pi^6}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\sin(\pi x(2n+1))\,\sin(\pi y(2m+1))\,\cos\left(\pi t\sqrt{(2n+1)^2+(2m+1)^2}\right)}{(2n+1)^3(2m+1)^3}. \tag{30}$$

An approximate solution of the problem (27)–(29) is obtained by three variants of Finite Element Method.

4.1. Nodeless FEM

Considered area was divided into smaller subareas (cuboid shape). In each subarea an approximate solution is sought as a linear combination of wave functions (wave polynomials). This combination consists of all polynomials up to the required degree. Unknown coefficients of this linear combination are determined by minimization (in the mean-square sense) of the difference between an approximate solution and given initial-boundary conditions. Moreover, the difference between solutions in the neighbouring elements and their normal derivatives on the common edge must be take into account.

Figure 1 shows the exact and the approximate solution obtained at the middle of a membrane, i.e. at point $x = y = 0.5$, for different time intervals Δt when the area was divided into 16 subareas. For getting an approximate solution, 25 wave functions are used. It means the approximation of polynomials of degree up to 4.

The error of approximation in the maximum norm was calculated in successive time intervals for a different length of the time step, what shows Table 1.

The content of Table 1 shows a good accuracy of the approximation solution given by the use of the nodeless FEM. Quite a small error is achieved for a relatively small number of elements. To get the same result we can increase the accuracy of the approximate solution in elements (more wave functions) or increase the number of elements.

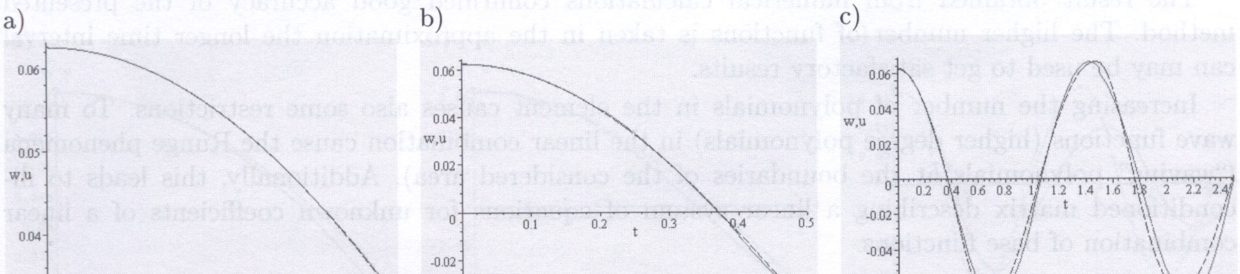


Fig. 1. The exact solution (broken line) and the approximate one (continuous line) at point $x = y = 0.5$ obtained by dividing considered area into 16 elements for different time intervals: $\Delta t = 0.05$ (a), $\Delta t = 0.1$ (b), $\Delta t = 0.5$ (c)

Table 1. The error of approximation in the maximum norm for a different length of the time step obtained by dividing considered area into 16 elements

	0–Δt	Δt–$2\Delta t$	$2\Delta t$–$3\Delta t$	$3\Delta t$–$4\Delta t$	$4\Delta t$–$5\Delta t$
$\Delta t = 0.005$	$1.04\cdot10^{-6}$	$1.09\cdot10^{-6}$	$1.15\cdot10^{-6}$	$1.48\cdot10^{-6}$	$2.33\cdot10^{-6}$
$\Delta t = 0.05$	$3\cdot10^{-6}$	$1.8\cdot10^{-5}$	$4.4\cdot10^{-5}$	$2.5\cdot10^{-5}$	$1.7\cdot10^{-4}$
$\Delta t = 0.1$	$8.25\cdot10^{-5}$	$9\cdot10^{-5}$	$2.5\cdot10^{-4}$	$6.5\cdot10^{-4}$	$1.6\cdot10^{-3}$
$\Delta t = 0.5$	$5.5\cdot10^{-3}$	$8\cdot10^{-3}$	$4.3\cdot10^{-3}$	$1.2\cdot10^{-2}$	$1.4\cdot10^{-2}$

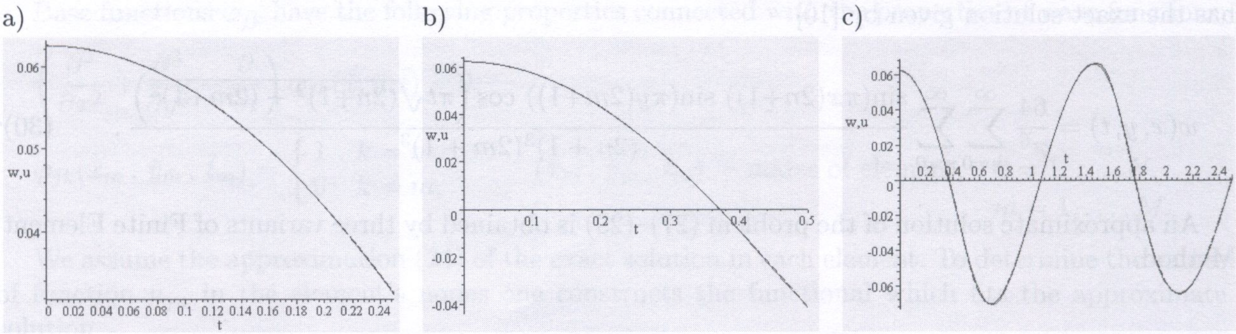


Fig. 2. The exact solution (broken line) and the approximate one (continuous line) at point $x = y = 0.5$ obtained by dividing considered area into 4 elements for different time intervals: $\Delta t = 0.05$ (a), $\Delta t = 0.1$ (b), $\Delta t = 0.5$ (c)

Table 2. The error of approximation in the maximum norm for a different length of the time step obtained by dividing considered area into 4 elements

	0–Δt	Δt–$2\Delta t$	$2\Delta t$–$3\Delta t$	$3\Delta t$–$4\Delta t$	$4\Delta t$–$5\Delta t$
$\Delta t = 0.005$	$2.8 \cdot 10^{-6}$	$2.5 \cdot 10^{-6}$	$2.4 \cdot 10^{-6}$	$1.2 \cdot 10^{-6}$	$4.5 \cdot 10^{-6}$
$\Delta t = 0.05$	$7.6 \cdot 10^{-6}$	$4.5 \cdot 10^{-6}$	$5.8 \cdot 10^{-5}$	$7 \cdot 10^{-5}$	$1.6 \cdot 10^{-4}$
$\Delta t = 0.1$	$9.5 \cdot 10^{-6}$	$5.1 \cdot 10^{-5}$	$1.6 \cdot 10^{-4}$	$3.5 \cdot 10^{-4}$	$6.2 \cdot 10^{-4}$
$\Delta t = 0.5$	$1 \cdot 10^{-3}$	$1.6 \cdot 10^{-3}$	$1.4 \cdot 10^{-3}$	$2 \cdot 10^{-3}$	$1.4 \cdot 10^{-3}$

Figure 2 shows the exact and the approximate solution obtained at the middle of a membrane i.e. at point $x = y = 0.5$, for different length time intervals Δt, and for only 4 elements. The approximate solution is given as the linear combination of 81 wave functions. It means that polynomials up to degree 8 are used.

As above for 4 elements, the error of approximation in the maximum norm was calculated in successive time intervals for a different length of the time step, what shows Table 2.

The result obtained from numerical calculations confirmed good accuracy of the presented method. The higher number of functions is taken in the approximation the longer time interval can may be used to get satisfactory results.

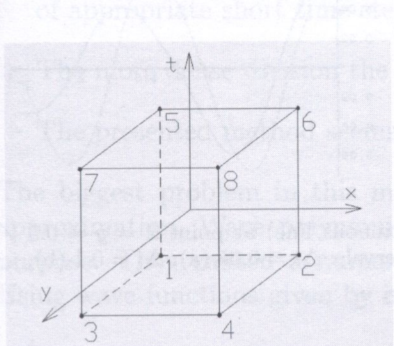
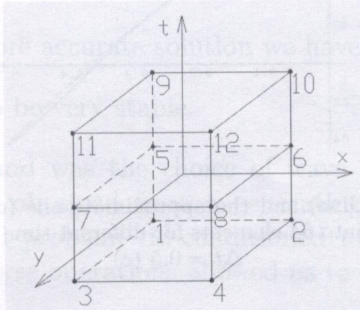
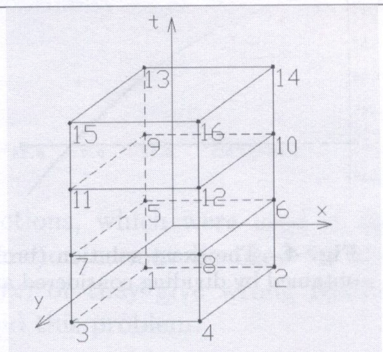
Increasing the number of polynomials in the element causes also some restrictions. To many wave functions (higher degree polynomials) in the linear combination cause the Runge phenomena ("waving" polynomials at the boundaries of the considered area). Additionally, this leads to ill-conditioned matrix describing a linear system of equations for unknown coefficients of a linear combination of base functions.

4.2. Continuity FEM

The considered area is divided into space-time elements. In each of them a local coordinates system as well as a mesh of nodes distribution are introduced. Base functions are created for each element. Their number depends on the number of nodes in each element (the same for different elements). Linear combinations of base functions (in which coefficients are unknown values of the function at nodes of the element) became the approximate solution of the considered problem in the element. Examples of nodes location in element are shown in Table 3.

The problem (27)–(29) is solved in a sequential manner in consecutive time intervals. In the first time interval the initial condition has the form (29), whereas in the next time step the initial condition is given as the value of the approximate solution at the moment ending previous time interval.

Table 3. Location of nodes in the element

8 nodes in the element	12 nodes in the element	16 nodes in the element
		
Degree of the approximate solution: – with respect to time value $\sim t$ – with respect to space value $\sim x^2$	Degree of the approximate solution: – with respect to time value $\sim t^2$ – with respect to space value $\sim x^4$	Degree of the approximate solution: – with respect to time value $\sim t^3$ – with respect to space value $\sim x^6$

The unknown coefficients (the values of the function at nodes of the element) are determined by a minimization of the functional which adjust (in mean-square sense) an approximate solution to initial and boundary conditions. This functional also fits solutions and their derivatives in the common edge of elements.

Figure 3 shows the approximate solution in case of 16 elements. Each element has 12 nodes (polynomials of degree up to 4 are used). All calculations are made for different time intervals in successive five time intervals.

Similarly as for the previous method the error of the approximation in the maximum norm was calculated in successive time intervals for a different length of the time step, what shows Table 4.

a) b) c)

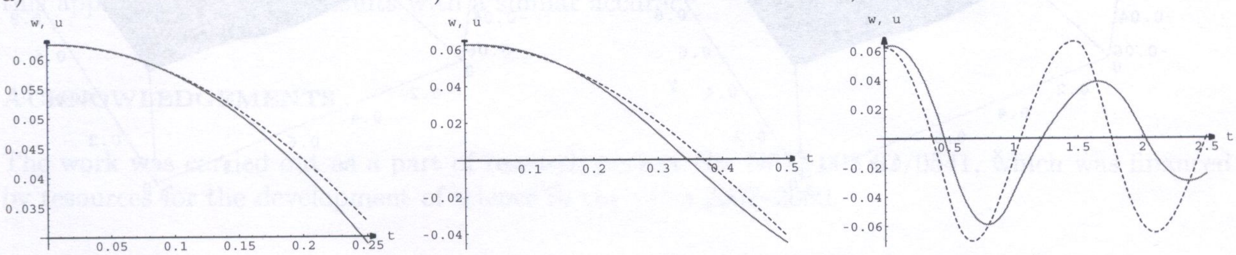


Fig. 3. The exact solution (broken line) and the approximate one (continuous line) at point $x = y = 0.5$ obtained by dividing considered area into 16 elements for different time intervals: $\Delta t = 0.05$ (a), $\Delta t = 0.1$ (b), $\Delta t = 0.5$ (c)

Table 4. The error of approximation in the maximum norm for a different length of the time step obtained by dividing considered area into 16 elements

	0–Δt	Δt–$2\Delta t$	$2\Delta t$–$3\Delta t$	$3\Delta t$–$4\Delta t$	$4\Delta t$–$5\Delta t$
$\Delta t = 0.005$	$2.08 \cdot 10^{-6}$	$9.17 \cdot 10^{-6}$	$2.15 \cdot 10^{-5}$	$3.91 \cdot 10^{-5}$	$6.23 \cdot 10^{-5}$
$\Delta t = 0.05$	$1.09 \cdot 10^{-4}$	$8.55 \cdot 10^{-5}$	$7.63 \cdot 10^{-4}$	$1.95 \cdot 10^{-3}$	$3.67 \cdot 10^{-3}$
Δt 0.1	$7.1 \cdot 10^{-4}$	$1.3 \cdot 10^{-3}$	$6.02 \cdot 10^{-3}$	$8.85 \cdot 10^{-3}$	$8.85 \cdot 10^{-3}$
$\Delta t = 0.5$	$2.9 \cdot 10^{-2}$	$3.0 \cdot 10^{-2}$	$4.28 \cdot 10^{-2}$	$6.5 \cdot 10^{-2}$	$6.33 \cdot 10^{-2}$

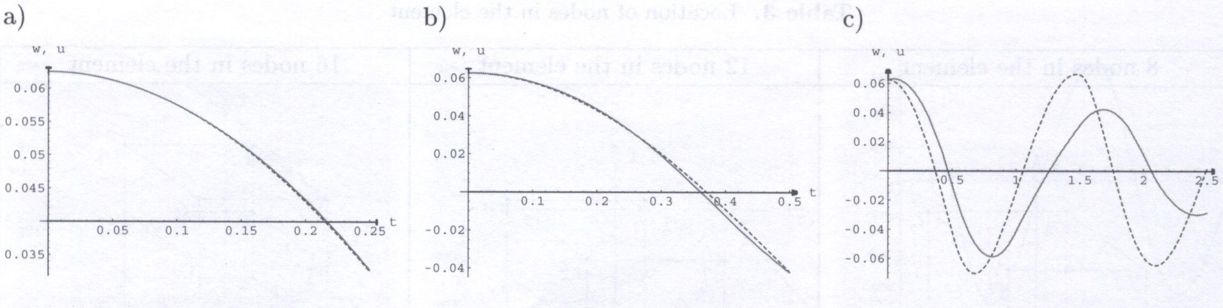


Fig. 4. The exact solution (broken line) and the approximate one (continuous line) at point $x = y = 0.5$ obtained by dividing considered area into 64 elements for different time intervals: $\Delta t = 0.05$ (a), $\Delta t = 0.1$ (b), $\Delta t = 0.5$ (c)

Table 5. The error of approximation in the maximum norm for a different length of the time step obtained by dividing considered area into 64 elements

	0–Δt	Δt–$2\Delta t$	$2\Delta t$–$3\Delta t$	$3\Delta t$–$4\Delta t$	$4\Delta t$–$5\Delta t$
$\Delta t = 0.005$	$4.22 \cdot 10^{-7}$	$1.79 \cdot 10^{-6}$	$3.9 \cdot 10^{-6}$	$6.87 \cdot 10^{-6}$	$1.08 \cdot 10^{-5}$
$\Delta t = 0.05$	$7.26 \cdot 10^{-6}$	$4.14 \cdot 10^{-7}$	$1.17 \cdot 10^{-4}$	$1.17 \cdot 10^{-4}$	$3.11 \cdot 10^{-4}$
$\Delta t = 0.1$	$2.95 \cdot 10^{-4}$	$6.84 \cdot 10^{-4}$	$8.86 \cdot 10^{-4}$	$3.47 \cdot 10^{-3}$	$3.59 \cdot 10^{-3}$
$\Delta t = 0.5$	$3.24 \cdot 10^{-2}$	$3.25 \cdot 10^{-2}$	$4.75 \cdot 10^{-2}$		$6.98 \cdot 10^{-2}$

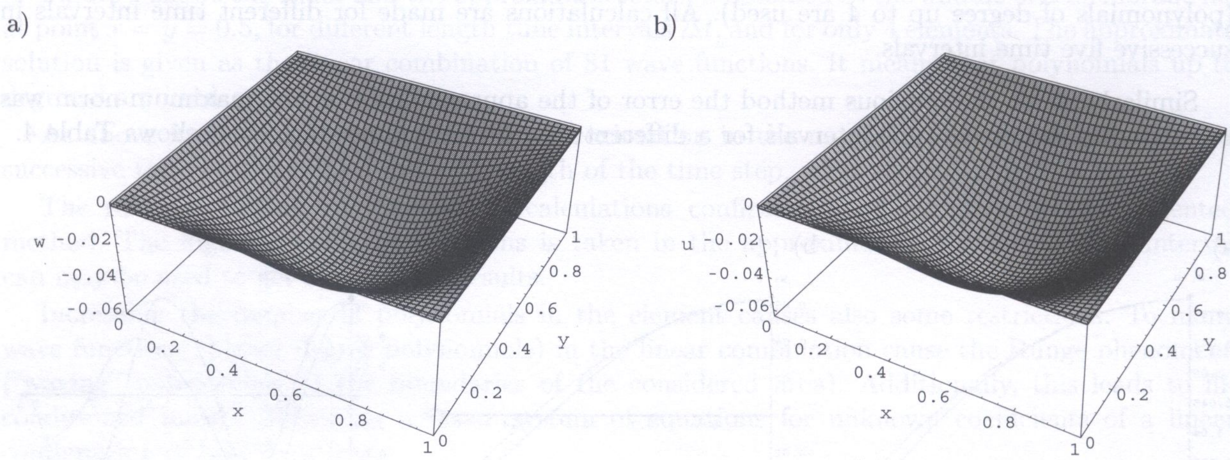


Fig. 5. Exact solution (a) and approximate solution (b) for time $t = 2$ obtained for the time step equals $\Delta t = 0.005$

It is difficult to take results received in this case as accurate, so to improve the approximate solution a more dense division was introduced. All calculations are repeated for 64 elements. Number of nodes in each element remained the same. The obtained approximations are shown in Fig. 4.

For such dense division, the error of approximation in the maximum norm was calculated in successive time intervals for a different length of the time step, what shows Table 5.

In order to check the influence of the number of iteration on the stability of the approximate solution for time interval $\Delta t = 0.005$, a membrane deflection is calculated for 400 successive time intervals. The results of these calculations are presented in Figure 5.

The analysis of the results presented above leads to following conclusions:

- Good accuracy of the approximation is obtained by the use of continuity FEM only in the case of appropriate short time step.

- The more dense division the more accurate solution we have.

- The presented method seems to be very stable.

The biggest problem in this method was the choice of wave functions, which were used in the approximation. Wave polynomials obtained from a generating function cannot be used for this purpose. They caused a numerical problem (ill-conditioned matrix) or they give wrong results. Using wave functions given by inverse operations allowed us to avoid this problem.

4.3. Discontinuity FEM

Solution of the considered problem (27)–(29) with the use of the discontinuity FEM did not give expected results. All calculations were carried out for the division of the domain into 16 and 64 elements with 12 nodes. Calculations were done for a different length of time's interval. To generate base functions in each element both kind of wave functions were used: given by inverse operations and obtained from generating functions. In each of these cases wrong results were obtained. Because an approximate solution differ substantially from the exact one, this method (in presented form) is not fitted to solve the problem described by the wave equation.

Perhaps wrong results were caused by the ill-conditioned matrix in the linear system of equations for the unknown coefficient of the approximation. Maybe a wrong criterion for the adjustment to given conditions were taken. A discontinuous FEM requires further research. In order to improve these results, probably the physical aspect of the problem has to be taken into account. New definition of a functional (see works [3, 4]) should be taken under consideration.

Because the numerical calculations show that nodeless FEM brings better results then other methods presented here, it seems to be profitable to use such an approach to solve the problems in elasticity. It does no matter which kind of wave functions is used to get an approximate solution in this approach. Both give results with a similar accuracy.

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