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On the stability of hybrid equilibrium and Trefftz finite element models for plate bending problems

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This paper is concerned with hybrid stress elements in the context of modelling the behaviour of plates subject to out of plane loading and based on Reissner–Mindlin assumptions. These elements are considered as equilibrium elements with statically admissible stress fields of which Trefftz fields form a special case. The existence of spurious kinematic modes in star patches of triangular elements is reviewed when boundary displacement fields are defined independently for each side. It is shown that for elements of moment degree > 1, the spurious modes for stars only exist at specific locations and/or for certain configurations. The kinematic properties of these modes are used to define sufficient conditions for the stability of a complete mesh of triangular elements. A method is proposed to check mesh stability, and introduce local modifications to ensure overall stability.

1. INTRODUCTION

The simulation of plate bending problems, or slabs, by approximate finite element models has lead to the development of many different types of plate element in order to strike a balance between accuracy and computational effort. Although displacement based elements have tended to dominate in general finite element applications, the benefits of hybrid stress elements have long been recognised in the context of plates.

This paper is concerned with hybrid stress elements for plates governed by Reissner-Mindlin theory where displacements are described by transverse deflections and rotations of the normals through the plate thickness, and internal actions are described by transverse shear forces and moments as stress-resultants, which, for brevity, will be collectively referred to as stresses. The stress fields generally satisfy the equilibrium conditions within an element, and can be classified as discontinuous but statically admissible (SA), hyperstatic within a particular element, continuous (SA) and/or of the Trefftz type. In the latter case compatibility of the corresponding deformations is also satisfied.

Hybrid elements allow the freedom to formulate elements with general polygonal boundaries, and to define independently the fields of stress and boundary displacements. The earliest hybrid elements due to Pian [9] assumed continuous frame functions for boundary displacements, but later formulations by Moitinho de Almeida *et al.* [8] were based on independent side functions which are discontinuous at the ends of the sides.

However spurious kinematic, or zero energy, modes have always been problematic for such elements. Recent work on hybrid equilibrium elements has focused on establishing general conditions for the stability of single triangular elements and patches of triangular elements forming stars, where stability implies freedom from spurious kinematic modes [5, 6]. The patches of elements can be considered as a generalisation of the macro-element concept first proposed by the Liege school [1]. It has been shown that when statically admissible internal moment fields are of degree greater than or equal to two, the spurious kinematic modes of single elements may not propagate between elements, and may thus lead to stable configurations.

The possible presence of spurious kinematic modes in a mesh of triangular equilibrium elements has lead to two strategies for meshing and/or solution. Almeida *et al.* [8] have utilised a special solver which accounts for singularities in a system of simultaneous algebraic equations if the loads are admissible, Maunder *et al.* [7] pursued and generalised the Liege technique of pre-assembling triangular elements into quadrilateral macro-elements which are free of spurious modes. What is of principal concern in this paper is to propose a method of stabilising a general unstructured mesh of triangular equilibrium elements so that arbitrary loads can be applied and conventional solvers can be used.

The structure of the remainder of this paper is as follows: Section 2 reviews hybrid equilibrium elements for plates in general terms; Section 3 then summarises the conditions for spurious modes for star patches containing one or more triangular elements; and Section 4 details a general method for mesh stabilisation based on the kinematic properties of star patches. The paper is concluded in Section 5.

2. Hybrid equilibrium elements

Included for completeness is a short review of hybrid equilibrium elements including some types of Trefftz element which can be regarded as special cases of equilibrium elements. For a more detailed review the reader is referred to [4].

Such elements may have polygonal boundaries with straight sides, and stress and displacement fields are normally in polynomial form. Equilibrium between elements is fully achieved when the polynomial forms for displacement are complete and the degree of the displacement fields is not less than the degree of the stresses. For plate bending elements, the degree of the lateral deflections is one less than that of the rotations of the transverse normals, and the degree of the shear forces which are equilibrated with the moments is also one less than the degree of the moments.

When the stresses are restricted so as to satisfy the compatibility conditions they become Trefftz fields, and such elements have been proposed in effect in [3] (termed HT-T elements), although the independent side functions were assumed to be distributions of traction which could be regarded as dual to the displacement fields as discussed in [10] (termed HTS D elements).

The term "weak Trefftz" has also been proposed [4] when the stresses are defined from complete polynomials of a certain degree after removal of the hyperstatic fields, which depend on the shape of an element. The remaining stresses have corresponding deformations (generalised strains) which are orthogonal in the energy sense with the local element hyperstatic fields, and thereby satisfy in a weak sense the compatibility conditions for deformations. These can be strongly expressed by the condition that the integral of the work done by deformations over an element should be zero for all possible hyperstatic fields.

Triangular elements are used extensively due to the convenience of using available mesh generators, e.g. based on Delaunay triangulation for unstructured meshes, although more general convex polygonal shapes can be generated by using Voronoi diagrams.

In algebraic terms, side displacements and admissible tractions for a mesh of hybrid equilibrium elements do work as defined by the scalar product in Eq. (1),

$$\boldsymbol{\delta} = \boldsymbol{V} \cdot \boldsymbol{v}; \qquad \boldsymbol{t} = \bar{\boldsymbol{S}} \cdot \boldsymbol{s}; \qquad \boldsymbol{D} = \int\limits_{sides} \boldsymbol{V}^T \bar{\boldsymbol{S}} \cdot \mathrm{d}\Gamma; \qquad \boldsymbol{t}^T \boldsymbol{\delta} = \boldsymbol{s}^T \boldsymbol{D}^T \boldsymbol{v}, \tag{1}$$

where $\boldsymbol{\delta}$ represents side displacements with n_{dof} degrees of freedom, and $\boldsymbol{t}$ represents tractions which equilibrates with internal stress fields with n_s degrees of freedom. These fields are determined from parameters $\boldsymbol{v}$ and $\boldsymbol{s}$ respectively. The matrix $\boldsymbol{D}$ has dimensions n_{dof} by n_s for a complete mesh, and $\boldsymbol{D}^T \boldsymbol{v} = 0$ when displacement modes $\boldsymbol{v}$ represent rigid body modes or spurious kinematic modes.

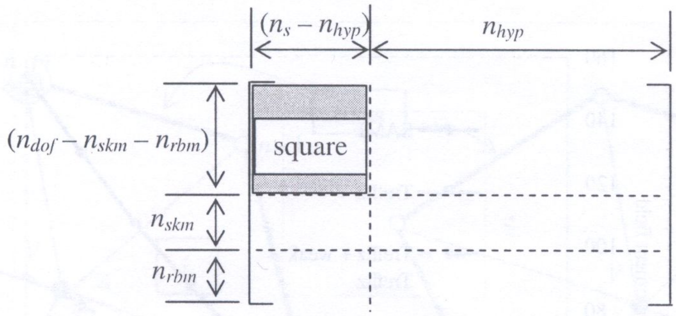


Fig. 1. Diagrammatic form of matrix D, with its non-singular square submatrix shown shaded

The numbers of independent modes are denoted by n_{rbm} and n_{skm}, respectively. The numbers of independent fields are related by the rank of $\boldsymbol{D}$, as indicated in Eq. (2) and Fig. 1 for a single element or a mesh of elements,

$$\text{rank}[\boldsymbol{D}] = (n_s - n_{hyp}) = (n_{dof} - n_{skm} - n_{rbm}). \tag{2}$$

Spurious kinematic (or zero energy) modes are a potential problem for hybrid elements whether of the equilibrium or Trefftz type. Spurious modes may propagate beyond a single element, and their presence also leads to a rank deficient stiffness matrix. It should be noted that the hyperstatic fields are here defined as those that do zero work with any of the side displacements. Thus generally this implies a traction free boundary, but may also include modes of traction of higher degree than the displacements that are not zero but do zero work. The spurious kinematic modes are those boundary displacements that do zero work for all tractions that equilibrate with internal stresses.

True hyperstatic fields within an element cannot be Trefftz fields since they have incompatible deformations, and an off-quoted necessary, but not sufficient, condition for the absence of spurious kinematic modes in hybrid Trefftz elements is given in Eq. (3) [2],

$$n_s \geq n_{dof} - n_{rbm}\,. \tag{3}$$

It has been recognised that this condition does not always lead to the absence of spurious modes, but experience has suggested that these modes do not propagate in practice and that stiffness matrices of complete meshes are not necessarily rank deficient. However this experience is not universal, and special solvers for singular matrices have been invoked in the context of hybrid equilibrium elements.

It has also been proposed that spurious modes at element level for Trefftz elements can always be avoided by using a sufficiently high number of internal stress fields. However this inevitably implies that, for polynomial fields, the internal fields have higher degree than the side displacements and consequently equilibrium of stresses between elements cannot generally be satisfied.

The numbers of independent fields of stresses for a triangular element are dependent on the degree [4] and they are illustrated in Fig. 2, where it can be seen that for elements of high degree the numbers of hyperstatic fields tend to dominate for the equilibrium type of element. SAMF denotes statically admissible moment fields. The number of weak Trefftz stress fields is dependent on the shape of the element, and so this number would be different for more general polygonal elements.

The remaining sections of this paper consider the question of existence and avoidance of spurious modes for a complete triangulated mesh of hybrid equilibrium elements, however the techniques considered are equally applicable to elements which are weak Trefftz.

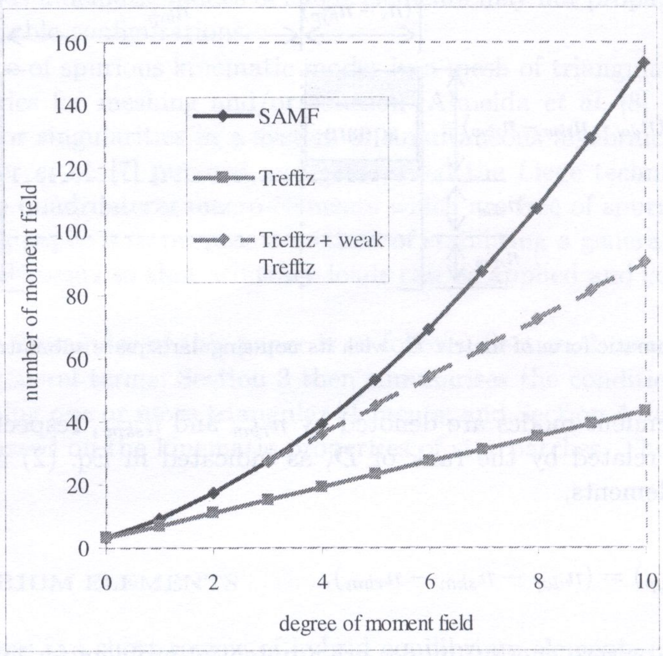


Fig. 2. Maximum numbers of moment fields versus degree for a triangular element

3. SPURIOUS KINEMATIC MODES FOR PATCHES OF TRIANGULAR EQUILIBRIUM ELEMENTS

3.1. Spurious modes of a single element

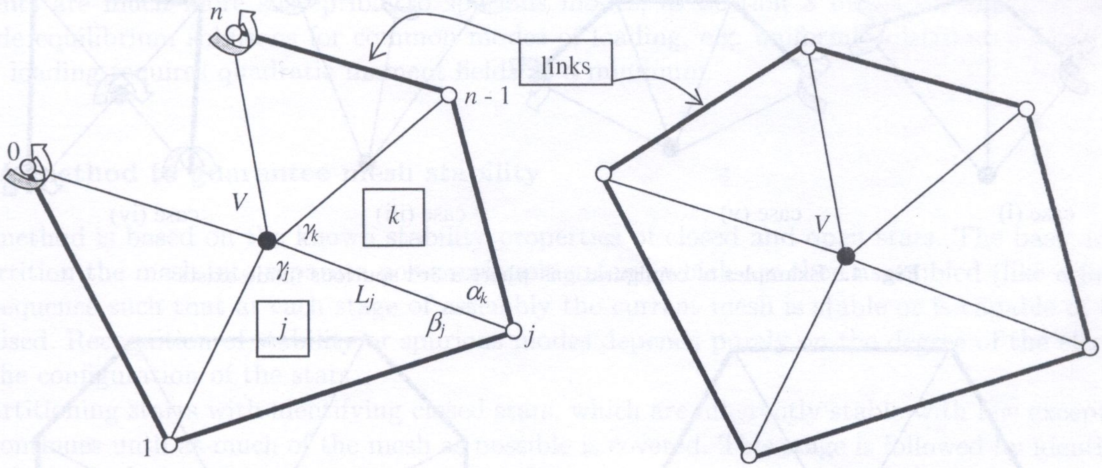
The numbers of spurious kinematic modes and their descriptions (shapes) for a single element of arbitrary degree have been detailed in [5]. It is shown in [5] that there exist 3 spurious modes for all degrees, each of which involves rotational deformations of pairs of sides incident with a corner node. An additional mode exists for elements of degree 2, which includes rotational displacements and transverse deflections of all 3 sides.

3.2. Spurious modes of a star patch of elements

A star patch consists of n elements that share a common corner node or vertex V. When a star patch is considered as a mesh in its own right, its "link" belongs to its boundary and refers to the sides that are not incident with its vertex. This term is borrowed from concepts used to describe simplicial complexes in topology. A star is defined as closed or open according to whether its link forms a closed circuit or a chain with two end nodes as illustrated in Fig. 3, where the links are shown with bold lines.

More recent work [6] has studied the stability of such stars, and the nature and number of spurious kinematic modes when they exist. The most important features that have emerged are that when the degree of the element is greater than 1, open stars generally contain 2 spurious modes which involve rotational deformations of the sides incident with the end nodes of the link, and indicated by curved arrows in Fig. 3; and closed stars are generally free of spurious modes.

There are certain exceptions to these features, and Tables 1 and 2 summarise the numbers of spurious kinematic modes for all stars dependent on degree, number of elements, and configuration. n_{deg} refers to the number of degenerate nodes, i.e. those where $(\beta_j + \alpha_k) = \pi$ radians. Case (i) can be considered as a special form of case (v) for degree 2, and these cases are the only open stars where transverse deflections are included in the spurious modes. Although not proven as yet,



Open star with $n = 4$ Closed star with $n = 5$

Fig. 3. Open and closed stars based on vertex V

Table 1. Spurious kinematic modes for open star patches containing 2 or more elements

Degree	Number of spurious kinematic modes
0 or 1	$(n+2)$
2	Case (i): 3 when $n = 2$, modes are dependent on configuration, otherwise for $n > 2$:
	Case (ii): 2 for a general configuration, except when $n_{deg} > 0$,
	Case (iii): $(2 + n_{deg})$, or
	Case (iv): 3 when $n = 3$ and the sum of adjacent angles at the common vertex $= \pi$ rads, i.e. $(\cot\gamma_j + \cot\gamma_k) = 0$ for $1 \le j \le (n-1)$.
	Case (v): 3 for other special configurations, i.e. $(\cot\gamma_j + \cot\gamma_k) = r(\cot\beta_j + \cot\alpha_k)$ for $1 \le j \le (n-1)$.
> 2	As for degree 2, but excluding cases (i) and (v).

Table 2. Spurious kinematic modes for closed star patches

Degree	Number of spurious kinematic modes
0	n
1	$(n-3)$
2	Case (i): 0 for a general configuration, except when $n_{deg} > 0$, then n_{deg};
	Case (ii): 1 when $n = 4$ and internal sides form the diagonals of a quadrilateral;
	Case (iii): 1 when n is even and > 4 for special configurations, i.e. $(\cot\gamma_j + \cot\gamma_k) = r(\cot\beta_j + \cot\alpha_k)$ for all j and $\sum_{j=1}^{n}(-1)^j W_j x_j = 0 = \sum_{j=1}^{n}(-1)^j W_j y_j$ where $(x_j\,,\,y_j)$ are the coordinates of the star nodes when the origin occurs at the internal vertex of the star.
> 2	As for degree 2, but excluding case (iii).

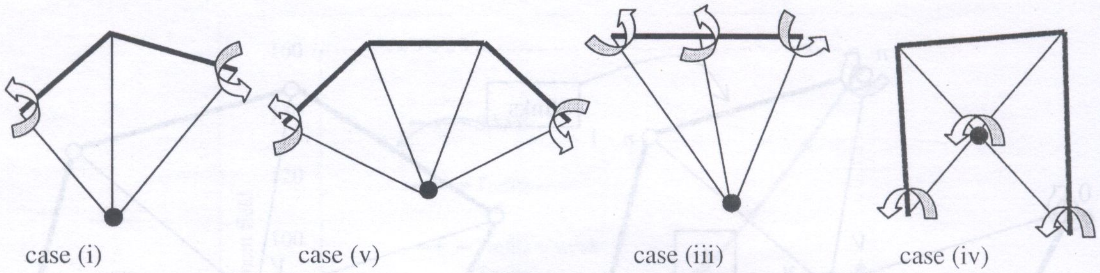


Fig. 4. Examples of configurations where a 3rd spurious mode exists

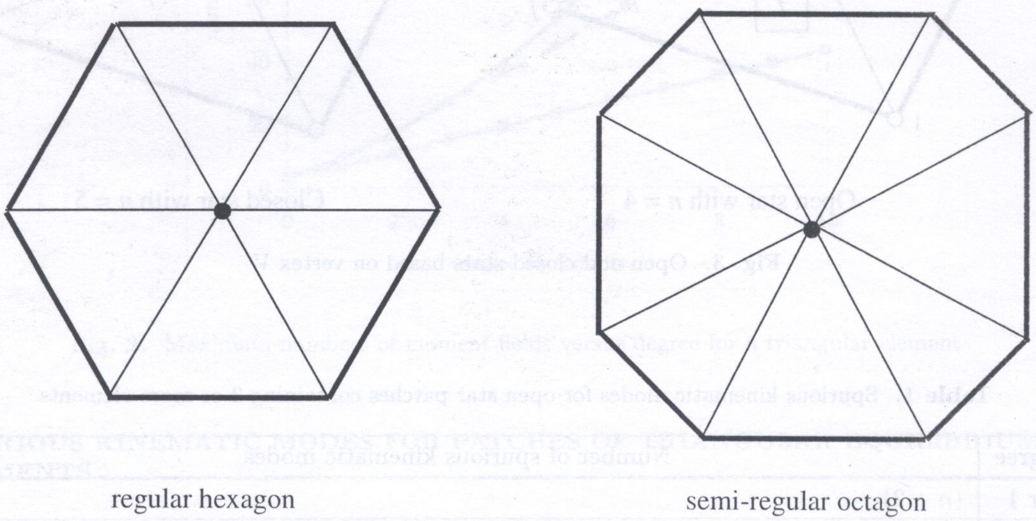


Fig. 5. Examples of polygonal elements with a single spurious kinematic mode for degree 2

experience indicates that all sides are deformed in these modes. All other forms of spurious mode only involve rotations of the normals through the thickness. Special configurations, or pathological cases, are illustrated in Fig. 4 when a third spurious mode exists. The configuration shown as an example of case (v) consists of three identical isosceles triangles. The curved arrows indicate that a spurious mode involves rotational deformations of those sides incident with the node within an arrow.

In Case (iii) of Table 2 for degree 2, $W_j = \frac{(\cot\gamma_j+\cot\gamma_k)}{L_j^2}$, and examples of special configurations which could arise in meshes based on regular tessellations are shown in Fig. 5. In such cases the spurious modes again appear to involve deformations of all sides, and they includes transverse deflections as well as rotations of the normals through the thickness.

4. Stability of a General Unstructured Mesh of Triangular Elements

The question of stability of, or absence of spurious kinematic modes from, a general mesh of triangular hybrid elements is now considered. Stability depends on three aspects of a mesh:

(i) the geometrical arrangement or configuration of the elements;

(ii) the degree of the elements;

(iii) the distribution of the loads and kinematic constraints.

Sufficient conditions for stability are proposed when the degree is greater than one, together with means of stabilisation if the conditions are not satisfied, although such means may not be necessary.

Although elements with constant or linear moment fields are popular in certain applications such elements are much more susceptible to spurious modes, as Section 3 indicates, and they fail to provide equilibrium solutions for common modes of loading, e.g. uniformly distributed loads. The latter loading requires quadratic moment fields as a minimum.

4.1. A method to guarantee mesh stability

The method is based on the known stability properties of closed and open stars. The basic idea is to partition the mesh into separate non-overlapping stars, which are then assembled (like a jigsaw) in a sequence such that at each stage of assembly the current mesh is stable or is capable of being stabilised. Recognition of stability or spurious modes depends purely on the degree of the elements and the configuration of the stars.

Partitioning starts with identifying closed stars, which are inherently stable with few exceptions, and continues until as much of the mesh as possible is covered. This stage is followed by identifying a sequence of open or closed stars from the remaining elements, so that each star is attached to the current assembly in a stable manner. The sides that are attached are constrained by the current assembly so as to remain rigid and undeformed. Thus any spurious mode in an open star which includes such a side is blocked. Consequently a single element should be attached on two or three sides, and an open star should be attached on sides that include $V0$ or 01, and Vn or $(n-1, n)$, with reference to Fig. 3.

Individual stars are unstable for certain configurations, i.e. when their links contain degenerate nodes, or they form quadrilaterals with diagonal subdivision, or in the case of degree 2 their configurations satisfy additional geometric conditions. Stabilisation of an individual star may be achieved by attaching it to another star that is stable, termed buttressing, or if that is not possible then internal subdivision may be necessary. Examples of such means of stabilisation are illustrated in Fig. 6.

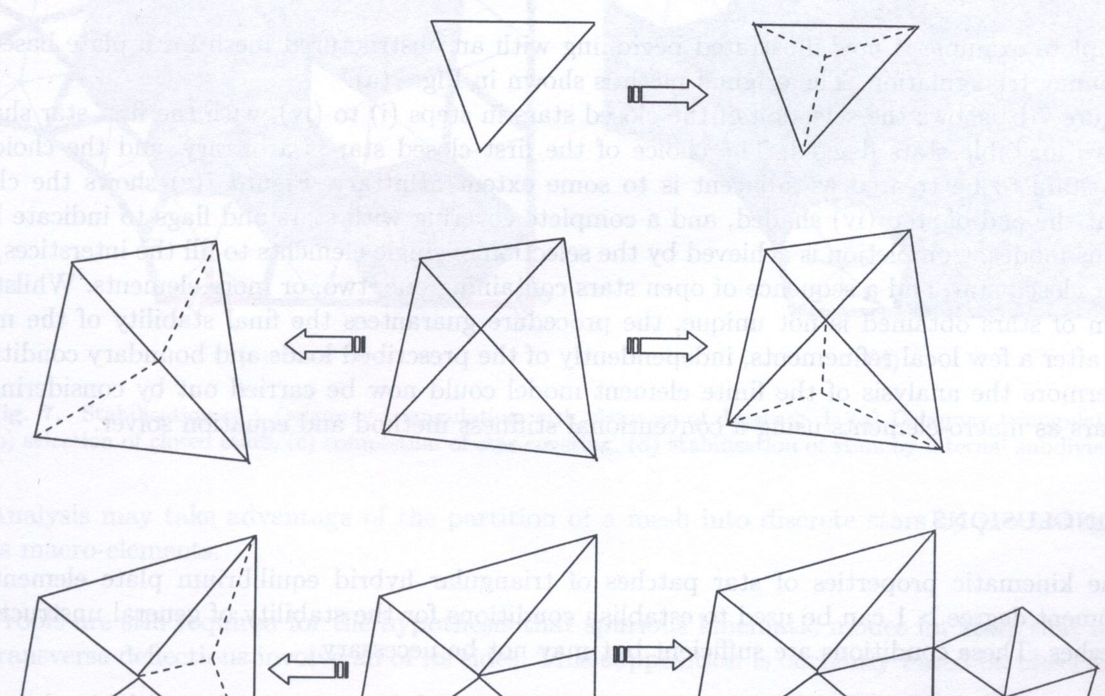


Fig. 6. Examples of stabilisation by subdivision or buttressing

A proposed procedure for partitioning a mesh, assembling stars, recognising spurious kinematic modes, and introducing local stabilisation, is outlined in the following algorithm:

(i) Select a vertex with a closed star from the mesh;

(ii) Check its stability by identifying any degenerate nodes or the other unstable configurations as defined in Table 2;

(iii) Flag the star if unstable – for subsequent checks to see if an adjacent star can provide stabilisation by acting as a buttress;

(iv) Select an adjacent closed star if one exists, i.e. a closed star whose link contains one or more sides in common, and return to step (ii).

(v) If no such adjacent star exists, then select an open star for attachment from the remaining elements. This may consist of a single element;

(vi) Check its stability subject to the kinematic constraints imposed by the interfaces with the current assembly, and return to step (iii);

(vii) If no adjacent star exists, then the mesh has been fully covered;

(viii) Check unstable stars for possible stabilisation by buttressing with adjacent stars;

(ix) If stabilisation by buttressing is not possible, then stabilise by internal subdivision with one or more macro-elements.

This algorithm provides sufficient but not necessary conditions for a stable mesh.

4.2. An example based on a Delaunay triangulation

A complete example is now illustrated beginning with an unstructured mesh for a plate based on a Delaunay triangulation. The original mesh is shown in Fig. 7(a).

Figure 7(b) shows the selection of the closed stars in steps (i) to (iv), with the first star shaded and two unstable stars flagged. The choice of the first closed star is arbitrary, and the choice of closed stars to be treated as adjacent is to some extent arbitrary. Figure 7(c) shows the closed stars at the end of step (iv) shaded, and a complete covering with stars and flags to indicate local spurious modes. Completion is achieved by the selection of single elements to fill the interstices, one further closed star, and a sequence of open stars containing one, two, or more elements. Whilst the pattern of stars obtained is not unique, the procedure guarantees the final stability of the mesh, albeit after a few local refinements, independently of the prescribed loads and boundary conditions. Furthermore the analysis of the finite element model could now be carried out by considering all the stars as macro-elements using a conventional stiffness method and equation solver.

5. Conclusions

- The kinematic properties of star patches of triangular hybrid equilibrium plate elements of moment degree > 1 can be used to establish conditions for the stability of general unstructured meshes. These conditions are sufficient but may not be necessary.
- The conditions rely solely on the geometric properties of the stars together with simple means of stabilising stars by buttressing or subdivision as they are assembled.
- Stable meshes can then be analysed by conventional solvers for any set of boundary conditions or loading without incurring rank deficiency.

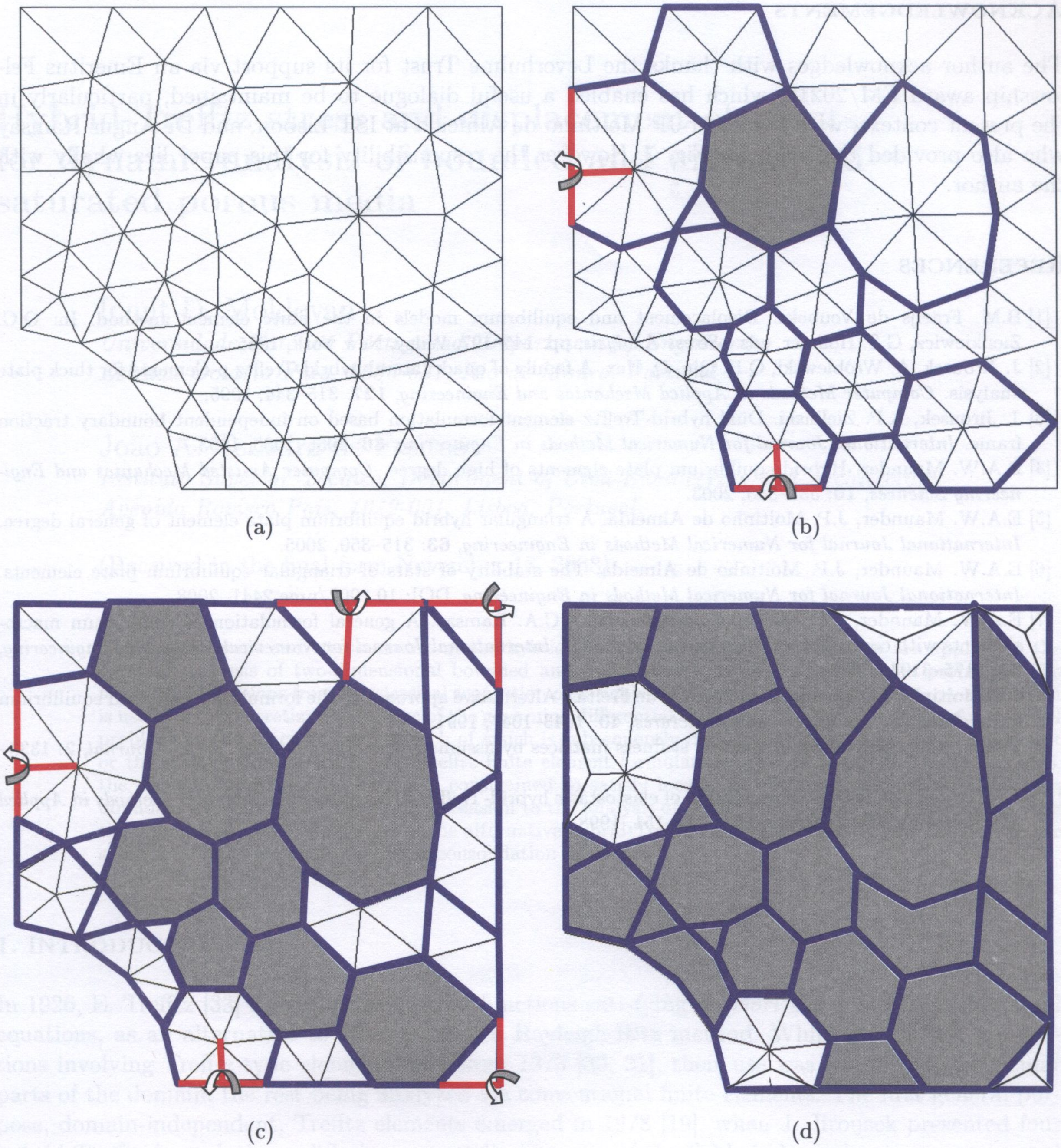


Fig. 7. Stabilisation of a Delaunay triangulation with elements of degree > 1; (a) Delaunay triangulation, (b) selection of closed stars, (c) completion of star covering, (d) stabilisation of stars by internal subdivision

- Analysis may take advantage of the partition of a mesh into discrete stars by processing them as macro-elements.
- Proofs are still required for the hypothesis that spurious kinematic modes for stars that include transverse deflections involve all of its sides. This supposition is currently based on observations.
- The method of mesh stabilisation needs to be evaluated from the computational point of view.
- The possibility of extending concepts to solid meshes of tetrahedral elements should be considered.

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