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Application of the Trefftz method to nonlinear potential problems

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In this paper some types of nonlinear potential problems are discussed and some of these problems are solved by the Trefftz method. The attention is paid to Fundamental Solutions Method (FSM) supported by Radial Basis Functions (RBF) approximation. Application of FSM to nonlinear boundary problem requires certain modifications and special algorithms. In this paper two methods of treating the nonlinearity are proposed. One on them is Picard iteration. Due to some problems of application of this method the Homotopy Analysis Method (HAM) is implemented for nonlinear boundary-value problems. The results of numerical experiment are presented and discussed. The conclusion is that the method based on FSM for solving nonlinear boundary-value problem gives result with demanded accuracy.

1. Introduction

The world where we are living is essentially nonlinear. Most of phenomenon in the whole cosmos, either inside or outside us, are described, although more or less approximately, by nonlinear equations. There exist some types of nonlinearity in mechanical problems. For example, the equation for isotropic heat conduction with spatially varying thermal conductivity yields the material nonlinearity. The other type of nonlinearity, is caused by internal sources. Moreover in some problems the geometrical nonlinearity appears. These are problems with a boundary changing in time. The other nonlinear problems are the systems with nonlinear boundary conditions. How to solve these nonlinear equations has been the hot point, especially today when we have supercomputers with high performance. Although the rapid development of digital computers makes it easier and easier to numerically solve nonlinear problems, it is still rather difficult to give analytical expressions of them. Most of nonlinear analytical techniques are unsatisfactory. Therefore, the numerical methods are used to solve such nonlinear problems. Traditionally, these problems have been solved numerically using finite difference method, finite volume method, or finite element method. Recently, boundary element method is applied intensely for nonlinear problems.

The literature treating solving the nonlinear problems by numerical methods is very wide. Above some problems, solved by using meshless methods, are mentioned. The authors of [4] have proposed quasilinear boundary element method for nonlinear Poisson type problems. The inhomogeneous part of the equation is approximated by Taylor's series. Also, the authors of [14] have formulated and solved inhomogeneous-nonlinear problem by boundary element analysis. The approximation by polynomials is applied for the inhomogeneous part of the equation. But the method gives the results with demanded accuracy only for few problems. A meshless numerical method based on the local boundary integral equation to solve linear and non-linear boundary value problems is proposed in [15]. One of the most characteristic nonlinear problems is nonlinear heat conduction. For the problem of heat conduction in material with spatially varying thermal conductivity the solution is found by homotopy analysis method supported by boundary element method by the authors of [7]. The other case of nonlinear heat conduction is for the thermal conductivity depended on the temperature. The authors of [5] applied Kirchhoff's transform to obtain pseudo-linear equations, which have been solved using boundary element method. The described problem has been solved by the author of [1] by implementing Trefftz indirect method. The other nonlinear problem solved by meshless methods, described in literature, is a large deflection of a thin elastic plates. The problem is described by the system of two coupled nonlinear biharmonic equations and the boundary conditions. This problem is solved in [13] with applying an iterative fashion. For each step of iterations a dual reciprocity boundary element method is implemented. The method of boundary element has been applied to solve such a problem in papers [3, 10, 11]. The Kansa's method has been also implemented to find the solution for the problem of large deflection of a thin plates by the author of [9]. The Picard iterations and the method of fundamental solutions is applied for solving the described problem in [12]. The anisotropic thin plate bending problems is solved by Trefftz boundary collocation method by the authors of [2]. Of course, these are very few examples of papers treating nonlinear problems. but, in the literature there are very few papers about implementation of the fundamental solutions method to nonlinear problems.

Application of the method of fundamental solutions to nonlinear problems requires modifications and special algorithm, which are subjects of this paper.

2. METHOD OF FUNDAMENTAL SOLUTIONS FOR NON-LINEAR PROBLEMS

2.1. General equations

The nonlinear boundary-value problem can be written in a general form

$$A_n \mathbf{u}(\mathbf{x}) = f_n(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \Omega, \tag{1}$$

where $n = 1, \ldots, N_e$, N_e is a number of equations, A_n $(n = 1, \ldots, N_e)$ are nonlinear partial differential operators, f_n $(n = 1, \ldots, N_e)$ are know functions, Ω is the region, which the differential equations are determined in. The coordinates of the points are given in the form $\mathbf{x} = (x_1, \ldots, x_N)$. Unknown functions to calculate are $\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), \ldots, u_{N_e}(\mathbf{x}))$.

For the considered problem boundary conditions are given by

$$B_l \mathbf{u}(\mathbf{x}) = g_l(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \Gamma, \tag{2}$$

for $l = 1, \ldots, N_{bc}$, where N_{bc} is a number of the boundary conditions under consideration, Γ is the boundary of the region Ω. If in the considered problem, the nonlinear operator can be written as a sum of linear and nonlinear operator, the Picard iterations method is applied.

2.2. Picard iterations

The nonlinear operator A_n is rewritten as

$$A_n = L_n + N_n \qquad \text{for } n = 1, \ldots, N_e, \tag{3}$$

where L_n is linear partial differential operator, A_n, N_n are nonlinear partial differential operators.

The system of differential equations (1) is written as a system of linear differential equations. The nonlinearity of equation is added to inhomogeneous part of equation. Therefore, the considered system of equations has the form

$$L_n \mathbf{u}(\mathbf{x}) = f_n(\mathbf{x}) - N_n \mathbf{u}(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \Omega, \tag{4}$$

where $n = 1, \ldots, N_e$. Of course, the boundary conditions (2) are still valid. Proposed transformation of the system of coupled nonlinear equations gives the system of quasilinear equations in implicit form. To solve such a system of equations the Picard iterations are implemented. The iterative fashion of the considered system of equations is given as

$$L_n \mathbf{u}^{(k)}(\mathbf{x}) = f_n(\mathbf{x}) - N_n \mathbf{u}^{(k-1)}(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \Omega, \tag{5}$$

where $n = 1, \ldots, N_e$, for $k = 1, 2, \ldots$ Each of the equations determined in k-th iteration step is solved by Method of Fundamental Solutions with boundary conditions

$$B_l \mathbf{u}^{(k)}(\mathbf{x}) = g_l(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \Gamma, \tag{6}$$

for $l = 1, \ldots, N_{bc}$. The inhomogeneous part of each equation is approximated by Radial Basis Functions and polynomials. The iterative process begins with initial approximations of the solution, which is obtained by solving the auxiliary boundary value problem:

$$L_n \mathbf{u}^{(0)}(\mathbf{x}) = f_n(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \Omega, \tag{7}$$

where $n = 1, \ldots, N_e$. The set of equations (7) is the system of uncoupled linear equations. Each equation with proper boundary condition

$$B_l \mathbf{u}^{(0)}(\mathbf{x}) = g_l(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \Gamma, \tag{8}$$

for $l = 1, \ldots, N_{bc}$ is solved by Fundamental Solutions Method. If the functions $f_n(\mathbf{x})$ are not equal to zero, they are approximated by Radial Basis Functions and polynomials.

The iterative process has to be stopped when the obtained results reach demanded accuracy. There are some criteria to decide to stop calculations. In this paper the error of obtained solution is defined by

$$E_s = \max_{1 \le n \le N_e} \frac{1}{N_c} \sqrt{\sum_{i=1}^{N_c} \left(u_n^{(k)}(\mathbf{x}_i^c) - u_n^{(k-1)}(\mathbf{x}_i^c) \right)^2}, \tag{9}$$

where $\{\mathbf{x}_i^c\}_{i=1}^{N_c}$ is a set of trial points with arbitrary chosen number of trial points N_c.

Inconvenience of Picard method, in some cases, is difficulty with reaching the convergence of iterative process. The convergence depends on the initial approximation of the solution. In the algorithm given above, the initial approximation of the solution is taken as a solution of the boundary-value problem given by Eqs. (7), (8). It is also possible to take the initial approximation by solving the modified Eq. (7), i.e. the right-hand side function is taken to be equal to zero. Of course, the boundary conditions (8) are still valid. The Picard method is based on an assumption that the nonlinear differential operator in Eq. (1) can be rewritten in the form (3). If the considered operator has not this feature or the iterative process divergences, the implementation of Picard iteration is not possible. In such a case the Homotopy Analysis Method, which is more general than Picard Iterations, is applied. The HAM does not require that the nonlinear operator includes the linear part.

2.3. Homotopy Analysis Method

The HAM treats the considered problem, given by Eqs. (1)–(2), as the one parameter family of problems. The parameter λ has values from the interval $[0, 1]$. Let us construct a homotopy

$$\mathbf{H}(\mathbf{x}, \lambda) = (H_1(\mathbf{x}, \lambda), \ldots, H_{N_e}(\mathbf{x}, \lambda)) \tag{10}$$

which satisfies

$$H_n(\mathbf{x}, \lambda) = \lambda \left(A_n \mathbf{U}(\mathbf{x}, \lambda) - f_n(\mathbf{x}) \right) + (1 - \lambda) L_{0n} \mathbf{U}(\mathbf{x}, \lambda) \tag{11}$$

for $n = 1, \ldots, N_e$. The operator L_{0n} has to be selected. L_{0n} is a simple linear operator, whose fundamental solution is familiar to the user of HAM and which may be different from L_n (described by Eq. (3)) even if L_n exists.

The considered Eq. (1) is equivalent to the homotopy (11) for parameter $\lambda = 1$, therefore

$$H_n(\mathbf{x}, 1) = A_n \mathbf{u}(\mathbf{x}) - f_n(\mathbf{x}) \tag{12}$$

for $n = 1, \ldots, N_e$. Moreover, the homotopy (11) for $\lambda = 0$ gives

$$H_n(\mathbf{x}, 0) = L_{0n}\mathbf{u}_0(\mathbf{x}) \tag{13}$$

for $n = 1, \ldots, N_e$, where $\mathbf{u}_0(\mathbf{x}) = (u_{0,1}(\mathbf{x}), \ldots, u_{0,N_e}(\mathbf{x}))$ is known solution of the boundary-value problem with the operator L_{0n} for $n = 1, \ldots, N_e$. For Eq. (1) the homotopy can be determined as

$$(1-\lambda)L_{0n}\left(\mathbf{U}(\mathbf{x},\lambda) - \mathbf{u}_0(\mathbf{x})\right) = \lambda\left(A_n\mathbf{U}(\mathbf{x},\lambda) - f_n(\mathbf{x})\right) \tag{14}$$

for $n = 1, \ldots, N_e$, where

$$\mathbf{U}(\mathbf{x},\lambda) = (U_1(\mathbf{x},\lambda), \ldots, U_{N_e}(\mathbf{x},\lambda)) \tag{15}$$

is the solution of Eq. (14) depended on parameter λ. The functions $U_n(\mathbf{x},\lambda_1)$, $U_n(\mathbf{x},\lambda_2)$ for $n = 1, \ldots, N_e$ and arbitrary chosen λ_1, $\lambda_2 \in [0,1]$ are called homotopic.

The solution of the considered problem (1) is suitable to the function (14) for parameter $\lambda = 1$. Therefore

$$\mathbf{U}(\mathbf{x}, 1) = \mathbf{u}(\mathbf{x}). \tag{16}$$

Moreover the function (14) for $\lambda = 0$,

$$\mathbf{U}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \tag{17}$$

is to be determined as the solution of an auxiliary boundary-value problem with the operator L_{0n} for $n = 1, \ldots, N_e$. The function $\mathbf{u}_0(\mathbf{x})$ is used to find solution of the problem (1) in the next steps of the algorithm. The homotopy (14) satisfies Eqs. (16) and (17). It means that the homotopy (14) is the family of problems and the solution of one of the problems (for $\lambda = 0$) is known. Using this known solution, the solution of other member of the homotopy family (for $\lambda = 1$) is calculated. This is the solution of the considered boundary-value problem (1)–(2). The homotopy defined by Eq. (14) should satisfy the boundary condition (2), therefore

$$B_l\mathbf{U}(\mathbf{x},\lambda) = (1-\lambda)B_l\mathbf{u}_0(\mathbf{x}) + \lambda g_l(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \Gamma \tag{18}$$

for $l = 1, \ldots, N_{bc}$. Let us assume that the *continuous deformation* $U_n(\mathbf{x},\lambda)$ (for $n = 1, \ldots, N_e$) is smooth enough about λ so that

$$u_{0,n}^{[m]}(\mathbf{x}) = \left.\frac{\partial^m U_n(\mathbf{x},\lambda)}{\partial \lambda^m}\right|_{\lambda=0}, \qquad m = 1, 2, 3, \ldots \tag{19}$$

for $n = 1, \ldots, N_e$, called mth-order deformation derivative, exists. Then, according to the theory of Taylor's series, from Eq. (17) the next formula is obtained,

$$U_n(\mathbf{x},\lambda) = U_n(\mathbf{x},0) + \sum_{m=1}^{\infty} \frac{\lambda^m}{m!} \left.\frac{\partial^m U_n(\mathbf{x},\lambda)}{\partial \lambda^m}\right|_{\lambda=0} = u_{0,n}(\mathbf{x}) + \sum_{m=1}^{\infty} \frac{u_{0,n}^{[m]}(\mathbf{x})}{m!}. \tag{20}$$

The expression introduced above is called the Taylor's homotopy series. The following notation

$$\mathbf{u}_0^{[m]} = \left(u_{0,1}^{[m]}, \ldots, u_{0,N_e}^{[m]}\right) \tag{21}$$

is introduced. Then the boundary-value problem describing the deformation derivatives have the form

$$L_{0n}\mathbf{u}_0^{[m]}(\mathbf{x}) = R_{m,n}(\mathbf{x}) \tag{22}$$

for $n = 1, \ldots, N_e$, where

$$R_{1,n}(\mathbf{x}) = A_n\mathbf{u}_0 - f_n(\mathbf{x}), \tag{23}$$

$$R_{m,n}(\mathbf{x}) = m\left(L_{0n}\mathbf{u}_0^{[m-1]} + \left.\frac{\partial^{m-1}A_n\mathbf{U}(\mathbf{x},\lambda)}{\partial \lambda^{m-1}}\right|_{\lambda=0}\right), \tag{24}$$

for $m = 2, 3, \ldots$ The boundary conditions for Eqs. (22) are given by

$$B_l \mathbf{u}_0^{[m]}(\mathbf{x}) = \delta_{m,1}\left(g_l(\mathbf{x}) - B_l \mathbf{u}_0(\mathbf{x})\right) \qquad \text{for } \mathbf{x} \in \Gamma \tag{25}$$

for $l = 1, \ldots, N_{bc}$, $m = 1, 2, \ldots$ Moreover $\delta_{1,1} = 1$ and $\delta_{m,1} = 0$ for $m \geq 2$.

Each of Eqs. (22)–(24) with the boundary condition (25) for $m = 1, 2, \ldots$ is numerically solved. In the literature, there are some suggestions to such problem by Boundary Element Method. This paper proposes to use the Method of Fundamental Solutions for finding the solution of Eqs. (22)–(24) with the boundary conditions (24) for $m = 1, 2, \ldots$ Having the obtained solutions of the problems (22)–(24) with the conditions (25) and using Eq. (19) for $\lambda = 1$, the solution of the considered problem (1) with the boundary conditions (2) is obtained as

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) + \sum_{m=1}^{\infty} \frac{\mathbf{u}_0^{[m]}(\mathbf{x})}{m!}. \tag{26}$$

It is obvious that for the numerical implementation the series in Eq. (26) has to be truncated. This way the approximation of the solutions is calculated with demanded accuracy. Moreover, in the numerical calculations the convergence of the computed solution is under control of the user. The parameter h is introduced to control the convergence. Hence, the homotopy depends on this parameter, too,

$$(1-\lambda) L_{0n}\left(\mathbf{U}(\mathbf{x},\lambda,h) - \mathbf{u}_0(\mathbf{x})\right) = h\lambda\left(A_n \mathbf{U}(\mathbf{x},\lambda,h) - f_n(\mathbf{x})\right) \tag{27}$$

for $n = 1, \ldots, N_e$ and

$$\mathbf{U}(\mathbf{x},\lambda,h) = \left(U_1(\mathbf{x},\lambda,h), \ldots, U_{N_e}(\mathbf{x},\lambda,h)\right) \tag{28}$$

is the solution of the problem, depended on the homotopy parameters $\lambda \in [0,1]$ and $h \neq 0$.

So, the partial differential equation (22) with the boundary conditions (25) is to be modified. The new version of the boundary-value problem has the form

$$L_{0n} \mathbf{u}_0^{[m]}(\mathbf{x}) = R_{m,n}(\mathbf{x}) \tag{29}$$

for $n = 1, \ldots, N_e$, where

$$R_{1,n}(\mathbf{x}) = h\left(A_n \mathbf{u}_0 - f_n, (\mathbf{x})\right) \tag{30}$$

$$R_{m,n}(\mathbf{x}) = m\left(L_{0n}\mathbf{u}_0^{[m-1]} + h \left.\frac{\partial^{m-1} A_n \mathbf{U}(\mathbf{x},\lambda,h)}{\partial \lambda^{m-1}}\right|_{\lambda=0}\right), \tag{31}$$

for $m = 2, 3, \ldots$ The boundary conditions for the equation given above are

$$B_l \mathbf{u}_0^{[m]}(\mathbf{x}) = \delta_{m,1}\left(g_l(\mathbf{x}) - B_l \mathbf{u}_0(\mathbf{x})\right) \qquad \text{for } \mathbf{x} \in \Gamma \tag{32}$$

for $l = 1, \ldots, N_{bc}$, $m = 1, 2, \ldots$ Moreover $\delta_{1,1} = 1$ and $\delta_{m,1} = 0$ for $m \geq 2$.

Finally, the solutions of the boundary-value problem (1), (2) is described by the formula

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) + \sum_{m=1}^{M} \frac{\mathbf{u}_0^{[m]}(\mathbf{x})}{m!} \tag{33}$$

where M is a natural number, chosen properly to obtain the solution with demanded accuracy.

The series (33) is truncated and the calculations stopped if the mean-square error E_s is less than small parameter i.e. $E_s < \varepsilon$, where ε is the parameter determining the accuracy of computing and error is defined as

$$E_s = \max_{1 \leq n \leq N_e} \frac{1}{N_c} \sqrt{\sum_{i=1}^{N_c} \left(u_{0,n}^{[m]}(\mathbf{x}_i^c)\right)^2}. \tag{34}$$

In both Picard iterations and HAM the crucial point is to solve the linear boundary problem (given by Eqs. (5)–(6) or (7)–(8) or (29)–(32)). In this paper the implementation of Method of Fundamental Solutions is proposed. The considered equations are mostly inhomogeneous, therefore, the approximation by Radial Basis Functions is used.

2.4. Method of Fundamental Solutions

Let us rewrite Eqs. (5), (7), (29)–(31) in a general form

$$L_n\mathbf{u}(\mathbf{x}) = f_n(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \Omega \tag{35}$$

and the boundary conditions (6), (8), (32) have a general form as

$$B_l\mathbf{u}(\mathbf{x}) = g_l(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \Gamma \tag{36}$$

for $l = 1, \dots, N_{bc}$. The operator L_n ($n = 1, \dots, N_e$) is a linear partial differential operator of arbitrary type.

Solution of the problem is a sum of a particular solution $\mathbf{u}_p(\mathbf{x}) = (u_{p,1}(\mathbf{x}), \dots, u_{p,N_e}(\mathbf{x}))$ of Eq. (35) and a homogeneous solution $\mathbf{u}_h(\mathbf{x}) = (u_{h,1}(\mathbf{x}), \dots, u_{h,N_e}(\mathbf{x}))$ of the considered boundary-value problem (35), (36)

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}_p(\mathbf{x}) + \mathbf{u}_h(\mathbf{x}). \tag{37}$$

The particular solution is obtained using the approximation by Radial Basis Functions and monomials. The set of trial points $\{\mathbf{x}_i^g\}_{i=1}^{N_g}$ is chosen. These points lie in the region and on the boundary of the considered region (see Fig. 1) and are called *grid points.*

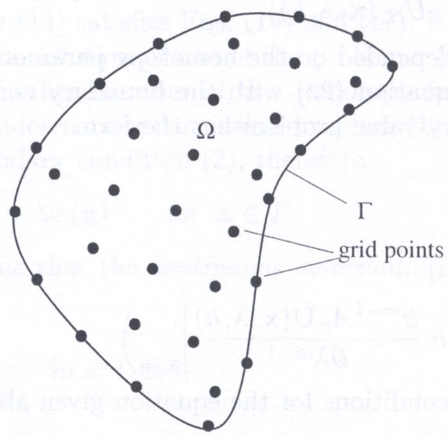

Fig. 1. Grid points in the region Ω and on the boundary Γ

The inhomogeneous part of Eq. (35) is approximated by linear combination of Radial Basis Functions and monomials,

$$f_n(\mathbf{x}) = \sum_{j=1}^{N_g} a_{n,j}\varphi_n(r_j^g) + \sum_{j=1}^{N_p} b_{n,j}p_{n,j}(\mathbf{x}), \tag{38}$$

where $n = 1, \dots, N_e$, $\varphi_n(r_j^g)$ is a Radial Basis Function chosen for the operator L_n, $r_j^g = \sqrt{\sum_{l=1}^{N}(x_l - x_{l,j}^g)^2}$ for $j = 1, \dots, N_g$, $p_{n,j}(\mathbf{x})$ is a monomial chosen for the operator L_n, coefficients $a_{n,j}$ for $j = 1, \dots, N_g$ and $b_{n,j}$ for $j = 1, \dots, N_p$ are real numbers. Equation (38) is written for each grid point, which gives a system of linear algebraic equations

$$\sum_{j=1}^{N_g} a_{n,j}\varphi_n(r_{ij}^g) + \sum_{j=1}^{N_p} b_{n,j}p_{n,j}(\mathbf{x}_i^g) = f_n(\mathbf{x}_i^g) \tag{39}$$

for $i = 1, \ldots, N_g$ and $n = 1, \ldots, N_e$, where $r^g_{ij} = \sqrt{\sum_{l=1}^{N} (x^g_{l,i} - x^g_{l,j})^2}$. Moreover, the following relationship has to be held

$$\sum_{j=1}^{N_g} a_{n,j} p_{n,i}(\mathbf{x}^g_j) = 0 \qquad \text{for } i = 1, \ldots, N_p\,, \quad n = 1, \ldots, N_e \tag{40}$$

The solution of the system of linear equations (39) and (40) gives the particular solution of the problem (35), (36) in the form

$$u_{p,n}(\mathbf{x}) = \sum_{j=1}^{N_g} a_{n,j}\phi_n(r^g_j) + \sum_{j=1}^{N_p} b_{n,j}\psi_{n,j}(\mathbf{x}), \tag{41}$$

where the functions ϕ_n and $\psi_{n,j}$ for $j = 1, \ldots, N_p$ are the particular solutions of the following equations,

$$L_n\phi_n = \varphi_n\,, \tag{42}$$

$$L_n\psi_{n,j} = p_{n,j}\,, \tag{43}$$

for $n = 1, \ldots, N_e$.

This procedure completes the computation the particular solution. Next part of this chapter is the description of implementation of MFS to obtain the homogeneous solution.

The solution given in the form (37) fulfils the boundary condition (36). So, Eq. (37) can be rewritten in the form

$$B_l\left(\mathbf{u}_h(\mathbf{x}) + \mathbf{u}_p(\mathbf{x})\right) = g_l(\mathbf{x}) \tag{44}$$

for $l = 1, \ldots, N_{bc}$. And rearranging Eq. (44) gives

$$B_l\mathbf{u}_h(\mathbf{x}) = g_l(\mathbf{x}) - B_l\mathbf{u}_p(\mathbf{x}) \tag{45}$$

for $l = 1, \ldots, N_{bc}$. The homogeneous solution $u_{h,n}(\mathbf{x})$ is approximated by linear combination of fundamental solutions

$$u_{h,n}(\mathbf{x}) = \sum_{k=1}^{N_{fs}}\sum_{j=1}^{N_s} c_{n,(k-1)N_s+j} f_{Sk,n}(r^s_j), \tag{46}$$

where $n = 1, \ldots, N_e$, $r^s_j = \sqrt{\sum_{l=1}^{N} (x_l - x^s_{l,j})^2}$ for $j = 1, \ldots, N_s$, which yields the system of linear algebraic equations

$$\sum_{k=1}^{N_{fs}}\sum_{j=1}^{N_s} c_{n,(k-1)N_s+j} B_l f_{Sk,n}(r^s_j) = g_l(\mathbf{x}) - B_l u_{p,n}(\mathbf{x}) \tag{47}$$

for $n = 1, \ldots, N_e$, $l = 1, \ldots, N_{bc}$. The boundary of the considered region is discretized, i.e. the set of boundary points is chosen as $\{\mathbf{x}^b_i\}_{i=1}^{N_b}$. To avoid the singularity of the fundamental solutions the set of source points $\{\mathbf{x}^s_i\}_{i=1}^{N_s}$ is chosen. The sets of points are presented in Fig. 2.

Equation (47) is written for each boundary point

$$\sum_{k=1}^{N_{fs}}\sum_{j=1}^{N_s} c_{n,(k-1)N_s+j} B_l f_{Sk,n}(r^s_{ij}) = g_l(\mathbf{x}^b_i) - B_l u_{p,n}(\mathbf{x}^b_i) \tag{48}$$

for $n = 1, \ldots, N_e$, $l = 1, \ldots, N_{bc}$, where $r^s_{ij} = \sqrt{\sum_{l=1}^{N} \left(x^b_{l,i} - x^s_{l,j}\right)^2}$ for $i = 1, \ldots, N_b$, $j = 1, \ldots, N_s$.

In the general case the system of equations given above consists of $N_b N_{bc}$ linear algebraic equations of $N_s N_{fs}$ unknowns. The solution of the set of equations (48) gives the approximation of the homogeneous solution which is expressed as the linear combination of the fundamental solutions.

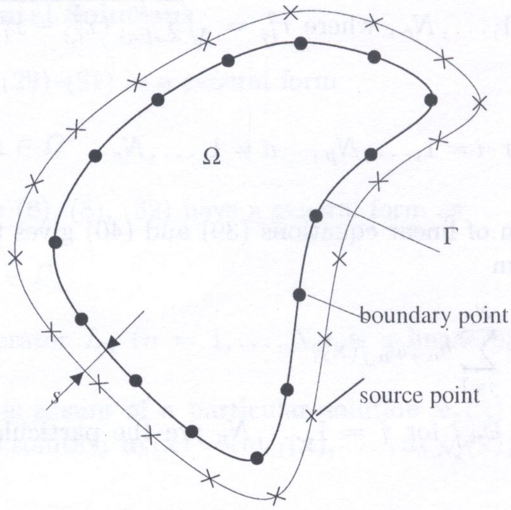

Fig. 2. The boundary point on the boundary Γ of the region Ω and the source points on the contour similar to the boundary of the region

3. Numerical implementation – HAM

3.1. HAM for the equation of second order

The inhomogeneous equation of second order is written in the general form as

$$Au(\mathbf{x}) = f(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \Omega \tag{49}$$

where A is a nonlinear partial differential operator of second order, f is a known function, Ω is a region, which the equation is determined in. The points of the region are written as $\mathbf{x} = (x_1, \ldots, x_N)$. The function to calculate is $u(\mathbf{x})$. In the considered problem the boundary conditions is in the form

$$Bu(\mathbf{x}) = g(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \Gamma \tag{50}$$

where Γ is a boundary of the region Ω. The HAM for Eq. (49) with the boundary condition (50) introduces the homotopy in the form

$$(1-\lambda)\, L_0\left(U(\mathbf{x},\lambda,h) - u_0(\mathbf{x})\right) = h\lambda\left(AU(\mathbf{x},\lambda,h) - f(\mathbf{x})\right). \tag{51}$$

The auxiliary operator L_0 should be linear and the fundamental solution of the homogeneous equation with the operator L_0 should be known. For the equation of second order (49) the Laplace's operator is chosen,

$$L_0 = \nabla^2. \tag{52}$$

So, the homotopy (51) has the form

$$(1-t)\,\nabla^2\left(U(\mathbf{x},t,h) - u_0(\mathbf{x})\right) = ht\left(AU(\mathbf{x},t,h) - f(\mathbf{x})\right). \tag{53}$$

To obtain the deformation derivatives it is necessary to solve the following boundary problems,

$$\nabla^2 \mathbf{u}_0^{[m]}(\mathbf{x}) = R_m(\mathbf{x}), \tag{54}$$

where

$$R_1(\mathbf{x}) = h\left(A\mathbf{u}_0 - f(\mathbf{x})\right), \tag{55}$$

$$R_m(\mathbf{x}) = m\left(\nabla^2 \mathbf{u}_0^{[m-1]} + h\left.\frac{\partial^{m-1} A\mathbf{U}(\mathbf{x},\lambda,h)}{\partial\lambda^{m-1}}\right|_{\lambda=0}\right), \tag{56}$$

for $m = 2, 3, \ldots$ The boundary condition for each of the given above equation is

$$Bu_0^{[m]}(\mathbf{x}) = \delta_{m,1}\left(g(\mathbf{x}) - Bu_0(\mathbf{x})\right) \quad \text{for } \mathbf{x} \in \Gamma \tag{57}$$

$m = 1, 2, \ldots$ and $\delta_{1,1} = 1$, $\delta_{m,1} = 0$ for $m \geq 2$.

The solution of the boundary problems (54)–(57) is calculated in [6, 8] using the boundary element method. In this paper for this purpose the method of fundamental solutions is applied.

3.1.1. HAM for problems with spatially varying coefficients

In this subsection the spatially varying coefficients of Eq. (49) are considered. Therefore, the operator A has the form

$$A = \nabla \cdot (\kappa(\mathbf{x})\nabla) \tag{58}$$

where $\kappa(\mathbf{x})$ is a function of the spatial variables, describing the material nonlinearity.

For the operator (58), Eq. (19) is

$$\left.\frac{\partial^m A\mathbf{U}(\mathbf{x},\lambda,h)}{\partial\lambda^m}\right|_{\lambda=0} = \left.\frac{\partial^m \left(\nabla \cdot (\kappa(\mathbf{x})\nabla\mathbf{U}(\mathbf{x},\lambda,h))\right)}{\partial\lambda^m}\right|_{\lambda=0}. \tag{59}$$

By making m differentials of q. (59) and using $\lambda = 0$, the following formula is obtained,

$$\left.\frac{\partial^m A\mathbf{U}(\mathbf{x},\lambda,h)}{\partial\lambda^m}\right|_{\lambda=0} = m\nabla \cdot \left(\kappa(\mathbf{x})\nabla\mathbf{u}_0^{(m)}(\mathbf{x},h)\right). \tag{60}$$

Therefore

$$\left.\frac{\partial^m A\mathbf{U}(\mathbf{x},\lambda,h)}{\partial\lambda^m}\right|_{\lambda=0} = mA\mathbf{u}_0^{[m]}(\mathbf{x},h). \tag{61}$$

Equation (61) shows, that obtaining Eq. (59) for the operator (60) is just applying the operator (60) to the function $\mathbf{u}_0^{[m]}(\mathbf{x},h)$.

3.1.2. HAM for problems with coefficients varying with computed quantity

The procedure to obtain Eq. (49) in the case of the coefficients varying with computed quantity is presented below. The nonlinear operator under consideration is in the form

$$\mathbf{A}\mathbf{u}(\mathbf{x}) = \nabla \cdot \left(\kappa\left(\mathbf{u}(\mathbf{x})\right)\nabla\mathbf{u}(\mathbf{x})\right) \tag{62}$$

where $\mathbf{u}(\mathbf{x})$ is an unknown function.

Let us assume that the coefficient is linearly dependent on the unknown function

$$\kappa(\mathbf{u}(\mathbf{x})) = \alpha_0 + \alpha_1\mathbf{u}(\mathbf{x}). \tag{63}$$

Therefore, Eq. (19) for the operator (62) with the function $\mathbf{u}(\mathbf{x}) = \mathbf{U}(\mathbf{x},\lambda,h)$ has the form

$$\left.\frac{\partial^m A\mathbf{U}(\mathbf{x},\lambda,h)}{\partial\lambda^m}\right|_{\lambda=0} = \left.\frac{\partial^m \left(\nabla \cdot (\kappa(\mathbf{U}(\mathbf{x},\lambda,h))\nabla\mathbf{U}(\mathbf{x},\lambda,h))\right)}{\partial\lambda^m}\right|_{\lambda=0}. \tag{64}$$

The mth order differential deformation is given by

$$\left.\frac{\partial^m A\mathbf{U}(\mathbf{x},\lambda,h)}{\partial\lambda^m}\right|_{\lambda=0} = m!\left(\alpha_0\nabla^2\mathbf{u}_0(\mathbf{x}) + \alpha_1\sum_{i=1}^{m}\left(\mathbf{u}_0^{[m-1-i]}(\mathbf{x})\,\nabla^2\mathbf{u}_0^{[i]}(\mathbf{x}) + \frac{\partial\mathbf{u}_0^{[m-1-i]}}{\partial x}(\mathbf{x})\frac{\partial\mathbf{u}_0^{[i]}(\mathbf{x})}{\partial x} + \frac{\partial\mathbf{u}_0^{[m-1-i]}}{\partial y}(\mathbf{x})\frac{\partial\mathbf{u}_0^{[i]}(\mathbf{x})}{\partial y}\right)\right). \tag{65}$$

3.2. HAM for the equation of fourth order

The Homotopy Analysis Method is implemented to the problem described by two nonlinear coupled biharmonic equations. The system of equation is in the form

$$A_1\mathbf{u}(\mathbf{x}) = f_1(\mathbf{x}), \qquad A_2\mathbf{u}(\mathbf{x}) = f_2(\mathbf{x}), \tag{66}$$

for $\mathbf{x} \in \Omega$, where A_1, A_2 are nonlinear partial differential operators of order 4, f_1, f_2 are known functions, Ω is the region, which the equations are described in. The points in the region are in the form $\mathbf{x} = (x_1, \dots, x_N)$. The functions to calculate are: $\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}))$. The boundary conditions are as follows,

$$\begin{aligned} B_1u_1(\mathbf{x}) &= g_1(\mathbf{x}) && \text{for } \mathbf{x} \in \Gamma, \\ B_2u_1(\mathbf{x}) &= g_2(\mathbf{x}) && \text{for } \mathbf{x} \in \Gamma, \\ B_3u_2(\mathbf{x}) &= g_3(\mathbf{x}) && \text{for } \mathbf{x} \in \Gamma, \\ B_4u_2(\mathbf{x}) &= g_4(\mathbf{x}) && \text{for } \mathbf{x} \in \Gamma, \end{aligned} \tag{67}$$

where Γ is the boundary of Ω.

To implement the Homotopy Analysis Method in Eq. (27) they are rewritten in the form

$$\begin{aligned} (1-\lambda)\, L_{01}\left(\mathbf{U}(\mathbf{x},\lambda,h) - \mathbf{u}_0(\mathbf{x})\right) &= h\lambda\left(A_1\mathbf{U}(\mathbf{x},\lambda,h) - f_1(\mathbf{x})\right), \\ (1-\lambda)\, L_{02}\left(\mathbf{U}(\mathbf{x},\lambda,h) - \mathbf{u}_0(\mathbf{x})\right) &= h\lambda\left(A_2\mathbf{U}(\mathbf{x},\lambda,h) - f_2(\mathbf{x})\right), \end{aligned} \tag{68}$$

where the operators L_{01}, L_{02} have to be determined. For the equations of fourth order it is chosen that: $L_{01} = \nabla^4$ and $L_{02} = \nabla^4$, where ∇^4 is the biharmonic operator.

Therefore Eqs. (68) are as follows,

$$\begin{aligned} (1-\lambda)\, \nabla^4\left(\mathbf{U}(\mathbf{x},\lambda,h) - \mathbf{u}_0(\mathbf{x})\right) &= h\lambda\left(A_1\mathbf{U}(\mathbf{x},\lambda,h) - f_1(\mathbf{x})\right), \\ (1-\lambda)\, \nabla^4\left(\mathbf{U}(\mathbf{x},\lambda,h) - \mathbf{u}_0(\mathbf{x})\right) &= h\lambda\left(A_2\mathbf{U}(\mathbf{x},\lambda,h) - f_2(\mathbf{x})\right). \end{aligned} \tag{69}$$

To obtain the deformation derivative the boundary-value problems have to be solved

$$\nabla^4\mathbf{u}_0^{[m]}(\mathbf{x}) = R_{m,n}(\mathbf{x}) \tag{70}$$

where

$$R_{1,n}(\mathbf{x}) = h(A_n\mathbf{u}_0 - f_n(\mathbf{x})), \tag{71}$$

$$R_{m,n}(\mathbf{x}) = m\left(\nabla^4\mathbf{u}_0^{[m-1]} + h\left.\frac{\partial^{m-1}A_n\mathbf{U}(\mathbf{x},\lambda,h)}{\partial\lambda^{m-1}}\right|_{\lambda=0}\right), \tag{72}$$

for $n = 1, 2$, $m = 2, 3, \dots$, with the boundary conditions

$$\begin{aligned} B_1u_{0,1}^{[m]}(\mathbf{x},t) &= \delta_{m,1}\left(g_1(\mathbf{x}) - B_1u_{0,1}(\mathbf{x})\right), \\ B_2u_{0,1}^{[m]}(\mathbf{x},t) &= \delta_{m,1}\left(g_2(\mathbf{x}) - B_2u_{0,1}(\mathbf{x})\right), \\ B_3u_{0,2}^{[m]}(\mathbf{x},t) &= \delta_{m,1}\left(g_3(\mathbf{x}) - B_3u_{0,2}(\mathbf{x})\right), \\ B_4u_{0,2}^{[m]}(\mathbf{x},t) &= \delta_{m,1}\left(g_4(\mathbf{x}) - B_4u_{0,2}(\mathbf{x})\right), \end{aligned} \tag{73}$$

for $\mathbf{x} \in \Gamma$ and $m = 1, 2, \dots$, $\delta_{1,1} = 1$, $\delta_{m,1} = 0$ for $m \geq 2$.

One of the numerical experiments presented below is based on the nonlinear operator in the form

$$A\mathbf{u}(\mathbf{x}) = \nabla^2\nabla^2\mathbf{u}(\mathbf{x}) - a_0G\left(\mathbf{u}(\mathbf{x})\right) \tag{74}$$

where $\mathbf{u}(\mathbf{x})$ is a vector of two unknowns. It is assumed that the problem is two-dimensional, so $\mathbf{x} = (x_1\,, x_2)$. The operator G is defined as

$$G(\mathbf{u}(\mathbf{x})) = G(u_1(\mathbf{x}), u_2(\mathbf{x})) = \frac{\partial^2 u_1(\mathbf{x})}{\partial x_1^2}\frac{\partial^2 u_2(\mathbf{x})}{\partial x_2^2} + \frac{\partial^2 u_1(\mathbf{x})}{\partial x_2^2}\frac{\partial^2 u_2(\mathbf{x})}{\partial x_1^2} - 2\frac{\partial^2 u_1(\mathbf{x})}{\partial x_1 \partial x_2}\frac{\partial^2 u_2(\mathbf{x})}{\partial x_1 \partial x_2}\,. \tag{75}$$

For operator (74), Eq. (19) has the form

$$\left.\frac{\partial^m A\mathbf{U}(\mathbf{x}, \lambda, h)}{\partial \lambda^m}\right|_{\lambda=0} = \left.\frac{\partial^m}{\partial \lambda^m}\left(\nabla^2\nabla^2 \mathbf{u}(\mathbf{x}) - a_0 G\mathbf{u}(\mathbf{x})\right)\right|_{\lambda=0}. \tag{76}$$

Realisation of m derivatives in Eq. (76) and introducing $\lambda = 0$ gives

$$\left.\frac{\partial^m A\mathbf{U}(\mathbf{x}, \lambda, h)}{\partial \lambda^m}\right|_{\lambda=0} = \nabla^2\nabla^2 \mathbf{u}^{[m]}(\mathbf{x}) - a_0 \sum_{i=0}^{m} \binom{m}{i} G\left(u_1^{[m]}(\mathbf{x}), u_2^{[i]}(\mathbf{x})\right), \tag{77}$$

$m = 1, 2, \ldots$ and $\delta_{1,1} = 1$, $\delta_{m,1} = 0$ for $m \geq 2$.

So, the procedure of obtaining the solution by HAM supported by FSM is concluded.

4. Numerical experiment

4.1. Steady-state heat conduction in material with spatially varying conductivity

The equation for the steady-state heat conduction in material with spatially varying conductivity is

$$\nabla \cdot (\kappa(\mathbf{x}) \nabla T(\mathbf{x})) = q(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \Omega \tag{78}$$

where T is temperature, considered region is $\Omega = \{\, (x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1 \,\}$, conductivity coefficient is taken as

$$\kappa(\mathbf{x}) = 1 + x + y + xy \tag{79}$$

and the function

$$q(\mathbf{x}) = \pi \cos(\pi x y)\,(x(1 + x) + y(1 + y)) - \pi^2 \sin(\pi x y)(x^2 + y^2)(1 + x + y + xy). \tag{80}$$

The boundary condition is

$$T(\mathbf{x}) = 1 + \sin(\pi x y) \qquad \text{for } \mathbf{x} \in \Gamma \tag{81}$$

where

$$\Gamma = \{\, (x, y) \mid 0 \leq x \leq 1,\ y = 0 \,\} \cup \{\, (x, y) \mid 0 \leq x \leq 1,\ y = 1 \,\}$$
$$\cup \{\, (x, y) \mid y = 0,\ 0 \leq y \leq 1 \,\} \cup \{\, (x, y) \mid x = 1,\ 0 \leq y \leq 1 \,\}.$$

The considered problem has been solved by two methods: Picard iteration and Homotopy Analysis Method. Both are based on the fundamental solution method. The used formulas for the HAM are presented in Subsection 3.1.1. To implement method of Picard iterations the following formula is written,

$$\nabla^2 T^{(k)}(\mathbf{x}) = \frac{1}{1 + x + y + xy}\left(q(\mathbf{x}) - \left((1 + y)\frac{\partial T^{(k-1)}(\mathbf{x})}{\partial x} + (1 + x)\frac{\partial T^{(k-1)}(\mathbf{x})}{\partial y}\right)\right), \tag{82}$$

for $k = 1, 2, \ldots$

The results are presented in Table 1. Both methods give results with the same demanded accuracy. In considered case the Picard iterations leads to the required solutions performs less calculations than the Method of Homotopy Analysis.

Table 1. Mean square error of results obtained by Picard iterations and the Homotopy Analysis Method

k	E_s Picard iterations	E_s HAM
1	0.00199321	0.00321878
2	0.000148845	0.00111504
3	0.0000209764	0.000395614
4	$1.66785 \cdot 10^{-6}$	0.000155713
5	$2.38598 \cdot 10^{-7}$	0.0000650707
6		0.0000288127
7		0.0000133659
8		$6.45732 \cdot 10^{-6}$
9		$3.25019 \cdot 10^{-6}$

4.2. Steady-state heat conduction in material with conductivity varying with temperature

The equation describing the heat conduction in the material with conductivity varying with temperature is

$$\nabla \cdot (\kappa(T(\mathbf{x}))\nabla T(\mathbf{x})) + q(\mathbf{x}) = 0, \tag{83}$$

the considered region is a square $\Omega = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$. It is assumed that in the region there are no sources, therefore $q(\mathbf{x}) = 0$.

The boundary conditions are as follows,

$$\begin{aligned} &T(x) = T_p && \text{for } \Gamma_1 = \{(x, y) \mid x = 0, 0 \leq y \leq 1\}, \\ &T(x) = T_k && \text{for } \Gamma_2 = \{(x, y) \mid x = 1, 0 \leq y \leq 1\}, \\ &\frac{\partial}{\partial n} T(x) = 0 && \text{for } \Gamma_3 = \{(x, y) \mid x = 0, 0 \leq y \leq 1\} \cup \{(x, y) \mid x = 1, 0 \leq y \leq 1\}, \end{aligned} \tag{84}$$

where $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$ is the boundary of the region.

The conductivity is described by the formula

$$\kappa(T(\mathbf{x})) = \alpha_0 + \alpha_1 T(\mathbf{x}). \tag{85}$$

The considered problem has been solved by two methods: Picard iteration and Homotopy Analysis Method. Both are based on the fundamental solution method. The used formulas for the HAM are presented in Subsection 3.1.2. To implement method of Picard iterations the following formula is written

$$\nabla^2 T^{(k)}(\mathbf{x}) = \frac{1}{\kappa(T^{(k-1)}(\mathbf{x}))} \left(\frac{\partial \kappa(T^{(k-1)}(\mathbf{x}))}{\partial T(\mathbf{x})} \left(\left(\frac{\partial T^{(k-1)}(\mathbf{x})}{\partial x} \right)^2 + \left(\frac{\partial T^{(k-1)}(\mathbf{x})}{\partial y} \right)^2 \right) \right) \tag{86}$$

In this experiment the solutions have been calculated for a set of values of parameters α_0, α_1. The solutions are obtained with accuracy $\varepsilon < 10^{-5}$. Table 2 consists of the number of iterations (for Picard iterations) and the number of deformation derivatives (in the Homotopy Analysis Method) calculated to obtained the results with demanded accuracy. It looks that the HAM needs a few more calculations than the Picard iterations.

The experiment has been performed also for other values of parameters α_0, α_1 that it is put in Table 2. But for some values (i.e. $\alpha_0 = 0.075$, $\alpha_1 = 0.925$) the iteration process diverges, so the solution has been not reached. The homotopy analysis method in this case gives results with demanded accuracy. The error of the results for the parameters $\alpha_0 = 0.075$, $\alpha_1 = 0.925$ is presented in Table 3. The error of the calculations performed by Picard iterations increases at every next step. Finally, the solution is not reached. But calculations provided by HAM converge to the solution. The error defined by the next deformation derivative decreases.

Table 2. A number of iterations (for Picard iterations) and number of derivative deformations (in the HAM)

	number of iterations	number of deformation derivatives
$\alpha_0 = 0.9$ $\alpha_1 = 0.1$	2	2
$\alpha_0 = 0.8$ $\alpha_1 = 0.2$	3	4
$\alpha_0 = 0.7$ $\alpha_1 = 0.3$	3	5
$\alpha_0 = 0.6$ $\alpha_1 = 0.4$	4	5
$\alpha_0 = 0.5$ $\alpha_1 = 0.5$	4	6
$\alpha_0 = 0.4$ $\alpha_1 = 0.6$	5	7
$\alpha_0 = 0.3$ $\alpha_1 = 0.7$	7	8
$\alpha_0 = 0.2$ $\alpha_1 = 0.8$	8	9

Table 3. The number of iterations and elements in Taylor series used in calculation for $\alpha_0 = 0.075$, $\alpha_1 = 0.925$

k	Mean-square error	
	Picard iterations	HAM
1	0.010202	0.0100024
2	0.002952	0.00374584
3	0.002124	0.00208754
4	0.002026	0.000996911
5	0.011988	0.000478022
6	0.183888	0.00023584
7	0.010202	0.000114273
8	–	$5.19416 \cdot 10^{-5}$
9	–	$2.14415 \cdot 10^{-5}$
10		$7.77 \cdot 10^{-6}$

4.3. Large displacement of an elastic plate

The static large displacement of an elastic plate is described by the system of von Karman's equations

$$\nabla^4 w = \frac{g}{D} + \frac{h}{D} G(w, F), \tag{87}$$

$$\nabla^4 F = -\frac{E}{2} G(w, w) \quad \text{in } \Re^2 \tag{88}$$

where $w = w(x, y)$ is a plate displacement, $F = F(x, y)$ is an Airy's type function, ∇^4 is the biharmonic operator, $D = \frac{Eh^3}{12(1-\nu^2)}$ is a rigidity of the plate of thickness h; E, ν are, respectively, Young's modulus, Poisson's ratio, $\Re^2$ is a region 2-D occupied by the plate. The operator G is defined by Eq. (75)

At every boundary point the set of boundary conditions is determined. For clamped edge of the plate the conditions are

$$w(s) = 0, \qquad \frac{\partial w}{\partial n}(s) = 0, \qquad F(s) = 0, \qquad \frac{\partial F}{\partial n}(s) = 0, \tag{89}$$

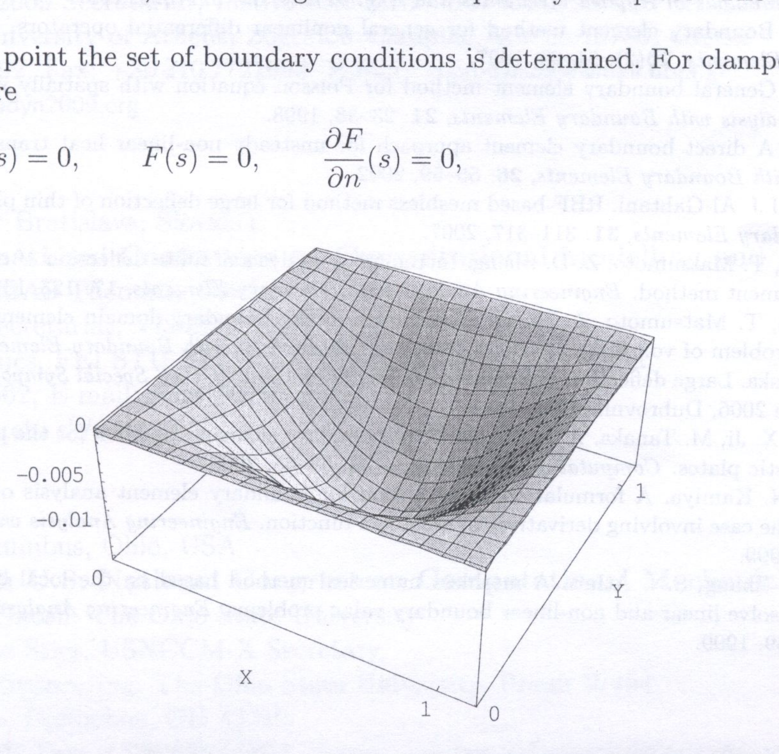

Fig. 3. The large deflection of the plate

where s is the boundary point.

For the considered example the parameters of the plate are chosen as $h = 0.02\,\mathrm{m}$, $E = 2.068 \cdot 10^7\,\mathrm{Pa}$, $\nu = 0.3$ and the external loading $q = 2.5\,\mathrm{kg/m^2}$.

The result presented in Fig. 3 shows the deflection of the plate obtained by HAM. The number of used derivative deformations is equal to 5. The large deflection of the plate is compliant the expected one.

5. Conclusions

In this paper two methods are proposed to solve nonlinear problems. Both are based on the method of fundamental solutions. The obtained results show that the methods are good tool to solve such systems. Unfortunately, the Picard iteration procedure has some defects. The use of this method is limited to the problems which contain linear part in the operator. For some cases the method is not convergent. At the other side the homotopy analysis method has no such limitations as iterations. But, due to luck of literature, in most cases user is obliged to determined deformation derivatives by him self. Concluding the results of numerical experiment we can say that the method of homotopy analysis gives results for the considered examples with demanded accuracy.

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