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Optimal parameters of method of fundamental solutions for Poisson problems in heat transfer by means of genetic algorithms

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This paper describes the application of the method of fundamental solutions to the solution of the boundary value problems of the two-dimensional steady heat transfer with heat sources. For interpolation of an inhomogeneous term in Poisson equation the radial basis functions are used. Three cases of boundary value problems are solved and five cases of radial basis functions are used. For comparison purposes the boundary value problems for which exact solution exists were chosen. Application of method of fundamental solutions with boundary collocation and radial basis function for solution of inhomogeneous boundary value problems introduces some number of parameters related with these tools. For optimal choosing of these parameters the genetic algorithm is used. The results of numerical experiences related to optimal parameters are presented.

Keywords: method of fundamental solutions; meshless method; radial basis functions; steady heat transfer, Poisson's equation; genetic algorithm; particular solution

1. Introduction

Poisson equation is encountered by numerous problems regarding heat conduction at the presence of heat sources. Classical numerical methods (FDM, FEM) for solution of this equation make use of some form of discretization. In the last decade research on numerous types of meshfree methods has made significant progress in science and engineering. One of them is some version of Trefftz method, namely the version without boundary integration. Trefftz method can be understood as a method in which differential equation is fulfilled exactly whereas the boundary condition is fulfilled approximately. There are few possibilities in approximate fulfilment of the boundary condition within the Trefftz method. Among others one of this is the Boundary Collocation Method (BCM) [21] in which the boundary conditions are fulfilled in collocation manner. Two basic different sets of functions, which fulfil exactly differential equations, are used in frame of BCM: T-complete Herrera functions [15] and fundamental solutions of governing equation [9] (also known as the Method of Fundamental Solutions — MFS). In the MFS, the solution of a homogeneous governing equation subject to a nonhomogeneous boundary condition, is approximated by a linear combination of fundamental solutions of differential equation. In this method one placed the considered domain with boundary into a large fictitious domain with boundary. The singularities (source points) of fundamental solutions are placed on Ω with boundary $\partial\Omega$ into a large fictitious domain Ω with boundary $\partial\Omega$. The singularities (source points) of fundamental solutions are placed on $\partial\Omega$.

In case of Inhomogeneous Boundary Volume Problem (IBVP) the application of MFS is not so easy. In literature one can find three different methods of solution of IBVP by MFS. First one was proposed by Burges and Mahajerin [4] and is based on direct numerical domain integration. In the second one, proposed by Poullikkas *et al.* [31] the particular solution is taken to be Newton potential and difficulties involved in evaluation of domain integration is avoided by the Atkinson method [1]. A third method proposed by Golberg [12] is based on dual reciprocity method. In this method, radial basis functions are employed to interpolate the inhomogeneous term of given differential equation. Based on this interpolation, an approximate analytic particular solution, which is a key to the solution process, can be obtained. This version of Trefftz method for solution of inhomogeneous boundary volume problem was called in [2] as the particular solution Trefftz method.

The application of particular solution Trefftz method introduces some number of parameters related with this tool. These parameters can be the following:

1. Number of source points in MFS;
2. Distance between boundary of problem considered and fictitious boundary if they are geometrically similar or coordinates of source points in more general case;
3. Number of collocation points in BCM;
4. Kind and shape parameter of radial basis function for interpolation of inhomogeneous term.

It is well known that the choice of these parameters can either greatly enhance or degrade the quality of IBVP. Investigation in the optimisation of some of these parameters in not completely new, but as yet optimal choices were considered separately. For example in papers [6, 18–20, 26–29] in frame of MFS for solution of homogeneous BVP the investigation of optimal position of source points was considered. Another examples, not in all cases related with solution of BVP, are investigation of optimal value of shape parameters in interpolation by radial basis function, given in papers [5, 10, 11, 13, 14, 16, 33, 35–40]. Because of the "uncertainty principle" [34], the problem of optimal values of method parameters in frame of MFS and RBF is not trivial task. In the theory of MFS exists theorem [3] that approximate solution tends to exact solution if radius of source contour tends to infinity. But when this radius is increased, in the limit, the influence of a source on one collocation point is indistinguishable from another, and the collocation matrix becomes nearly singular. Similarly in application of RBF for interpolation if the shape parameter is increased theoretically the interpolation is more and more accurate [17]. But in such case, as shape parameter gets large, the shape of basis function becomes flat and is insensitive to the difference in radial distance; the interpolation matrix becomes more ill-conditioned. Again, we have the curious result the solution actually gets more accurate until it reaches the breakdown point caused by the machine roundoff error.

The purpose of this paper is the solution of boundary value problems with 2-D Poisson's equation by means of particular solution Trefftz method with optimal choosing method parameters.

We note that once the problem has been formulated as an optimisation problem, then various optimisation algorithms may be used in order to locate the optimum of the objective function. The efficiency of a particular optimisation method clearly depends on the form of the objective function. In the problem considered in this paper, the objective function is maximal local error in satisfaction of boundary condition and has a complex non-linear structure. Moreover, an analytical expression for the objective function for every possible solution of the problem is unknown. Therefore genetic algorithms appear to be very suitable for optimizing the objective fun:tion of the problem since they do not require knowledge of the gradient of the objective functions, which makes them particularly suited to optimization problems for which an analytical expression for the fitness function is unknown.

2. Particular solution Trefftz method

The particular solution Trefftz method for 2-D Poisson equation has been well described by many authors but not in all cases with this name. For the sake of completeness, we outline the basic methodology of considered method. Consider a problem described by 2D Poisson equation

$$\nabla^2 u = b(x,y) \quad \text{in } \Omega \tag{1}$$

with an inhomogeneous boundary conditions

$$Bu = g(x,y) \qquad \text{on } \partial\Omega \tag{2}$$

where $b(x,y)$ and $g(x,y)$ are known functions, B is an operator of boundary conditions. In physical interpretation for case of heat transfer u is temperature and $b(x,y)$ is function of heat generation.

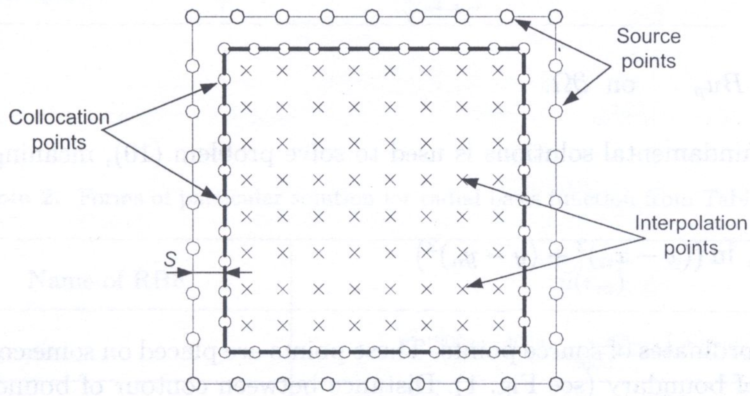

Fig. 1. The distributions of collocation, source, and interpolation points

Let $\{P_i(x_i,y_i)\}_{i=1}^{M_1}$ denote the set of M interpolation points in Ω, of which $\{P_i(x_i,y_i)\}_{i=1}^{M_1}$ are interior points and $\{(x_i,y_i)\}_{M_1+1}^{M}$ are boundary points (see Fig. 1). The right-hand side function in Eq. (1) is approximated by radial basis functions (RBFs) as

$$b(x,y) = \sum_{m=1}^{M} \alpha_m \hat{\varphi}_m(r_m) + \sum_{k=1}^{K} \beta_k \tilde{\varphi}_k(x,y) \tag{3}$$

where $\varphi_m(r_m) = \sqrt[2]{(x-x_m)^2 + (y-y_m)^2}$ is a RBF, $\{\varphi_k(x,y)\}_{k=1}^{K}$ is a basis for P_{m-1}, the set of d-variate polynomials of degree $\leq m-1$, and $K = \{ {m+d-1 \atop d} \}$ is the dimension of P_{m-1}. The coefficients α_m and β_k can be found by solving the system of linear equations,

$$\sum_{i=1}^{M} \alpha_m \hat{\varphi}_m(r_{mi}) + \sum_{k=1}^{K} \beta_k \tilde{\varphi}_k(x,y) = b(x_i,y_i), \qquad 1 \leq i \leq M, \tag{4}$$

$$\sum_{m=1}^{M} \alpha_m \tilde{\varphi}_k(x_m,y_m) = 0, \qquad 1 \leq k \leq K, \tag{5}$$

where $r_{i,j} = \sqrt[2]{(x_i-x_j)^2 + (y_i-y_j)^2}$, and $\{(x_i,y_i)\}_{i=1}^{M}$ are the interpolation points on $\Omega \cup \partial\Omega$. The approximate particular solutions u_p of Eq. (1) can be obtained using coefficients α_m and β_k,

$$u_p = \sum_{m=1}^{M} \alpha_m \hat{u}_m(r_m) + \sum_{k=1}^{K} \beta_k \tilde{u}_k(x,y), \tag{6}$$

where

$$\nabla^2 \hat{u}_j(x,y) = \hat{\varphi}_j(x,y), \qquad j = 1,2,\ldots,M, \tag{7}$$

$$\nabla^2 \tilde{u}_k(x,y) = \tilde{\varphi}_k(x,y), \qquad k = 1,2,\ldots,K. \tag{8}$$

The general solution of the differential equation (1) now can be given as

$$u = u_h + u_p \tag{9}$$

where u_h is the solution of the boundary value problem in the form

$$\nabla^2 \tilde{u}_h = 0 \qquad \text{in } \Omega \tag{10}$$

and

$$B u_h = g(x,y) - B u_p \qquad \text{on } \partial\Omega. \tag{11}$$

The method of fundamental solutions is used to solve problem (10), meaning that

$$u_h(x,y) = \sum_{n=1}^{N} c_n \ln\left((x-x_n)^2 + (y-y_n)^2\right) \tag{12}$$

where x_n , y_n are coordinates of source points. These points are placed on some contour geometrically similar to contour of boundary (see Fig. 1). Distance between contour of boundary and contour of sources is the method parameter and equals S. Enforcement of the boundary conditions yields

$$\sum_{n=1}^{N} cB \ln\left((x_i - x_n)^2 + (y_i - y_n)^2\right) = g(x_i,y_i) - Bu_p(x_i,y_i) \tag{13}$$

where NC is the number of collocation points on the boundary. If $NC > N$ the system of equations (13) is solved in least square sense.

3. Radial basis functions, their shape parameter, and their particular solutions

During the past two decades radial basis functions have been intensively studied and widely applied to many areas of science and engineering. Various types of radial basis functions can be used for interpolation of right-hand side function in Eq. (1). In our numerical experiments we chose some of these functions, commonly used by another authors and given in Table 1. Here r_m represents the Euclidean distance of the given point from a fixed point m in the domain.

All considered radial basis functions possess the shape parameters. The quality of interpolation strongly depends on the values of these parameters. Once one chooses the radial basis function $\hat{\varphi}(r_m)$ one has to determine $\hat{u}(r_m)$, which is evaluated by integrating Eq. (7).

The governing equation for $\hat{u}(r_m)$ is therefore given by

$$\frac{1}{r_m}\frac{\mathrm{d}}{\mathrm{d}r_m}\left(r_m \frac{\mathrm{d}\hat{u}}{\mathrm{d}r_m}\right) = \hat{\varphi}(r_m). \tag{14}$$

Expressions for the particular solution for various basis functions are listed in Tables 2 and 3.

Table 1. Forms of radial basis function used in numerical experiments

Name of RBF	$\hat{\varphi}(r_m)$	Shape parameters
Polynomial	$a_0 + a_1 r_m + a_2 r_m^2 + a_3 r_m^3$	a_0 , a_1 , a_2 , a_3
Polyharmonic spline function	$\begin{cases} 0 & \text{for } r_m = 0 \\ r_m^n \ln r_m & \text{for } r_m \neq 0 \end{cases}$	n
Multiquadrics	$\sqrt[2]{r_m^2 + c^2}$	c
Wendtland's function	$\begin{cases} (1 - \frac{r_m}{a_0})^4 (1 + \frac{4 r_m}{a_0}) & \text{for } r_m \leq a_0 \\ 0 & \text{for } r_m > a_0 \end{cases}$	a_0
Inverted multiquadrics	$\frac{1}{\sqrt[2]{r_m^2 + c^2}}$	c

Table 2. Forms of particular solution for radial basis function from Table 1

No	Name of RBF	$\hat{u}(r_m)$
1	Polynomial	$\frac{r_m^2 (900 a_0 + 400 a_1 r_m + 225 a_2 r_m^2 + 144 a_3 r_m^3)}{3600}$
2	Polyharmonic spline function	$\begin{cases} 0 & \text{for } r_m = 0 \\ \frac{r_m^{n+2}((n+2)\ln r_m - 2)}{(n+2)^3} & \text{for } r_m \neq 0 \end{cases}$
3	Multiquadrics	$-\frac{1}{3} c^3 \ln\left(c\sqrt{c^2 + r_m^2} + c\right) + \frac{1}{9}(4c^2 + r^2)\sqrt{c^2 + r_m^2}$
4	Wendtland's function	$\begin{cases} \frac{r_m^2}{4} - \frac{5 r_m^4}{8 a_0^2} + \frac{4 r_m^5}{5 a_0^3} - \frac{5 r_m^6}{12 a_0^5} + \frac{4 r_m^7}{49 a_0^5} & \text{for } r_m \leq a_0 \\ \frac{259 a_m^2}{5880} + \frac{a_0^2 \ln(r/a_0)}{14} & \text{for } r_m > a_0 \end{cases}$
5	Inverted multiquadrics	$-c \ln\left(\sqrt{c^2 + r_m^2} + c\right) + \sqrt{c^2 + r_m^2}$

Table 3. Polynomial functions and their particular solutions

k	$\tilde{\varphi}(x, y)$	$\tilde{u}(x, y)$
1	1	$\frac{1}{4}(x^2 + y^2)$
2	x	$\frac{1}{8}(x^3 + xy^2)$
3	y	$\frac{1}{8}(y^2 + yx^2)$
4	xy	$\frac{1}{12}(yx^3 + xy^3)$
5	x^2	$\frac{1}{14}(x^4 + x^2 y^2 - \frac{y^2}{6})$
6	y^2	$\frac{1}{14}(y^4 + x^2 y^2 - \frac{x^2}{6})$

4. Optimisation method by means of genetic algorithm

The optimization problem of suitable selection of parameters for MFS and RBF can be written as

$$\max \rightarrow F(X) \tag{15}$$

$$\text{Subject to: } g_j(X) \quad \text{for } j = 1, 2, \ldots \tag{16}$$

where $F(X)$ is the objective function, X is the design variables' vector, and $g_j(X)$ are constraints defining the searching space and o is the number of constraints. The X vector includes parameters of MFS and RBF in following order: number of collocation points – NC, number of source points – N, distance between boundary and source contour – S, and parameters of RBF: polynomial coefficients – a_0 , a_1 , a_2 , a_3 , power index of polyharmonic spline function – n, shape parameter of multiquadrics or inverted multiquadrics – c, and radius of Wendtland's function – a. Apart from design variables the optimization problem has a few of parameters such as: the number of interpolation's points – M, the number of elements of polynomial function – K. The values of these parameters was selected arbitrarily ($K = 6$, $M = 10$). The function $F(X)$ is defined as maximal relative error in consideration area for the case when the exact solution is known. When the exact solution is unknown the function $F(X)$ is defined as sum maximal relative error on the boundary and residual error (maximal interpolation error in considered case). The constraints $g_j(X)$ are rigid determined by MFS and RBF, so design variables have to have correctly values. The restrictions were defined as follows:

- The first is determined by mathematical rule that the number of collocation points must be less (or equal) than the number of source's points. This condition provides that the system of equations has clear-cut solution.
- The second concern is the places for source points. The theory of MFS requires that the source's points must be outside consideration area. The condition $S > 0$ assures this restriction.
- The third is defined as $NC > 0$ and $N > 0$, because otherwise the number of collocation or source points will be equal 0 and getting results from MFS will be impossible.
- The fourth constraint concerns the parameters of RBF. These parameters (for example n in thin plate spline function, or c in multiquadric function) must have suitable values is this way that the functions $\hat{\varphi}$ and $\hat{u}$ do not include singularities.

The objective function was defined as

$$F(X) = \frac{1}{\sqrt{\delta_{\max}(X)}} - E \cdot P_{\max} \tag{17}$$

where $\delta_{\max}$ is maximal error in satisfaction boundary conditions, E is penalty function, which equals 0 if all constrains and conditions are fulfilled or 1 in the opposite case, $P_{\max}$ is value of fitness function for the actual the best solution. When one of these constraints is not fulfilled, the calculations are not allowed to continue. For the method which doesn't consider all constraints the penalty function decreases value of objective function about value of fitness function for the actual the best solution. This process leads to deleting these from among individuals which are the worst adapted. The penalty function E has value 0 when all the constraints are fulfilled. As optimization method the real-coded genetic algorithm was chosen. More details about genetic algorithm can be found in [25].

5. Numerical results for some examples of BVPs

In this section, three examples of BVP with Poisson equation are numerically analysed by MFS with searching of optimal method parameter. These examples are chosen from literature for which exact solutions are known. Having the values of exact solution $u_d = \{u_d^i\}_{i=1...M}$ and the approximate solution $u_p = \{u_p^i\}_{i=1...M}$ for M points in domain (control points), the relative error is evaluated by

$$\delta = \{\delta_i\} = \frac{u_d^i - u_p^i}{u_{\max}^i} . \tag{18}$$

The maximal relative error $\delta_{\max}$ is chosen from $\{\delta_i\}$.

The second rate of estimation for obtained solutions is the error on the boundary. It is very valuable coefficient to assessment of received solution when the exact solution is unknown. The relative error on the boundary is calculated base on values of boundary conditions $u_d = \{u_d^i\}_{i=1...M}$ and approximated solution $u_p = \{u_p^i\}_{i=1...M}$ for M points which are placed on the boundary. The relative error δ_{bnd} on the boundary is solved similarly to relative error (18) but when for all boundaries the value of condition is equal 0 the boundary error is solved for $u_d^{\max} = 1$. This situation is occurred for second example presented in this paper. The interpolation error δ_{int} is defined similarly to both above. This coefficient is solved on the base: the right side of the governing equation (1) and obtained interpolation function, for certain number of control points. For calculations of $\delta_{\max}$ and δ_{int} the number of control points have been equal to 441, and for δ_{bnd} the control points have been 84. These points are placed evenly in domain Ω and the boundary $\partial\Omega$.

In all cases the considered domain is unit square $\Omega = [0,1] \times [0,1]$ and boundary conditions results from form of exact solution. In numerical experiences the following parameters of genetic algorithm were used in calculations: Population Size – 30, Probability Crossover – 0.9, Probability Mutation – 0.1, and Number of Generations – 30. For all three cases of BVPs five types of RBFs are used.

The optimisation problem has been used in two forms. In first the selection of parameters is treated as optimisation method with objective function in exact solution aspect. If the exact solution is unknown the objective function can be defined in the same way as multi-criteria optimization method called weighting method [41], when the individual objective functions (the error on the boundary and the interpolation error) have the same weights. The comparison of two approaches is presented below for three following examples.

Example 1

As the first example, we shall consider the BVP governed with the differential equation [23, 24]

$$\nabla^2 u = 13 \exp(-2x + 3y). \tag{19}$$

We shall consider that the boundary conditions result from exact solution in the form

$$u(x, y) = \exp(-2x + 3y). \tag{20}$$

The profile of right-hand side of Eq. (19) is presented in Fig. 2. The optimal parameters and maximal relative error for BVP formulated by Eq. (19) with boundary conditions resulting with exact solution (20) on the boundary are given in Tables 4 and 5.

As the objective function the exact solution has been used. These tables present results for each of five RBFs. The best solution has been obtained for RBF = 2.

Tables 6 and 7 present results for objective function without exact solution. The best solution has been received for the same RBF as previous occurrence but the $\delta_{\max}$ is a little worse in comparison with results in Table 5. However there are cases where the use of second definition of objective function resulted in better solutions (for example RBF: 1 and 4).

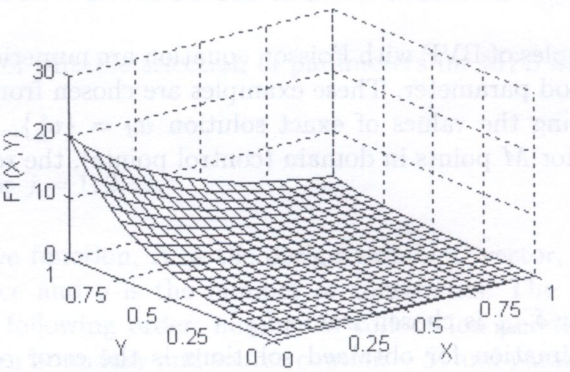

Fig. 2. The right hand side Poisson's equation (18)

Table 4. The optimal parameters for objective function in exact solution aspect IP = 1

RBF	a_0	a_2	a_2	$a_3/c/n$	NC	N	S
1	−1.139E+01	1.930E−01	3.328E+01	4.663E+01	40	34	2.155E+00
2				10	26	23	5.989E−01
3				1.032E+00	33	25	2.157E−01
4				1.989E+00	39	18	1.121E−01
5				1.176E+00	43	27	9.447E−02

Table 5. The errors for IP = 1 (continuation of Table 4)

RBF	δ_{int}	$\delta_{\max}$	δ_{bnd}
1	2.918E−03	4.308E−05	5.816E−06
2	1.065E−05	1.889E−07	4.503E−08
3	4.140E−05	3.544E−05	2.522E−06
4	5.785E−03	6.320E−05	6.847E−07
5	1.806E−02	1.541E−03	2.440E−04

Table 6. The optimal parameters for objective function without exact solution IP = 1

RBF	a_0	a_2	a_2	$a_3/c/n$	NC	N	S
1	−1.139E+01	1.930E−01	3.328E+01	4.663E+01	40	34	2.155E+00
2				10	26	23	5.989E−01
3				1.032E+00	33	25	2.157E−01
4				1.989E+00	39	18	1.121E−01
5				1.176E+00	43	27	9.447E−02

Table 7. The errors for IP = 1 (continuation of Table 6)

RBF	δ_{int}	$\delta_{\max}$	δ_{bnd}
1	3.821E−03	4.240E−05	2.717E−05
2	1.065E−05	1.922E−07	1.675E−09
3	4.031E−05	2.133E−02	1.119E−06
4	5.777E−03	6.318E−05	2.033E−06
5	3.964E−05	3.837E−01	1.440E−06

Example 2

In [22] the following equation was considered,

$$\nabla^2 u = -\frac{\sin(\pi x)\,\sin(\pi y)}{2\pi^2} \tag{21}$$

with exact solution

$$u(x, y) = \sin(\pi x)\,\sin(\pi y) \tag{22}$$

The profile of right-hand side of Eq. (21) is presented in Fig. 3. The optimal parameters and maximal relative error for BVP formulated by Eq. (21) with boundary conditions resulting with exact solution (22) on the boundary are given in Tables 8 and 9.

These results were obtained for objective function defined by exact solution. For this case the best solution was given for RBF = 2. Tables 10 and 11 show optimal values of parameters and received errors for objective function without exact solution.

For RBF = 1 when the second kind of objective function was used the value of is less than in other case. For RBF = 5 the least (optimal) values of boundary and interpolation errors have led to bad solution ($\delta_{\max} = 3.331$).

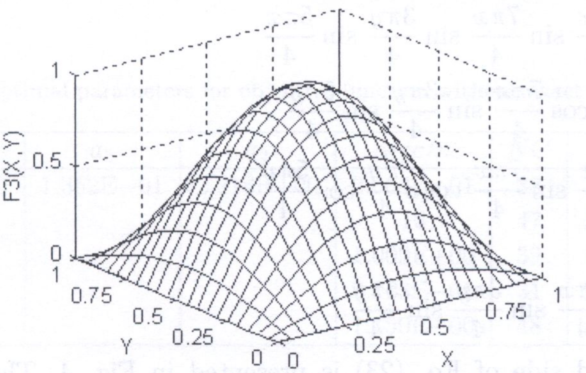

Fig. 3. The right hand side Poisson's equation (20)

Table 8. The optimal parameters for objective function in exact solution aspect IP = 2

RBF	a_0	a_2	a_2	$a_3/c/n$	NC	N	S
1	1.715353	0	0	1.038E+01	31	20	2.308E−01
2				9	48	35	1.197E+00
3				1.000E+00	50	40	2.429E+00
4				1.283E+00	48	29	1.049E+00
5				4.366E−01	45	39	5.966E−01

Table 9. The errors for IP = 1 (continuation of Table 8)

RBF	δ_{int}	$\delta_{\max}$	δ_{bnd}
1	4.265E−03	1.626E−04	1.411E−08
2	6.043E−05	1.524E−06	2.381E−08
3	2.970E−05	1.648E−03	1.668E−07
4	5.139E−03	1.663E−04	3.235E−06
5	2.270E−03	4.073E−03	2.408E−05

Table 10. The optimal parameters for objective function without exact solution IP = 2

RBF	a_0	a_2	a_2	$a_3/c/n$	NC	N	S
1	3.640E+00	7.959E−02	2.098E+01	2.679E+01	23	11	1.454E+00
2				9	34	19	3.722E−01
3				1.077E+00	42	28	3.491E−01
4				1.998E+00	19	13	1.364E+00
5				1.192E+00	23	14	5.291E−01

Table 11. The errors for IP = 1 (continuation of Table 10)

RBF	δ_{int}	δ_{max}	δ_{bnd}
1	3.662E−03	1.266E−04	2.440E−06
2	6.043E−05	1.570E−06	2.022E−10
3	1.268E−05	4.078E−01	2.519E−07
4	4.546E−03	1.694E−04	1.462E−06
5	1.445E−05	3.331E+00	2.349E−06

Example 3

In [8] the following equation was considered

$$\begin{aligned}\nabla^2 u = &-\frac{751\pi^2}{144}\sin\frac{\pi x}{6}\sin\frac{7\pi x}{4}\sin\frac{3\pi y}{4}\sin\frac{5\pi x}{4}\\ &+\frac{7\pi}{12}\cos\frac{\pi x}{6}\cos\frac{7\pi x}{4}\sin\frac{3\pi y}{4}\sin\frac{5\pi y}{4}\\ &+\frac{15\pi^2}{8}\sin\frac{\pi x}{6}\sin\frac{7\pi x}{4}\cos 3\frac{\pi y}{4}\cos\frac{5\pi y}{4}\end{aligned} \tag{23}$$

with the exact solution

$$u(x,y) = \sin\frac{\pi x}{6}\sin\frac{7\pi x}{4}\sin\frac{3\pi y}{4}\sin\frac{5\pi y}{4}. \tag{24}$$

The profile of right-hand side of Eq. (23) is presented in Fig. 4. The optimal parameters and maximal relative error for BVP formulated by Eq. (23) with boundary conditions resulting with exact solution (24) on the boundary are given in Tables 12 and 13.

Similarly to previous examples, the results in Tables 14 and 15 concern to optimization's results for objective function without exact solution. For both objective functions the best solutions have been obtained for RBF = 2. The worst solution has been received by RBF = 5.

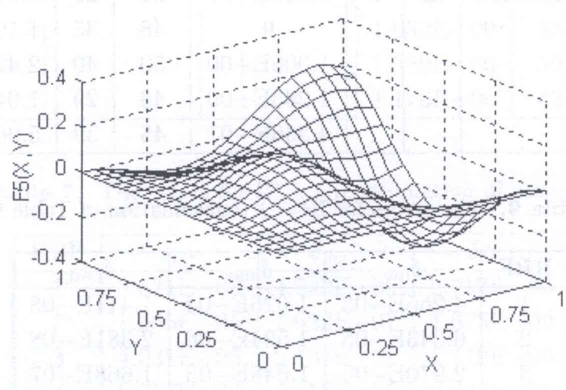

Fig. 4. The right hand side Poisson's equation (22)

Table 12. The optimal parameters for objective function in exact solution aspect IP = 3

RBF	a_0	a_2	a_2	$a_3/c/n$	NC	N	S
1	3.159E+01	0.000E+00	0.000E+00	3.876E+00	49	27	2.140E+00
2				10	48	34	1.690E+00
3				9.998E−01	38	33	7.618E−01
4				6.216E−01	49	13	9.416E−01
5				3.016E−01	24	17	3.463E−01

Table 13. The errors for IP = 1 (continuation of Table 12)

RBF	δ_{int}	δ_{max}	δ_{bnd}
1	2.348E−02	1.709E−03	3.018E−04
2	1.776E−03	1.906E−04	3.609E−05
3	4.717E−04	2.835E−03	4.543E−06
4	1.261E−02	1.188E−03	1.447E−04
5	1.350E−02	9.020E−02	8.997E−06

Table 14. The optimal parameters for objective function without exact solution IP = 3

RBF	a_0	a_2	a_2	$a_3/c/n$	NC	N	S
1	3.195E+00	1.362E−01	2.218E+00	2.664E+01	38	14	2.993E+00
2				10	47	23	1.991E−01
3				1.009E+00	39	16	4.729E−01
4				5.713E−01	31	23	3.351E−01
5				1.150E+00	48	31	6.405E−02

Table 15. The errors for IP = 1 (continuation of Table 14)

RBF	δ_{int}	δ_{max}	δ_{bnd}
1	1.340E−02	2.046E−03	7.784E−04
2	1.776E−03	2.075E−04	4.692E−09
3	4.644E−04	1.615E−01	2.588E−05
4	1.144E−02	1.205E−03	2.161E−06
5	2.476E−04	6.448E+00	2.631E−05

Table 16. Comparison obtained the best solution with other results

Task	[30]	[24]	[35]	[7]	here
1	2.80E−05	–	8.98E−02	–	1.89E−07
2	–	2.92E−03	–	–	1.52E−06
3	–	1.91E−03	–	3.00E−02	1.90E−04

Table 16 shows results from comparison obtained solutions and solutions presented by other authors. For each example the optimization approach provided better results. Figure 5 shows average objective function's values for individual generations of GA. The average values were solved for all optimization processes presented in this paper. It follows that evolution process was quickly convergent and 30 generations were sufficient.

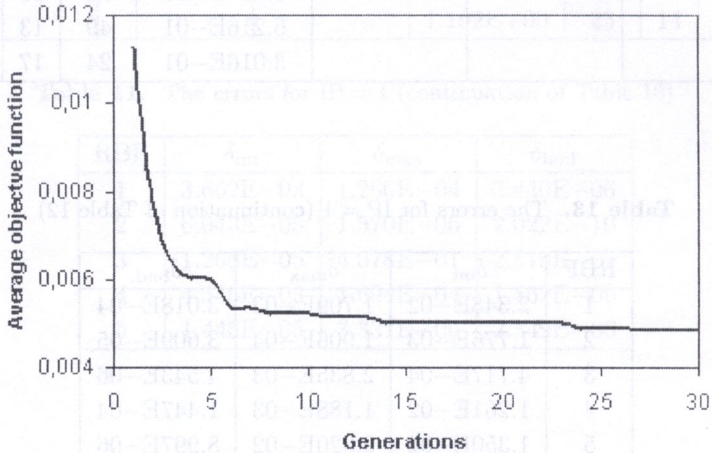

Fig. 5. Average objective function in all generations

The optimisation process consists of many calculations of BVP. If one BVP was solved in about 0.5 s, then one optimization process by genetic algorithm, for which population size equals 30 and generations equals 30, lasted around 8 min. All computations in this paper were performed in double precision on computer with processor Athlon 1700+ and 256 MB RAM. The programs have been written in Borland Delphi 3.0.

6. Conclusions

In the paper the evolutionary algorithm was successfully applied to find optimal parameters of method of fundamental solutions for three boundary value problems with Poisson equation related with steady heat conduction at presence of heat generation. We conclude this paper with the following observations:

- Application of RBF and MFSs allows solution of heat conduction problems with functions of steady heat generation.
- The genetic algorithm allows determination of optimal method parameters of MFS and RBF.
- The quality of solution of IBVP strongly depends on the values of method parameters in particular on shape parameters of RBF
- The optimal values of method parameters depend on case of IBVP and choice of RBF. Hence, one cannot say, for example that good value of parameter S is 0.1.
- Space created by parameters of MFS and RBF is multi-modal function in which variables are binary and floating point. This fact constrains choice of optimisation method.
- Essential influence on quality of solution has form of RBF. The best results are obtained for polyharmonic spline function.

- The main disadvantage of this proposal is the large computing time required to obtain the optimal method parameters of MFS and RBF because the genetic algorithm is a stochastic search method.
- Generally, both forms of objective functions (function on the base of interpolation error or function on the base of error on the boundary) bring similar values for each kind of errors. These results suggested that the problems for which the exact solution is unknown, the results obtained from optimisation process with second objective function may be near proper solutions.

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