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Application of Trefftz method for temperature rise analysis on human skin exposed to radiation

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This paper describes the application of the Trefftz method to the temperature rise in human skin exposed to radiation from a cellular phone. A governing equation is given as the Poisson equation. An inhomogeneous term of the equation is approximated with a polynomial function in Cartesian coordinates. The use of the approximated term transforms the original boundary-value problem to that governed with a homogeneous differential equation. The transformed problem can be solved by the traditional Trefftz formulation. Firstly, the present method is applied to a simple numerical example in order to confirm the formulation. The temperature rise in a skin exposed to radiation is considered as a second example.

Keywords: Trefftz method, Poisson equation, polynomial function

1. Introduction

There has been an increasing public concern regarding the possible health effects of human exposure to an electromagnetic radiation. The principal biological effect of the radiation has been considered to be dominantly thermal in nature. The hazardous electromagnetic field levels can be quantified analyzing the thermal response of the human body exposed to the radiation [7, 20].

In this paper, we will focus on the temperature rise in human skin exposed to the radiation. The problem to be solved is governed with the Poisson equation with adequate boundary conditions. There is an inhomogeneous term in the equation and therefore, in general, it is not adequate to use the boundary-type solution procedures such as the boundary element methods, method of fundamental solutions and the Trefftz methods because there exist domain integrals due to the inhomogeneous term in the integral equation. In order to transform the domain integrals due to boundary integrals, several formulations have been presented; dual reciprocity method [18, 19], multiple reciprocity method [16, 17], radial bases function approximation [3, 8, 11, 14, 21], polynomial function approximation method [23], boundary point interpolation method [13], method of fundamental solution [1, 5, 12, 15] and so on.

In this paper, the Trefftz method is applied to the analysis of the temperature rise in human skin exposed to the radiation [4, 9, 22]. The problem to be solved is governed with the Poisson equation. An inhomogeneous term of the equation is approximated with a polynomial function in Cartesian coordinates. The use of the approximated term transforms the boundary value problem of Poisson equation to that of Laplace equation. The boundary value problem governed with the Poisson equation can be solved easily according to the traditional Trefftz formulation. In the present method, the system of equations is derived according to the collocation-type formulation. It is not necessary to do boundary integrals as well as domain integrals and therefore, the formulation is much simpler than the other methods.

The remaining of the paper is organized as follows. In Section 2, the formulation of the problem to be solved is described. In Section 3, the present method is applied to some numerical examples. Section 4 summarizes the conclusions.

2. FORMULATION

2.1. Governing equation and boundary condition [7, 20]

For harmonically varying electromagnetic field, the temperature rise on the human skin can be calculated from the bio-heat equation

$$C\rho \frac{\partial u}{\partial t} = K\nabla^2 u + Q_{em} - B(u - u_b) \tag{1}$$

where u is the temperature of the tissue, u_b is the temperature of the blood, K is the thermal conductivity of the tissue, C is the heat capacity of the tissue, B is the term associated with blood flow, and Q_{em} is the electromagnetic power deposition.

The electromagnetic power deposition Q_{em} is given as

$$Q_{em} = \rho \cdot SAR \tag{2}$$

where ρ and SAR denote the tissue density and specific absorption rate (SAR), respectively. For harmonically varying electromagnetic field, SAR is defined as

$$SAR = \frac{\sigma}{2\rho} |E|^2 \tag{3}$$

where $|E|$ is the peak value of electromagnetic field, and σ is the conductivity of the tissue.

In addition, the boundary condition for Eq. (1) is given by

$$u = \bar{u}, \tag{4}$$

$$q \equiv \frac{\partial u}{\partial n} = \bar{q}, \tag{5}$$

and, on the skin surface,

$$H \cdot (u - u_a) = -Kq \tag{6}$$

where H and u_a are the convection coefficient and temperature of the air, respectively.

At the thermal steady state $\partial u / \partial t = 0$ and therefore Eq. (1) is reduced to

$$\nabla^2 u + b = 0 \tag{7}$$

where

$$b = \frac{1}{K} [\rho \cdot SAR - B(u - u_a)]. \tag{8}$$

The boundary conditions (4)–(6) are rewritten as

$$\alpha u + \beta q = \gamma \tag{9}$$

where α, β and γ denote the parameters derived from the boundary conditions.

2.2. Trefftz Formulation [10]

Now, we consider the boundary value problem defined by Eqs. (7) and (9). An inhomogeneous term (8) is approximated with a polynomial function in Cartesian coordinates

$$b = c^T r. \tag{10}$$

In this study, the fifth-order polynomial is adopted for the function $\boldsymbol{r}$. Therefore, $\boldsymbol{c}$ and $\boldsymbol{r}$ are defined respectively as follows

$$\begin{aligned} \boldsymbol{c}^T &= \{c_1\,, c_2\,, \ldots\,, c_{21}\}, \\ \boldsymbol{r}^T &= \{r_1\,, r_2\,, r_3\,, \ldots\,, r_{21}\} \\ &= \{1,\ \ x, y,\ \ x^2, xy, y^2,\ \ x^3, x^2y, xy^2, y^3,\ \ x^4, x^3y, x^2y^2, xy^3, y^4,\ \ x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5\}. \end{aligned} \tag{11, 12}$$

The use of Eq. (10) transforms the original governing equation into

$$\nabla^2 u + \boldsymbol{c}^T \boldsymbol{r} = 0. \tag{13}$$

Since the term r_i is a polynomial function, the related particular solution u_i^p can be determined easily. u_i^p satisfies the equation

$$\nabla^2 u_i^p + r_i = 0. \tag{14}$$

In the Trefftz method, the homogeneous solution of the governing equation u^h is approximated with the superposition of the related T-complete function u_i^* [6]. The unknown function u is approximated with the T-complete function u_i^* and the particular solution u_i^p as follows,

$$u = u^h + \boldsymbol{c}^T \boldsymbol{u}^p = \boldsymbol{a}^T \boldsymbol{u}^* + \boldsymbol{c}^T \boldsymbol{u}^p, \tag{15}$$

where $\boldsymbol{a}$ denotes the unknown parameter vector for approximating the homogeneous solution. Besides, u_i^* and u_i^p satisfy the equations

$$\begin{aligned} \nabla^2 u_i^* \quad &= 0, \\ \nabla^2 u_i^p + r_i &= 0. \end{aligned}$$

Equation (15) satisfies Eq. (7) but does not satisfy Eq. (9). Substituting Eq. (15) into Eq. (9) leads to residual expressions

$$\begin{aligned} R &= \alpha T + \beta q - \gamma \\ &= \alpha(\boldsymbol{a}^T \boldsymbol{u}^* + \boldsymbol{c}^T \boldsymbol{u}^p) + \beta(\boldsymbol{a}^T \boldsymbol{q}^* + \boldsymbol{c}^T \boldsymbol{q}^p) - \gamma \\ &= (\alpha \boldsymbol{u}^* + \beta \boldsymbol{q}^*)^T \boldsymbol{a} + (\alpha \boldsymbol{u}^p + \beta \boldsymbol{q}^p)^T \boldsymbol{c} - \gamma \end{aligned} \tag{16}$$

where

$$q_i^* = \frac{\partial u_i^*}{\partial \boldsymbol{n}}, \qquad q_i^p = \frac{\partial u_i^p}{\partial \boldsymbol{n}}. \tag{17}$$

Satisfying the residual equations at the boundary point P_m by means of the collocation formulation, we have

$$\boldsymbol{K a} = \boldsymbol{f} - \boldsymbol{B c}. \tag{18}$$

In the matrix $\boldsymbol{K}$, the total numbers of the rows and the columns are equal to the total number of the boundary collocation points and the T-complete functions, respectively. Therefore, we shall take more collocation points than the T-complete functions, i.e. $M > N$, and Eq. (18) is solved by using the singular value decomposition method of the LAPACK software [2].

2.3. Determination of parameter c

The unknown parameter vector $\boldsymbol{c}$ in Eq.(18) is determined by using the iterative process. Equation (10) held at the iteration steps (k) and $(k+1)$ are

$$\begin{aligned} b^{(k+1)} &= \boldsymbol{r}^T \boldsymbol{c}^{(k+1)}, \\ b^{(k)} &= \boldsymbol{r}^T \boldsymbol{c}^{(k)}. \end{aligned}$$

Subtracting both sides of the above equations leads to

$$\begin{aligned} b^{(k+1)} - b^{(k)} &= \boldsymbol{r}^T (\boldsymbol{c}^{(k+1)} - \boldsymbol{c}^{(k)}), \\ \Delta b &= \boldsymbol{r}^T \Delta \boldsymbol{c}, \end{aligned} \tag{19}$$

where the superscript (k) denotes the number of iteration.

The collocation points, which are referred as "the computing point", are placed on the boundary and within the domain. Holding Eq. (19) on the computing points and arranging them in the matrix form, we have

$$\boldsymbol{D} \Delta \boldsymbol{c} = \boldsymbol{f} \tag{20}$$

where $\boldsymbol{D}$ and $\boldsymbol{f}$ denote the coefficient matrix and vector, respectively. Equation (20) is solved for $\Delta \boldsymbol{c}$ with the singular value decomposition of LAPACK software [2]. The parameter $\boldsymbol{c}$ is updated with

$$\boldsymbol{c}^{(k+1)} = \boldsymbol{c}^{(k)} + \Delta \boldsymbol{c}. \tag{21}$$

The convergence criterion is defined as

$$\eta \equiv \frac{1}{M_c} \sum_{i=1}^{M_c} |\Delta b(Q_i)| < \eta_c \tag{22}$$

where M_c and η_c denote the total number of the computing points and the positive constant specified by a user, respectively. Q_i denotes the computing points placed on the boundary and within the domain.

3. Numerical examples

Example 1

A first example is considered for confirming the above formulation. An inhomogeneous term b is given as follows,

$$b = u. \tag{23}$$

The object under consideration is a square region of $-1 \leq x \leq 1$, $-1 \leq y \leq 1$ (Fig. 1). The boundary condition is given as follows,

$$\left.\begin{aligned} u + q &= \ \ 1 \quad (x = 1), \\ q &= \ \ 0 \quad (y = 1), \\ u &= -1 \quad (x = -1), \\ q &= \ \ 0 \quad (y = -1). \end{aligned}\right\}$$

Numerical results u are shown in Table 1. The theoretical solutions are obtained by Mathematica. We notice that the numerical solutions well agree with theoretical ones.

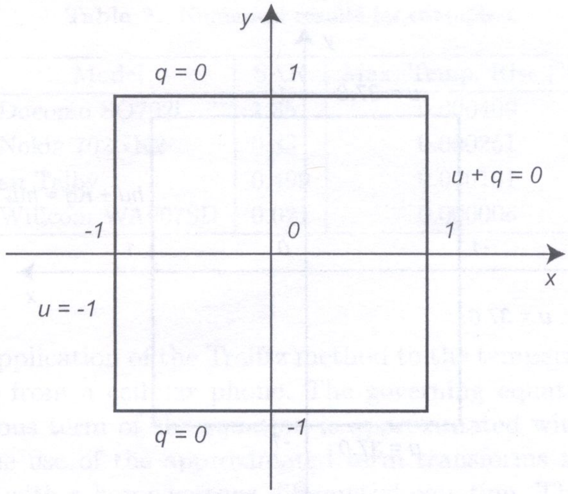

Fig. 1. Numerical example 1

Table 1. Numerical results for example 1

x-coordinate	Theoretical (u)	Present (u)	Error (%)
−1.	−1	−1	0
−0.75	−1.63385	−1.63386	0.00053
−0.5	−2.16612	−2.16614	0.00089
−0.25	−2.56370	−2.56373	0.00096
0.	−2.80189	−2.80193	0.00137
0.25	−2.86587	−2.86591	0.00119
0.5	−2.75167	−2.75171	0.00130
0.75	−2.46638	−2.46642	0.00156
1.	−2.02774	−2.02778	0.00180

Example 2

The temperature rise in a skin is considered in the second example. An inhomogeneous term is defined as follows,

$$b = \rho \cdot SAR - B(u - u_b). \tag{24}$$

The object under consideration is a square region of $-1 \le x \le 1, -1 \le y \le 1$ (Fig. 2). The physical parameters are given as follows: $\rho = 1010\,[\mathrm{kg/m^3}]$, $K = 0.50\,[\mathrm{W/m\,°C}]$, $B = 8650\,[\mathrm{W/m^3\,°C}]$, $u_b = 37.0\,[°\mathrm{C}]$, $h = 10.5\,[\mathrm{W/m^3\,°C}]$, and $u_a = 27.0\,[°\mathrm{C}]$.

The parameter SAR [W/kg] depends on cellular phones. We will consider four cellular phones; Docomo (SO702i), au (Talby), Softbank (Nokia 702NK2), and Willcom (WA007SH), which are popular in Japan. The boundary conditions are given as

$$\left.\begin{aligned} hu + Kq &= hu_a \quad (x = 1),\\ u &= 37.0 \quad (y = 1),\\ u &= 37.0 \quad (x = -1),\\ u &= 37.0 \quad (y = -1). \end{aligned}\right\}$$

Numerical results are shown in Table 2. Distributions of the temperature rise are shown in Figs. 3–6. The temperature rise increases according to the increase of the SAR value. They are 3G-type cellular phones, except for Willcom (WA007SH). However, from the view-point of the rise in the human skin, we notice that Willcom (WA007SH) is the most gentle and that the au(Talby) is gentler than the other 3G-type cellular phones.

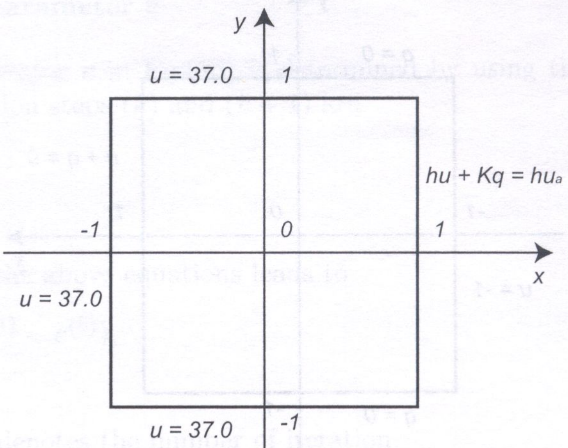

Fig. 2. Numerical example 2

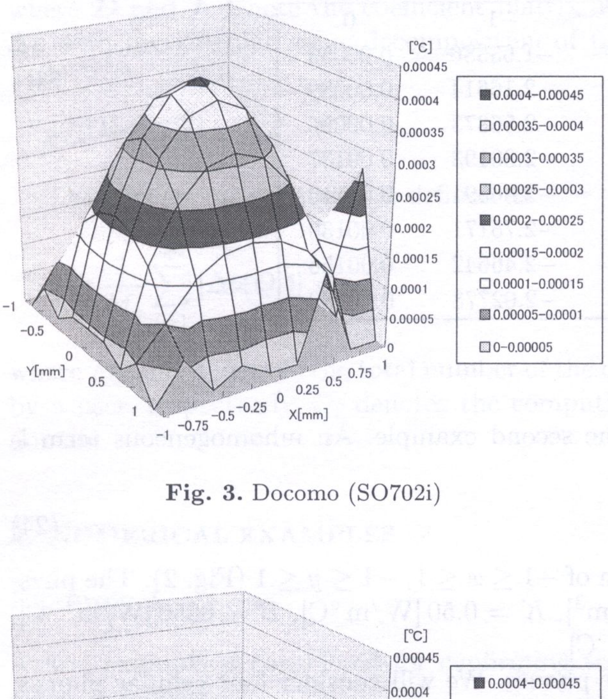

Fig. 3. Docomo (SO702i)

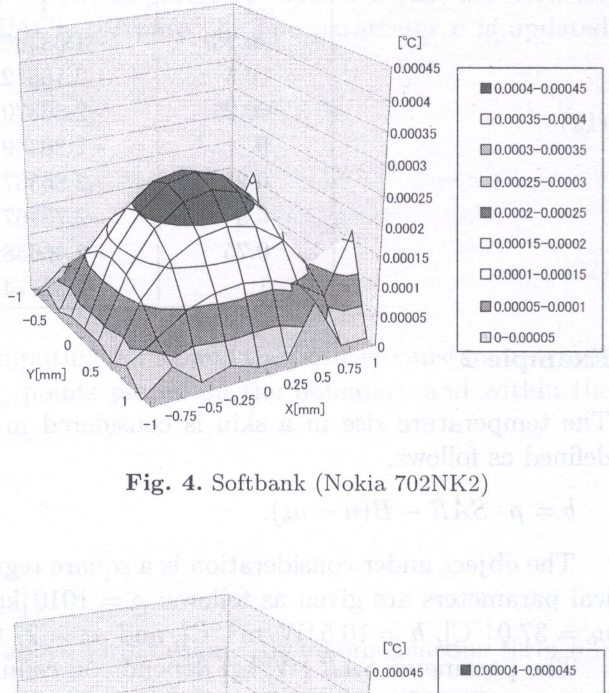

Fig. 4. Softbank (Nokia 702NK2)

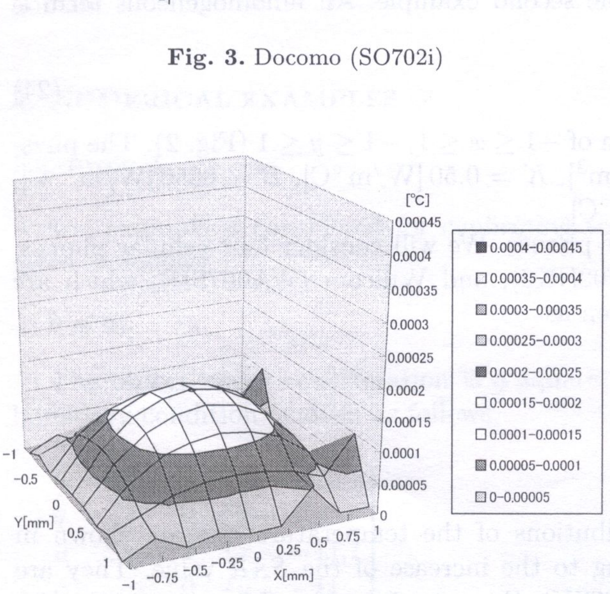

Fig. 5. au (Talby)

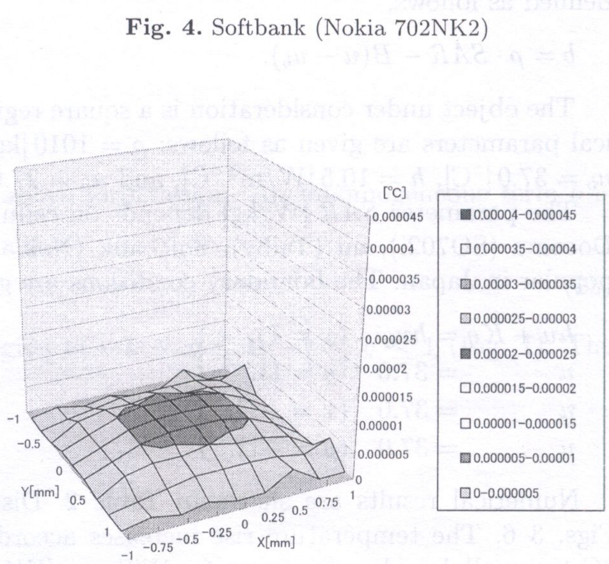

Fig. 6. Willcom (WA007SH)

Table 2. Numerical results for example 2

Model	SAR	Max. Temp. Rise [°C]
Docomo SO702i	1.35	0.000409
Nokia 702NK2	0.83	0.000251
au Talby	0.499	0.000151
Willcom WA007SH	0.021	0.000006

4. Conclusions

This paper describes the application of the Trefftz method to the temperature rise analysis in human skin exposed to radiation from a cellular phone. The governing equation is given as the Poisson equation. An inhomogeneous term of the equation is approximated with a polynomial function in Cartesian coordinates. The use of the approximated term transforms the original boundary-value problem to that governed with a homogeneous differential equation. The transformed problem can be solved by the traditional Trefftz formulation.

Firstly, the present method is applied to a boundary value problem of the Poisson equation. Numerical results well agreed with theoretical solutions. So, in a second example, the method was applied to the temperature rise in human skin exposed to radiation of a cellular phone. We compare four cellular phones; Docomo (SO702i), au (Talby), Softbank (Nokia 702NK2) and Willcom (WA007SH), which are popular in Japan. They are 3G-type cellular phones, except for Willcom (WA007SH). From the view-point of the rise in the human skin, we notice that Willcom (WA007SH) is the most gentle and that the au(Talby) is gentler than the other 3G-type cellular phones.

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