Computer Assisted Mechanics and Engineering Sciences, 14: 145–158, 2007.

Discrete-time impulsive Hopfield neural networks with finite distributed delays

Haydar Akça
Department of Mathematical Science, Faculty of Science
United Arab Emirates University, P.O. Box 17551, Al Ain, UAE

Rajai Alassar
Department of Mathematical Science, King Fahd University of Petroleum and Minerals,
P.O. Box 1071, Dhahran 31261, Saudi Arabia

Valéry Covachev
Department of Mathematics and Statistics, Sultan Qaboos University
Muscat 123, Sultanate of Oman;
Institute of Mathematics, Bulgarian Academy of Sciences
Sofia 1113, Bulgaria

H. Ali Yurtsever
Math Department, Georgetown University, Washington, DC 20057, USA;
Catholic University of America, Washington, DC 20064, USA

(Received in the final form March 20, 2007)

The discrete counterpart of a class of Hopfield neural networks with periodic impulses and finite distributed delays is introduced. A sufficient condition for the existence and global exponential stability of a unique periodic solution of the discrete system considered is obtained.

1. Introduction

A neural network is a network that performs computational tasks such as associative memory, pattern recognition, optimization, model identification, signal processing, etc. on a given pattern via interaction between a number of interconnected units characterized by simple functions. From the mathematical point of view, an artificial neural network corresponds to a nonlinear transformation of some inputs into certain outputs. Many types of neural networks have been proposed and studied in the literature and the Hopfield-type network has become an important one due to its potential for applications in various fields of daily life. The model proposed by Hopfield, also known as Hopfield's graded response neural network, is based on an analogue circuit consisting of capacitors, resistors and amplifiers. More details about artificial neural networks can be found in Section 2.

Hopfield neural networks have found applications in a broad range of disciplines [12–14] and have been studied both in the continuous and discrete time cases by many researchers. Most neural networks can be classified as either continuous or discrete. In spite of this broad classification, there are many real world systems and natural processes that behave in a piecewise continuous style interlaced with instantaneous and abrupt changes (impulses). Periodic dynamics of the Hopfield neural networks is one of the realistic and attractive modellings for the researchers. Signal transmission between the neurons causes time delays. Therefore the dynamics of Hopfield neural networks with discrete or distributed delays has a fundamental concern.

In the present paper, we introduce the discrete counterpart of a class of Hopfield neural networks with periodic impulses and finite distributed delays. Combining some ideas of [2, 26], we obtain a sufficient condition for the existence and global exponential stability of a unique periodic solution of the discrete system considered.

2. ARTIFICIAL NEURAL NETWORKS

An artificial neural network (ANN) is an information processing paradigm that is inspired by the way biological nervous systems, such as the brain, process information. The key element of this paradigm is the novel structure of the information processing system. It is composed of a large number of highly interconnected processing elements (neurons) working in unison to solve specific problems. ANNs, like people, learn by example. An ANN is configured for a specific application, such as pattern recognition or data classification, through a learning process. Learning in biological systems involves adjustments to the synaptic connections that exist between the neurons. This is true of ANNs as well.

Neural network simulations appear to be a recent development. However, this field was established before the advent of computers, and has survived at least one major setback and several eras. Many important advances have been boosted by the use of inexpensive computer emulations. Following an initial period of enthusiasm, the field survived a period of frustration and disrepute.

The first artificial neuron was produced in 1943 by the neurophysiologist Warren McCulloch and the logician Walter Pitts (see [20]). But the technology available at that time did not allow them to do too much. Neural networks process information in a similar way the human brain does. The network is composed of a large number of highly interconnected processing elements (neurons) working in parallel to solve a specific problem. Neural networks learn by example. Much is still unknown about how the brain trains itself to process information, so theories abound.

An artificial neuron is a device with many inputs and one output (Fig. 1). The neuron has two modes of operation; the training mode and the using mode. In the training mode, the neuron can be trained to fire (or not), for particular input patterns. In the using mode, when a taught input pattern is detected at the input, its associated output becomes the current output. If the input pattern does not belong in the taught list of input patterns, the firing rule is used to determine whether to fire or not.

An important application of neural networks is pattern recognition. Pattern recognition can be implemented by using a feed-forward (Fig. 2) neural network that has been trained accordingly. During training, the network is trained to associate outputs with input patterns. When the network is used, it identifies the input pattern and tries to output the associated output pattern. The power of neural networks comes to life when a pattern that has no output associated with it, is given as an input. In this case, the network gives the output that corresponds to a taught input pattern that is least different from the given pattern [4, 9, 10].

The above neuron does not do anything that conventional computers do not already do. A more sophisticated neuron (Fig. 3) is the McCulloch and Pitts model (MCP). The difference from the previous model is that the inputs are 'weighted', the effect that each input has at decision making is dependent on the weight of the particular input. The weight of an input is a number which when multiplied with the input gives the weighted input. These weighted inputs are then added together and if they exceed a pre-set threshold value, the neuron fires. In any other case the neuron does not fire. In mathematical terms, the neuron fires if and only if

$$X_1W_1 + X_2W_2 + X_3W_3 + \cdots > T,$$

where W_i, $i = 1, 2, \ldots$, are weights, X_i, $i = 1, 2, \ldots$, inputs, and T a threshold. The addition of input weights and of the threshold makes this neuron a very flexible and powerful one. The MCP

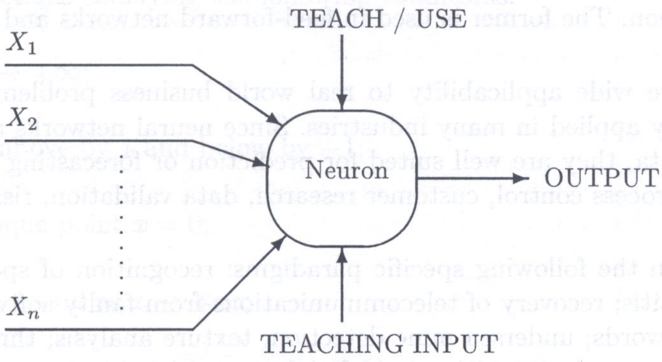

Fig. 1. A simple neuron

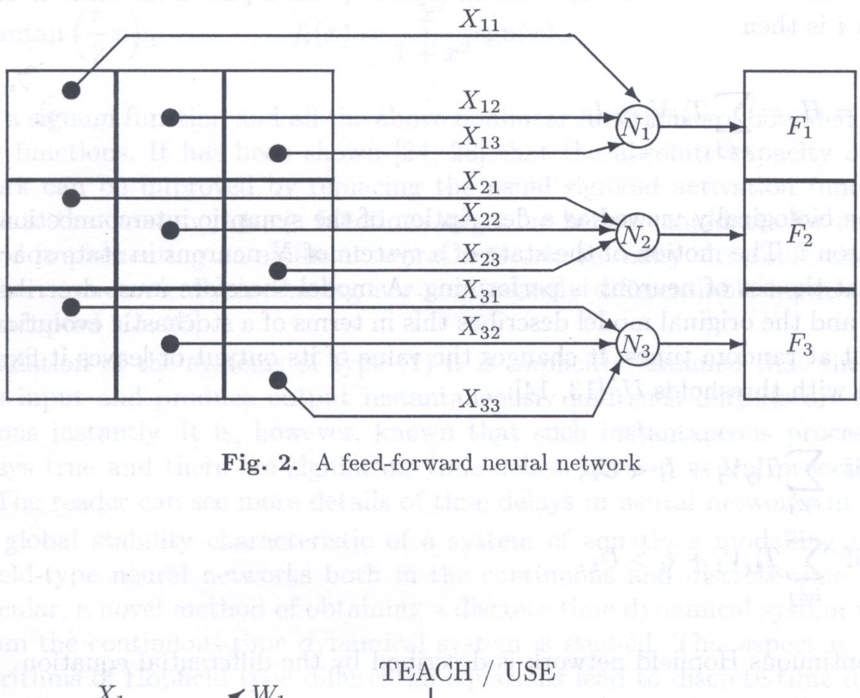

Fig. 2. A feed-forward neural network

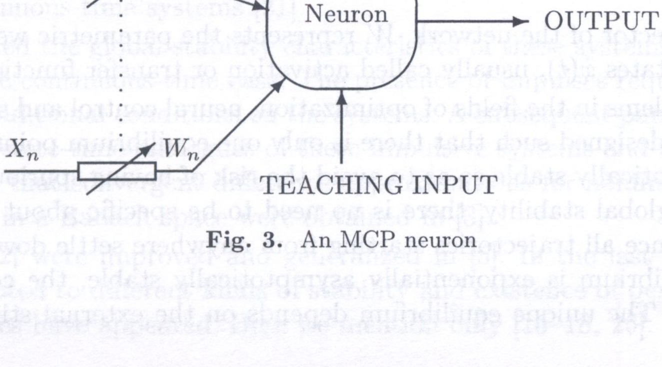

Fig. 3. An MCP neuron

neuron has the ability to adapt to a particular situation by changing its weights and/or threshold. Various algorithms exist that cause the neuron to 'adapt'; the most used ones are the Delta rule and the back error propagation. The former is used in feed-forward networks and the latter in feedback networks.

Neural networks have wide applicability to real world business problems. In fact, they have already been successfully applied in many industries. Since neural networks are best at identifying patterns or trends in data, they are well suited for prediction or forecasting needs including: sales forecasting, industrial process control, customer research, data validation, risk management, target marketing.

ANN are also used in the following specific paradigms: recognition of speakers in communications; diagnosis of hepatitis; recovery of telecommunications from faulty software; interpretation of multi-meaning Chinese words; undersea mine detection; texture analysis; three-dimensional object recognition; hand-written word recognition; and facial recognition.

Hopfield-type (additive) networks have been studied intensively during the last two decades and have been applied to optimization problems [6–8, 11, 13, 14, 23]. Their starting point was marked by the publication of two papers [12, 13] by Hopfield. The original model used two-state threshold 'neurons' that followed a stochastic algorithm: each model neuron i had two states, characterized by the values V_i^0 or V_i^1 (which may often be taken as 0 and 1, or -1 and 1, respectively). The input of each neuron came from two sources, external inputs I_i and inputs from other neurons. The total input to neuron i is then

$$\text{Input to}\quad i = H_i = \sum_{i \neq j} T_{ij} V_j + I_i$$

where T_{ij} can be biologically viewed as a description of the synaptic interconnection strength from neuron j to neuron i. The motion of the state of a system of N neurons in state space describes the computation that the set of neurons is performing. A model therefore must describe how the state evolves in time, and the original model describes this in terms of a stochastic evolution. Each neuron samples its input at random times. It changes the value of its output or leaves it fixed according to a threshold rule with thresholds U_i [13, 14]:

$$V_i \to V_i^0 \quad \text{if} \quad \sum_{i \neq j} T_{ij} V_j + I_i < U_i\,,$$

$$V_i \to V_i^1 \quad \text{if} \quad \sum_{i \neq j} T_{ij} V_j + I_i > U_i\,.$$

The simplest continuous Hopfield network is described by the differential equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t) + Wf[x(t)] \tag{1}$$

where $x(t)$ is the state vector of the network, W represents the parametric weights, and f is a nonlinearity acting on the states $x(t)$, usually called activation or transfer function.

In order to solve problems in the fields of optimization, neural control and signal processing, neural networks have to be designed such that there is only one equilibrium point and this equilibrium point is globally asymptotically stable so as to avoid the risk of having spurious equilibria and local minima. In the case of global stability, there is no need to be specific about the initial conditions for the neural circuits since all trajectories starting from anywhere settle down at the same unique equilibrium. If the equilibrium is exponentially asymptotically stable, the convergence is fast for real-time computations. The unique equilibrium depends on the external stimulus. The nonlinear neural activation functions $f_i(\cdot)$, $i \in \mathbb{Z}^+$, are usually chosen to be continuous and differentiable nonlinear sigmoid functions satisfying the following conditions:

(a) $f_i(x) \to \mp 1$ as $x \to \mp\infty$;

(b) $f_i(x)$ is bounded above by 1 and below by -1;

(c) $f_i(x) = 0$ at a unique point $x = 0$;

(d) $f_i'(x) > 0$ and $f_i'(x) \to 0$ as $x \to \mp\infty$;

(e) $f_i'(x)$ has a global maximum value of 1 at the unique point $x = 0$.

Some examples of activation functions $f_i(\cdot)$ are

$$f_i(x) = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad f_i(x) = \frac{1 - e^{-x}}{1 + e^{-x}} = \tanh(x/2),$$

$$f_i(x) = \frac{2}{\pi} \arctan\left(\frac{\pi}{2}x\right), \qquad f_i(x) = \frac{x^2}{1 + x^2}\operatorname{sgn}(x),$$

where $\operatorname{sgn}(\cdot)$ is a signum function and all the above nonlinear functions are bounded, monotonic and non-decreasing functions. It has been shown [24, 28] that the absolute capacity of an associative memory network can be improved by replacing the usual sigmoid activation functions. There, it seems appropriate that non-monotonic functions might be better candidates for neuron activation in designing and implementing an artificial neural network. In many electronic circuits, amplifiers that have neither monotonically increasing nor continuously differentiable input-output functions are frequently adopted [15, 25].

In the formulation of the systems of type (1) it is implicitly assumed that the neurons of the system process input and produce output instantaneously and such outputs are delivered to the receiving neurons instantly. It is, however, known that such instantaneous processing and delivery is not always true and there are significant time delays both in neural processing and axonal transmission. The reader can see more details of time delays in neural networks in [19].

In [22] the global stability characteristic of a system of equations modelling the dynamics of additive Hopfield-type neural networks both in the continuous and discrete-time cases is investigated. In particular, a novel method of obtaining a discrete-time dynamical system whose dynamics is inherited from the continuous-time dynamical system is studied. This aspect is important since numerical algorithms of Hopfield-type differential equations lead to discrete-time dynamic systems and such discrete-time systems should not give rise to any spurious behaviour if either system is to be used for coding equilibrium as associative memories corresponding to temporally uniform external stimuli obtained. The discrete-time models serve as global numerical methods on unbounded intervals for the continuous-time systems [21].

In [1] we investigated the global stability characteristics of these systems supplemented with impulse conditions in the continuous-time case. The presence of impulses required some modifications and the imposing of additional conditions on the systems. A subsequent paper [2] was devoted to the formulation of the discrete-time analogues of these impulsive systems and the investigation of their stability. Let us recall that convergent difference approximations for nonlinear impulsive systems of differential equations in a Banach space were obtained in [3].

The results of [1, 2] were improved and generalized in [5]. In the last three years (2004–2006) numerous papers devoted to different kinds of stability and existence of periodic solutions of neural networks with impulses have appeared. Here we mention only [16–18, 26].

3. Statement of the problem. Main result

In [26], the authors consider a class of Hopfield neural networks with periodic impulses and finite distributed delays, which are formulated in the form of a system of impulsive delay differential equations

$$\begin{cases} \dfrac{\mathrm{d}x_i}{\mathrm{d}t} = -a_i x_i(t) + \sum\limits_{j=1}^{m} b_{ij} g_j(x_j(t)) + \sum\limits_{j=1}^{m} \int_0^{\omega} c_{ij}(s)\, g_j(x_j(t-s))\,\mathrm{d}s + d_i(t), \\ t > 0, \quad t \neq t_k, \\ x_i(t_k + 0) = \beta_{ik} x_i(t_k), \qquad i = \overline{1,m}, \quad k \in \mathbb{Z}^+, \end{cases} \tag{2}$$

where m is the number of neurons in the network, $x_i(t)$ is the state of the i-th neuron at time t, $a_i > 0$ is the rate at which the i-th neuron resets the state when isolated from the system, b_{ij} is the connection strength from the j-th neuron to the i-th one, $g_j(\cdot)$ are the transfer functions, ω is the maximum transmission delay from one neuron to another, $c_{ij}(\cdot)$ is the delayed connection strength function from the j-th neuron to the i-th one, $d_i(t)$ is the ω-periodic external input to the i-th neuron, $\mathbb{Z}^+$ is the set of all positive integers, t_k ($k \in \mathbb{Z}^+$) are the instants of impulse effect, β_{ik} ($i = \overline{1,m}$, $k \in \mathbb{Z}^+$) are constants. Let us assume that:

H1. For $j = \overline{1,m}$, $g_j(\cdot)$ is globally Lipschitz continuous with Lipschitz constant L_j,

$$|g_j(x) - g_j(y)| \leq L_j |x - y| \qquad \text{for all } x, y \in \mathbb{R}.$$

H2. For $i, j = \overline{1,m}$, $c_{ij}(\cdot)$ is absolutely integrable on $[0, \omega]$.

H3. $0 = t_1 < t_2 < \cdots < t_p < \omega$, $t_{k+p} = t_k + \omega$, $\beta_{i,k+p} = \beta_{ik}$ for $i = \overline{1,m}$, $k \in \mathbb{Z}^+$.

H4. There exist positive numbers $\lambda_1, \ldots, \lambda_m$ such that

$$\lambda_i a_i > L_i \sum_{j=1}^{m} \lambda_j \left(|b_{ji}| + \int_0^{\omega} |c_{ji}(s)|\,\mathrm{d}s \right), \qquad i = \overline{1,m}.$$

Later in the paper [26] system (2) is assumed to be accompanied by the initial condition

$$x(r) = \psi(r), \qquad r \in [-\omega, 0], \tag{3}$$

where $\psi : [-\omega, 0] \to \mathbb{R}^m$ is piecewise continuous with discontinuities of the first kind at the points $t_k - \omega$, $k = \overline{2,p}$. Moreover, ψ is left-continuous at each discontinuity point and satisfies

$$\psi_i(t_k - \omega + 0) = \beta_{ik} \psi_i(t_k - \omega), \qquad i = \overline{1,m}, \quad k = \overline{2,p}.$$

The solution of the initial value problem (2)–(3) is denoted by $x(t, \psi)$. Under the assumption that $|\beta_{ik}| \leq 1$ for all $i = \overline{1,m}$ and $k = \overline{1,p}$, making use of the Contraction mapping principle in a suitable Banach space, in [26] it is proved that system (2) is globally exponentially periodic, that is, it possesses a periodic solution $x(t, \psi^*)$ and there exist positive constants α and β such that every solution $x(t, \psi)$ of Eq. (2) satisfies

$$\|x(t, \psi) - x(t, \psi^*)\| \leq \alpha \|\psi - \psi^*\| e^{-\beta t} \qquad \text{for all } t \geq 0.$$

Here

$$\|\psi\| = \sup_{-\omega \leq r \leq 0} \max_{i=\overline{1,m}} |\psi_i(r)|, \qquad \|x(t, \psi)\| = \max_{i=\overline{1,m}} |x_i(t, \psi)|.$$

Now we shall formulate the discrete counterpart of problem (2)–(3). For $N \in \mathbb{Z}^+$ we choose the discretization step $h = \omega/N$. For the moment we assume N so large that

$$h < \min_{k=\overline{1,p}} (t_{k+1} - t_k).$$

In this case each interval $[nh, (n+1)h]$ contains at most one instant of impulse effect t_k.

For convenience we denote $n = [t/h]$, the greatest integer in t/h, for $t \geq -\omega$, $n_k = [t_k/h]$. Also, by abuse of notation we write $x_i(nh) = x_i(n)$.

Let $n \in \mathbb{Z}^+$, $n \neq n_k$. This means that the interval $[nh, (n+1)h]$ contains no instant of impulse effect t_k. Following [22], we approximate the differential equation in Eq. (2) on the interval $[nh, (n+1)h]$ by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x_i(t)e^{a_i t}\right) = e^{a_i t}\left\{\sum_{j=1}^{m} b_{ij} g_j(x_j(n)) + \sum_{j=1}^{m}\sum_{\nu=1}^{N} C_{ij}(\nu)\, g_j(x_j(n-\nu)) + d_i(n)\right\}, \qquad i = \overline{1,m},$$

where the quantities $C_{ij}(\nu)$ are suitably chosen, say, $C_{ij}(\nu) = \int_{(\nu-1)h}^{\nu h} c_{ij}(s)\,\mathrm{d}s$ or $C_{ij}(\nu) = c_{ij}(\nu)h$. We prefer the first choice, so that $\sum_{\nu=1}^{N} C_{ij}(\nu) = \int_0^1 c_{ij}(s)\,\mathrm{d}s$ is independent of h.

We integrate this differential equation over the interval $[nh, (n+1)h]$ to obtain

$$x_i(n+1)e^{a_i(n+1)h} - x_i(n)e^{a_i nh} = \frac{e^{a_i(n+1)h} - e^{a_i nh}}{a_i}$$
$$\cdot\left\{\sum_{j=1}^{m} b_{ij} g_j(x_j(n)) + \sum_{j=1}^{m}\sum_{\nu=1}^{N} C_{ij}(\nu) g_j(x_j(n-\nu)) + d_i(n)\right\}.$$

If we denote by $\phi_i(h)$ the positive quantities

$$\phi_i(h) = \frac{1 - e^{-a_i h}}{a_i}, \qquad i = \overline{1,m},$$

we can rewrite the last equation in the form

$$x_i(n+1) = e^{-a_i h} x_i(n) + \phi_i(h)\left[\sum_{j=1}^{m} b_{ij} g_j(x_j(n)) + \sum_{j=1}^{m}\sum_{\nu=1}^{N} C_{ij}(\nu) g_j(x_j(n-\nu)) + d_i(n)\right],$$
$$i = \overline{1,m}, \qquad n \neq n_k. \tag{4}$$

Next, for $n = n_k$ the interval $[nh, (n+1)h]$ contains the instant of impulse effect t_k. On this interval we approximate the impulse condition in (2) by

$$x_i(n_k + 1) = \beta_{ik} x_i(n_k), \qquad i = \overline{1,m}, \quad k \in \mathbb{Z}^+. \tag{5}$$

Finally, the initial condition (3) is replaced by

$$x(n) = \psi(n), \qquad n = -N, -N+1, \ldots, 0, \tag{6}$$

where $\psi = (\psi_i, \ldots, \psi_m) : \{-N, -N+1, \ldots, 0\} \to \mathbb{R}^m$. We assume that ψ satisfies $\psi_i(-N) = \psi_i(0)$ and

$$\psi_i(n_k + 1 - N) = \beta_{ik}\psi_i(n_k - N), \qquad i = \overline{1,m}, \quad k = \overline{1,p}.$$

We can regard the initial functions ψ as elements of the vector space

$$C^* = \{\psi_i(n) \mid \; i = \overline{1,m},\; n = -N+1, \ldots, 0;$$
$$\psi_i(n_k + 1 - N) = \beta_{ik}\psi_i(n_k - N),\; k = \overline{2,p};$$
$$\psi_i(-N+1) = \beta_{i1}\psi_i(0),\; i = \overline{1,m}\} \subset \mathbb{R}^{mN}$$

equipped with the norm

$$\|\psi\| = \max_{-N+1\le\nu\le 0}\max_{i=\overline{1,m}}|\psi_i(\nu)|.$$

The solution of the discrete initial value problem (4), (5), (6) is denoted by $x(n,\psi)$, $n\in\mathbb{Z}$, $n\ge -N+1$. We shall use the norm

$$\|x(n,\psi)\| = \max_{i=\overline{1,m}}|x_i(n,\psi)|.$$

Conditions H3 and H4 are replaced respectively by

$\widetilde{\text{H3}}$. $0=n_1<n_2<\cdots<n_p<N$, $n_{k+p}=n_k+N$, $\beta_{i,k+p}=\beta_{ik}$ for $i=\overline{1,m}$, $k\in\mathbb{Z}^+$.

$\widetilde{\text{H4}}$. There exist positive numbers $\lambda_1,\dots,\lambda_m$ such that

$$\lambda_i a_i > L_i\sum_{j=1}^m\lambda_j\left(|b_{ji}|+\sum_{\nu=1}^N|C_{ji}(\nu)|\right),\quad i=\overline{1,m}.$$

Our main result is the following

Theorem 3.1. *Let system (4)–(5) satisfy the conditions H1, $\widetilde{\text{H3}}$, $\widetilde{\text{H4}}$. Then there exists a number N_0 such that for each integer $N\ge N_0$ system (4)–(5) is globally exponentially periodic. That is, there exists an N-periodic solution $x(n,\psi^*)$ of system (4)–(5) and positive constants α and $q<1$ such that every solution $x(n,\psi)$ of Eqs. (4)–(5) satisfies*

$$\|x(n,\psi)-x(n,\psi^*)\|\le\alpha\|\psi-\psi^*\|q^n\quad\text{for all } n\in\mathbb{Z}^+.$$

4. Proof of the Main Result

Lemma 4.1. *Let conditions $\widetilde{\text{H4}}$ hold. Then there exists a number $\bar\rho>1$ such that for $1<\rho\le\bar\rho$ and $i=\overline{1,m}$ we have*

$$\lambda_i(1-\rho e^{-a_ih})-L_i\phi_i(h)\sum_{j=1}^m\lambda_j\left[\rho|b_{ji}|+\sum_{\nu=1}^N|C_{ji}(\nu)|\rho^{\nu+1}\right]>0. \tag{7}$$

In particular,

$$\lambda_i-L_i\phi_i(h)\sum_{j=1}^m\lambda_j\sum_{\nu=1}^N|C_{ji}(\nu)|\,\rho^{\nu+1}>0. \tag{8}$$

Proof. Let us denote by $G_i(\rho)$, $i=\overline{1,m}$, the left-hand side of inequality (7). The functions $G_i(\rho)$ are continuous for $\rho\ge 1$ and

$$G_i(1)=\lambda_i(1-e^{-a_ih})-L_i\phi_i(h)\sum_{j=1}^m\lambda_j\left[|b_{ji}|+\sum_{\nu=1}^N|C_{ji}(\nu)|\right]$$

$$=\phi_i(h)\left[\lambda_ia_i-L_i\sum_{j=1}^m\lambda_j\left(|b_{ji}|+\sum_{\nu=1}^N|C_{ji}(\nu)|\right)\right]>0$$

by virtue of $\widetilde{\text{H4}}$. There exist numbers $\rho_i>1$ such that $G_i(\rho)>0$ for $\rho\in(1,\rho_i]$.

Let $\bar\rho=\min\limits_{i=\overline{1,m}}\rho_i$. Then for $1<\rho\le\bar\rho$ we have

$$G_i(\rho)>0,\qquad i=\overline{1,m}.$$ □

In order to prove Theorem 3.1, we need the following lemma.

Lemma 4.2. *Let $x(n,\psi)$, $x(n,\tilde{\psi})$ be a pair of solutions of system* (4)–(5). *If conditions* H1, $\widetilde{\text{H3}}$, $\widetilde{\text{H4}}$ *are satisfied and $\bar{\rho}$ is given by Lemma 4.1, then for any $\rho \in (1,\bar{\rho}]$ and all $n \in \mathbb{Z}^+$ we have*

$$\|x(n,\psi) - x(n,\tilde{\psi})\| \le K(N,\rho) \prod_{k=1}^{i(0,n-1)} (1+\rho B_k)\rho^{-n}\|\psi - \tilde{\psi}\|, \tag{9}$$

where

$$K(N,\rho) = \frac{1}{\min\limits_{j=\overline{1,m}}\{\lambda_j a_j\}} \sum_{i=1}^{m} \lambda_i \left\{ \frac{1}{\phi_i(h)} + \sum_{j=1}^{m} L_j \sum_{\nu=1}^{N} |C_{ij}(\nu)| \sum_{r=1}^{\nu} \rho^r \right\}, \tag{10}$$

$$B_k = \max_{i=\overline{1,m}} |\beta_{ik}|, \qquad i(0,n-1) = \max\{k:\ n_k \le n-1\}.$$

Proof. Let us denote

$$y_i(n) = \rho^n \frac{|x_i(n,\psi) - x_i(n,\tilde{\psi})|}{\phi_i(h)}, \qquad i=\overline{1,m}.$$

Then for $n \ne n_k$ we derive

$$y_i(n+1) \le \rho e^{-a_i h} y_i(n) + \rho \sum_{j=1}^{m} |b_{ij}| L_j \phi_j(h) y_j(n) + \sum_{j=1}^{m}\sum_{\nu=1}^{N} |C_{ij}(\nu)| L_j \phi_j(h)\, \rho^{\nu+1} y_j(n-\nu).$$

We define a Lyapunov functional

$$V(n) = \sum_{i=1}^{m} \lambda_i \left\{ y_i(n) + \sum_{j=1}^{m} L_j\phi_j(h) \sum_{\nu=1}^{N} |C_{ij}(\nu)|\, \rho^{\nu+1} \sum_{r=n-\nu}^{n-1} y_j(r) \right\}.$$

We can now estimate the difference $V(n+1) - V(n)$ along the solutions of (4) for $n \ne n_k$ as follows:

$$V(n+1) - V(n) \le \sum_{i=1}^{m} \lambda_i (\rho e^{-a_i h} - 1) y_i(n) + \sum_{i=1}^{m} \lambda_i \sum_{j=1}^{m} L_j \phi_j(h) \left[\rho |b_{ij}| + \sum_{\nu=1}^{N} |C_{ij}(\nu)|\, \rho^{\nu+1} \right] y_j(n).$$

In the second term we change the order of summation with respect to i and j, and then we replace i by j and *vice versa*. Thus we obtain

$$\begin{aligned} V(n+1) - V(n) &\le \sum_{i=1}^{m} \lambda_i (\rho e^{-a_i h} - 1) y_i(n) + \sum_{i=1}^{m} L_i \phi_i(h) \sum_{j=1}^{m} \lambda_j \left[\rho |b_{ji}| + \sum_{\nu=1}^{N} |C_{ji}(\nu)|\, \rho^{\nu+1} \right] y_i(n) \\ &= -\sum_{i=1}^{m} \left\{ \lambda_i (1 - \rho e^{-a_i h}) - L_i \phi_i(h) \sum_{j=1}^{m} \lambda_j \left[\rho |b_{ji}| + \sum_{\nu=1}^{N} |C_{ji}(\nu)|\, \rho^{\nu+1} \right] \right\} y_i(n) \\ &= -\sum_{i=1}^{m} G_i(\rho) y_i(n) \le 0 \qquad \text{for} \quad \rho \in (1,\bar{\rho}], \end{aligned}$$

that is,

$$V(n+1) \le V(n) \qquad \text{for} \quad n \in \mathbb{Z}^+ \setminus \{n_1, n_2, \ldots\}.$$

Next we find successively

$$|x_i(n_k+1,\psi)-x_i(n_k+1,\tilde{\psi})| = |\beta_{ik}||x_i(n_k,\psi)-x_i(n_k,\tilde{\psi})| \le B_k|x_i(n_k,\psi)-x_i(n_k,\tilde{\psi})|,$$

$$y_i(n_k+1) \le \rho B_k y_i(n_k),$$

$$V(n_k+1) \le \sum_{i=1}^{m} \lambda_i \left\{ \rho B_k y_i(n_k) + \sum_{j=1}^{m} L_j\phi_j(h) \sum_{\nu=1}^{N} |C_{ij}(\nu)|\, \rho^{\nu+1} \sum_{r=n_k+1-\nu}^{n_k} y_j(r) \right\}.$$

Thus, by virtue of Eq. (8), we find

$$\begin{aligned} V(n_k+1)-(1+\rho B_k)V(n_k) \le &-\sum_{i=1}^{m}\left\{\lambda_i - L_i\phi_i(h)\sum_{j=1}^{m}\lambda_j\sum_{\nu=1}^{N}|C_{ji}(\nu)|\,\rho^{\nu+1}\right\} y_i(n_k) \\ &-\rho B_k \sum_{i=1}^{m} L_i\phi_i(h)\sum_{j=1}^{m}\lambda_j\sum_{\nu=1}^{N}|C_{ji}(\nu)|\,\rho^{\nu+1}\sum_{r=n_k+1-\nu}^{n_k-1} y_i(n_k-\nu) \\ &-(1+\rho B_k)\sum_{i=1}^{m} L_i\phi_i(h)\sum_{j=1}^{m}\lambda_j\sum_{\nu=1}^{N}|C_{ji}(\nu)|\,\rho^{\nu+1} y_i(n_k-\nu) \;\le\; 0, \end{aligned}$$

that is,

$$V(n_k+1) \le (1+\rho B_k)V(n_k) \qquad \text{for} \quad k\in\mathbb{Z}^+.$$

Combining these estimates, we derive

$$V(n) \le V(0) \prod_{k=1}^{i(0,n-1)} (1+\rho B_k). \tag{11}$$

Further we notice that

$$\begin{aligned} V(n) &\ge \rho^n \sum_{i=1}^{m} \frac{\lambda_i}{\phi_i(h)}|x_i(n,\psi)-x_i(n,\tilde{\psi})| \\ &\ge \rho^n \sum_{i=1}^{m} \lambda_i a_i |x_i(n,\psi)-x_i(n,\tilde{\psi})| \\ &\ge \rho^n \min_{j=1,m}\{\lambda_j a_j\} \sum_{i=1}^{m} |x_i(n,\psi)-x_i(n,\tilde{\psi})| \\ &\ge \rho^n \min_{j=1,m}\{\lambda_j a_j\} \|x(n,\psi)-x(n,\tilde{\psi})\|. \end{aligned} \tag{12}$$

On the other hand,

$$\begin{aligned} V(0) &= \sum_{i=1}^{m}\lambda_i\left\{ y_i(0) + \sum_{j=1}^{m} L_j\phi_j(h)\sum_{\nu=1}^{N}|C_{ij}(\nu)|\,\rho^{\nu+1}\sum_{r=-\nu}^{-1} y_j(r)\right\} \\ &= \sum_{i=1}^{m}\lambda_i\left\{ \frac{|\psi_i(0)-\tilde{\psi}_i(0)|}{\phi_i(h)} + \sum_{j=1}^{m} L_j \sum_{\nu=1}^{N}|C_{ij}(\nu)|\rho^{\nu+1}\sum_{r=-\nu}^{-1}|\psi_j(r)-\tilde{\psi}_j(r)|\,\rho^r\right\} \\ &\le \sum_{i=1}^{m}\lambda_i\left\{\frac{1}{\phi_i(h)} + \sum_{j=1}^{m} L_j\sum_{\nu=1}^{N}|C_{ij}(\nu)|\sum_{r=1}^{\nu}\rho^r\right\}\|\psi-\tilde{\psi}\|. \end{aligned} \tag{13}$$

From the inequalities (11), (12), and (13) we derive the assertion of Lemma 4.2 with $K(N,\rho)$ given by Eq. (10). □

Proof of Theorem 3.1. Let $s \in \mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$ and $Ns+1 \le n \le N(s+1)$. Then $i(0, n-1) \le p(s+1)$ and from Lemma 4.2 we obtain

$$\|x(n,\psi) - x(n,\tilde{\psi})\| \le K(N,\rho) \prod_{k=1}^{p}(1+\rho B_k) \left[\rho^{-N} \prod_{k=1}^{p}(1+\rho B_k)\right]^s \|\psi - \tilde{\psi}\|. \tag{14}$$

Let $q \in (\rho^{-1}, 1)$. Then we can find N_0 such that for $N \ge N_0$ we have

$$\rho^{-N} \prod_{k=1}^{p}(1+\rho B_k) \le q^N.$$

Then (14) takes the form

$$\|x(n,\psi) - x(n,\tilde{\psi})\| \le \tilde{K}(N,\rho)\, q^{sN} \|\psi - \tilde{\psi}\| \tag{15}$$

for $Ns+1 \le n \le N(s+1)$ and $\tilde{K}(N,\rho) = K(N,\rho)\prod_{k=1}^{p}(1+\rho B_k)$.

Now we define an operator $\mathcal{P} : C^* \to C^*$ as follows: for $\psi = \{\psi_i(n-N);\ i = \overline{1,m},\ n = \overline{1,N}\}$ we set

$$\mathcal{P}\psi = \{x_i(n,\psi);\ i = \overline{1,m}, n = \overline{1,N}\}.$$

Then

$$\mathcal{P}^{s+1}\psi = \left\{x_i(n,\psi);\ i = \overline{1,m}, n = \overline{Ns+1, N(s+1)}\right\}$$

and according to (15) we have

$$\|\mathcal{P}^{s+1}\psi - \mathcal{P}^{s+1}\tilde{\psi}\| \le \tilde{K}(N,\rho)\, q^{sN} \|\psi - \tilde{\psi}\|.$$

If we choose s so large that $\tilde{K}(N,\rho)\, q^{sN} \le \tilde{q} < 1$, then $\mathcal{P}^{s+1}$ is a contraction, hence it has a unique fixed point ψ^* : $\mathcal{P}^{s+1}\psi^* = \psi^*$. On the other hand, $\mathcal{P}^{s+1}(\mathcal{P}\psi^*) = \mathcal{P}(\mathcal{P}^{s+1}\psi^*) = \mathcal{P}\psi^*$, i.e., $\mathcal{P}\psi^*$ is also a fixed point for $\mathcal{P}^{s+1}$. These two fixed points must coincide, so

$$\mathcal{P}\psi^* = \psi^*$$

and ψ^* is a fixed point for the operator $\mathcal{P}$. This means that

$$x_i(n,\psi^*) = \psi_i(n-N) \qquad \text{for} \quad n = \overline{1,N}$$

and $x(n,\psi^*)$ is a periodic solution of problem (4)–(5).

Now applying inequality (15) to $x(n,\psi^*)$ and an arbitrary solution $x(n,\psi)$ we have

$$\|x(n,\psi) - x(n,\psi^*)\| \le \tilde{K}(N,\rho) q^{sN} \|\psi - \psi^*\|$$

for $Ns+1 \le n \le N(s+1)$. If we put $K_1(N,\rho) = \tilde{K}(N,\rho)\, q^{-N}$, then $\tilde{K}(N,\rho)\, q^{sN} = K_1(N,\rho)\, q^{(s+1)N} \le K_1(N,\rho)\, q^n$ for $Ns+1 \le n \le N(s+1)$. Thus we have

$$\|x(n,\psi) - x(n,\psi^*)\| \le K_1(N,\rho)\, q^n \|\psi - \psi^*\| \quad \text{for all } n \in \mathbb{Z}^+.$$

This shows that any solution $x(n,\psi)$ exponentially tends to the periodic solution $x(n,\psi^*)$ as $n \to +\infty$. □

5. AN ILLUSTRATIVE EXAMPLE

This is a nontrivial modification of the examples given in [26, 27].

Consider the impulsive Hopfield neural network with finite distributed delays

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = -x_1(t) + 0.5\tanh(x_1(t)) + 0.2\tanh(x_2(t)) \\ \qquad + \displaystyle\int_0^1 (1-s)\,[0.1\tanh(x_1(t-s)) + 0.3\tanh(x_2(t-s))]\,\mathrm{d}s + \sin(2\pi t), \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = -x_2(t) + 0.3\tanh(x_1(t)) + 0.4\tanh(x_2(t)) \\ \qquad + \displaystyle\int_0^1 (1-s)\,[0.2\tanh(x_1(t-s)) + 0.3\tanh(x_2(t-s))]\,\mathrm{d}s + \cos(2\pi t), \quad t > 0,\ t \neq t_k, \\ x_i(t_k+0) = \beta_{ik} x_i(t_k), \qquad i = 1,2, \quad k \in \mathbb{Z}^+, \end{cases}$$

where

$$t_{3k+1} = k, \quad t_{3k+2} = k + 0.3, \quad t_{3k+3} = k + 0.6, \qquad k \in \mathbb{Z}_0^+,$$

$$\beta_{11} = 2, \quad \beta_{12} = -0.1, \quad \beta_{13} = 0.4, \quad \beta_{21} = -1.5, \quad \beta_{22} = 0.7, \quad \beta_{23} = -0.5.$$

Then

$$\omega = 1, \quad a_1 = a_2 = 1, \quad b_{11} = 0.5, \quad b_{12} = 0.2, \quad b_{21} = 0.3, \quad b_{22} = 0.4,$$

$$c_{11}(s) = 0.1(1-s), \quad c_{12}(s) = 0.3(1-s), \quad c_{21}(s) = 0.2(1-s), \quad c_{22}(s) = 0.3(1-s),$$

$$g_1(\cdot) = g_2(\cdot) = \tanh(\cdot), \qquad L_1 = L_2 = 1,$$

$$d_1(t) = \sin(2\pi t), \qquad d_2(t) = \cos(2\pi t), \qquad B_1 = 2, \qquad B_2 = 0.7, \qquad B_3 = 0.5.$$

For $N \geq 4$ the corresponding discrete-time system is

$$\begin{cases} x_1(n+1) = e^{-h}x_1(n) + (1-e^{-h})\Big\{0.5\tanh(x_1(n)) + 0.2\tanh(x_2(n)) \\ \qquad + \sum_{\nu=1}^{N} [C_{11}(\nu)\tanh(x_1(n-\nu)) + C_{12}(\nu)\tanh(x_2(n-\nu))] + \sin(2\pi n x)\Big\}, \\ x_2(n+1) = e^{-h}x_2(n) + (1-e^{-h})\Big\{0.3\tanh(x_1(n)) + 0.4\tanh(x_2(n)) \\ \qquad + \sum_{\nu=1}^{N} [C_{21}(\nu)\tanh(x_1(n-\nu)) + C_{22}(\nu)\tanh(x_2(n-\nu))] + \cos(2\pi n x)\Big\}, \\ \qquad\qquad n \neq n_k, \\ x_i(n_k+1) = \beta_{ik}x_i(n_k), \qquad i = 1,2, \quad k \in \mathbb{Z}^+, \end{cases} \tag{16}$$

where

$$h = 1/N, \quad C_{11}(\nu) = 0.1\Phi(\nu), \quad C_{12}(\nu) = 0.3\Phi(\nu), \quad C_{21}(\nu) = 0.2\Phi(\nu), \quad C_{22}(\nu) = 0.3\Phi(\nu),$$

$$\Phi(\nu) = \int_{(\nu-1)h}^{\nu h} (1-s)\,\mathrm{d}s = h - h^2(\nu - 1/2).$$

We have

$$\sum_{\nu=1}^{N} \Phi(\nu) = \int_0^1 (1-s)\,\mathrm{d}s = 0.5.$$

Since

$$L_1\left\{|b_{11}|+|b_{21}|+\sum_{\nu=1}^{N}[|C_{11}(\nu)|+|C_{21}(\nu)|]\right\}=0.5+0.3+(0.1+0.2)0.5=0.95<1=a_1\,,$$

$$L_2\left\{|b_{12}|+|b_{22}|+\sum_{\nu=1}^{N}[|C_{12}(\nu)|+|C_{22}(\nu)|]\right\}=0.2+0.4+(0.3+0.3)0.5=0.9<1\ \ =a_2\,,$$

condition $\widetilde{\mathrm{H4}}$ is satisfied with $\lambda_1=\lambda_2=1$ and Theorem 3.1 holds. More precisely, if $\rho\in(1,\bar\rho)$, where $\bar\rho$ is given by Lemma 4.1, and $q\in(\rho^{-1},1)$, we can choose N so large that $(\rho q)^N\geq(1+2\rho)(1+0.7\rho)(1+0.5\rho)$. Thus, system (16) has a unique N-periodic solution, which is globally exponentially stable.

6. Conclusions

In the present paper we introduced the discrete counterpart of a class of Hopfield neural networks with periodic impulses and finite distributed delays. We derived a sufficient condition for the existence and global exponential stability of a unique periodic solution of the discrete system considered.

Acknowledgments

Partially supported by SABIC and Fast Track Research Grants SAB/2004-46. The first two authors would like to thank King Fahd University of Petroleum and Minerals, Department of Mathematical Sciences for providing excellent research facilities. The authors thank the referee for his helpful comments and suggestions.

References

[1] H. Akça, R. Alassar, V. Covachev, Z. Covacheva, E.A. Al-Zahrani. Continuous-time additive Hopfield-type neural networks with impulses. *J. Math. Anal. Appl.*, **290**: 436–451, 2004.

[2] H. Akça, R. Alassar, V. Covachev, Z. Covacheva. Discrete counterparts of continuous-time additive Hopfield-type neural networks with impulses. *Dyn. Syst. Appl.*, **13**: 75–90, 2004.

[3] V. Covachev, H. Akça, F. Yeniçerioğlu. Difference approximations for impulsive differential equations. *Appl. Math. Comput.*, **121**: 383–390, 2001.

[4] J. Freeman, D. Sakura. *Neural Networks: Algorithms, Applications, and Programming Techniques.* Addison–Wesley, Reading, Mass., 2001.

[5] K. Gopalsamy. Stability of artificial neural networks with impulses. *Appl. Math. Comput.*, **154**: 783–813, 2004.

[6] K. Gopalsamy, X.Z. He. Stability in asymmetric Hopfield nets with transmission delays. *Physica D*, **76**: 344–358, 1994.

[7] K. Gopalsamy, K.C. Issic, I.K.C. Leung, P. Liu. Global Hopf-bifurcation in a neural netlet. *Appl. Math. Comput.*, **94**: 171–192, 1998.

[8] D. Gulick. *Encounters with Chaos.* McGraw-Hill, New York, 1992.

[9] S. Haykin. *Neural Networks, A Comprehensive Foundation.* MacMillan, New York, 2000.

[10] R. Hecht-Nielsen. *Neuro Computing.* Addison-Wesley, Reading, Mass., 2000.

[11] J. Hertz, A. Krogh, R.G. Palmer. *Introduction to the Theory of Neural Computation.* Addison-Wesley Publishing Company, 1991.

[12] J.J. Hopfield. Neural networks and physical systems with emergent collective computational abilities. *Proc. National Acad. Sci.*, **79**: 2554–2558, 1982.

[13] J.J. Hopfield. Neurons with graded response have collective computational properties like those of two-state neurons. *Proc. National Acad. Sci.*, **81**: 3088–3092, 1984.

[14] J.J. Hopfield, D.W. Tank. Computing with neural circuits; a model. *Science*, **233**: 625–633, 1986.

[15] J. Jang, S. Lee, S. Shin, An optimization network for matrix inversion. In: D.Z. Anderson, ed., *Neural Information Processing Systems*, 397–401. American Institute of Physics, New York, 1988.

[16] Y. Li, L. Lu. Global exponential stability and existence of periodic solutions of Hopfield-type neural networks with impulses. *Physics Letters A*, **333**: 62–71, 2004.

[17] Y. Li, W. Xing, L. Lu. Existence and global exponential stability of periodic solutions of a class of neural networks with impulses. *Chaos, Solitons and Fractals*, **27**: 437–445, 2006.
[18] Y. Li, C. Yang. Global exponential stability on impulsive BAM neural networks with distributed delays. *J. Math. Anal. Appl.*, **324**: 1125–1139, 2006.
[19] C.M. Marcus, R.M. Westervelt. Stability of analog neural networks with delay. *Phys. Rev. A*, **39**: 347–359, 1989.
[20] W. McCulloch, W. Pitts. A logical calculus of the ideas immanent in nervous activity. *Bull. Math. Biophysics*, **9**: 127–147, 1943.
[21] G. Meinardus, G. Nurnberger, eds. *Delay Equations, Approximation and Application*. Birkhäuser, Boston, 1985.
[22] S. Mohamad, K. Gopalsamy. Dynamics of a class of discrete-time neural networks and their continuous-time counterparts. *Math. Comput. Simul.*, **53**: 1–39, 2000.
[23] S. Mohamad, K. Gopalsamy. Neuronal dynamics in time varying environments: Continuous and discrete time models. *Discrete and Continuous Dynamical Systems*, **6**: 841–860.
[24] M. Morita. Associative memory with non-monotone dynamics. *Neural Networks*, **6**: 115–126, 1993.
[25] D.W. Tank, J.J. Hopfield. Simple neural optimization networks: An A/D converter, signal decision circuit and a linear programming circuit. *IEEE Trans. Circuit Syst.*, **33**: 533–541, 1986.
[26] X. Yang, X. Liao, D.J. Evans, Y. Tang. Existence and stability of periodic solution in impulsive Hopfield neural networks with finite distributed delays. *Physics Letters A*, **343**: 108–116, 2005.
[27] X. Yang, X. Liao, G.M. Megson, D.J. Evans. Global exponential periodicity of a class of neural networks with recent history distributed delays. *Chaos, Solitons and Fractals*, **25**: 441–447, 2005.
[28] S. Yoshizawa, M. Morita, S. Amari. Capacity of associative memory using a nonmonotonic neuron model. *Neural Networks*, **6**: 167–176, 1993.