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# Stability of two-DOF systems with clearances using FET

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The finite element in time method (FET) is a fast and reliable implicit numerical method for obtaining steady state solutions of the periodically forced dynamical systems with clearances. Delineation of the stable and unstable solutions could help in predicting regular and chaotic motions of such dynamical systems and transitions to either type of response. Stability of the FET solutions can be investigated via the Floquet theory, without any special effort for calculating the monodromy matrix. The applicability of the stability analysis is demonstrated through the study of two-degree-of-freedom systems with clearances. Close agreement is found between obtained results and published findings of the harmonic balance method and the piecewise full decoupling method.

## 1. Introduction

Gaps and clearances exist in many dynamical systems either by design or due to the manufacturing tolerances and wear. Vibration of mechanical systems with clearances can result in relative motion across the clearance space and impacting between the components. The vibro-impacts cause excessive levels of noise, large dynamic loads and large changes in the dynamic stiffness.

The characteristics of such systems include an abrupt variation of stiffness which can be assumed as piecewise linear. Determination of the steady state response of piecewise linear dynamical systems is essential in order to design against large-amplitude resonant motions. Exact solutions of piecewise linear equations of motion are very rare and almost all of the methods for their solving are only approximate [9, 11]. The most commonly used solution methods are the classical numerical time integration (Runge–Kutta, etc.), the harmonic balance method [2, 3, 13], the incremental harmonic balance method [8, 14] and the piecewise full decoupling method [6, 7].

An alternative approach is the finite element in time method (FET) [1, 4, 5, 12, 15] which is based on a weak form of Hamilton principle. Similar to the standard finite element technique, the time interval is divided into a finite number of time elements. The solution for all the spatial degrees of freedom at all time steps within a given time interval is sought through a set of algebraic equations. The conventional shape functions can be used in the time element because the time element does not impose the periodic boundary conditions.

A complete characterization of the dynamic behavior of piecewise linear dynamical systems requires, among other things, determination of stability of their steady state solutions. The stability of steady state solutions obtained by FET is investigated considering the Floquet theory. The monodromy matrix can be directly determined by applying a standard procedure of the static condensation of time elements.

The objective of this paper is to analyze the stability of two-degree-of-freedom systems with clearances by using the finite element in time method.

## 2. Problem formulation

The mechanical model of a three-degree-of-freedom semi-definite system with clearances is shown in Fig. 1. The model consists of three mass elements $m_1$, $m_2$, $m_3$, two linear viscous dampers $c_1$, $c_2$, and two piecewise linear stiffness elements $k_1\,h_1(\bar{x}_1)$, $k_2\,h_2(\bar{x}_2)$.

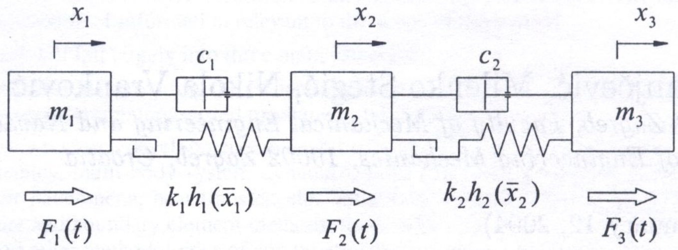


**Fig. 1.** Three-degree-of-freedom semi-definite system with clearances

With $\mathbf{x}^T = [x_1, x_2, x_3]$ and $\bar{\mathbf{x}}^T = [x_1 - x_2, x_2 - x_3] = [\bar{x}_1, \bar{x}_2]$ being the absolute and relative displacements, the equation of motion can be expressed as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\bar{\mathbf{x}}} + \mathbf{K}\mathbf{h}(\bar{\mathbf{x}}) = \mathbf{f}(t), \tag{1}$$

where

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} c_1 & 0 \\ -c_1 & c_2 \\ 0 & -c_2 \end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix} k_1 & 0 \\ -k_1 & k_2 \\ 0 & -k_2 \end{bmatrix}, \tag{2}$$

$$\mathbf{h}^T(\bar{\mathbf{x}}) = [h(\bar{x}_1), h(\bar{x}_2)], \qquad \mathbf{f}^T(t) = [F_1(t), F_2(t), F_3(t)]. \tag{3}$$

By introducing the non-dimensional relative displacement $\mathbf{q}$ and the non-dimensional time $\tau = \omega_k t$ as new independent variables, the equation of motion (1) is transformed into the non-dimensional form

$$\mathbf{q}'' + \mathbf{Z}\,\mathbf{q}' + \mathbf{\Omega}\,\mathbf{h}(\mathbf{q}) = \mathbf{f}_0 + \mathbf{f}_a \cos(\eta\tau), \tag{4}$$

where

$$\mathbf{q} = \frac{1}{b}\bar{\mathbf{x}}, \quad \omega_k = \sqrt{k_1\left(\frac{1}{m_1} + \frac{1}{m_2}\right)}, \quad \frac{\mathrm{d}}{\mathrm{d}\tau}() = ()' \tag{5}$$

$$\mathbf{Z} = 2\begin{bmatrix} \zeta_{11} & -\zeta_{12}\omega_{12} \\ -\zeta_{21}\omega_{21} & \zeta_{22}\omega_{22} \end{bmatrix}, \tag{6}$$

$$\mathbf{\Omega} = \begin{bmatrix} 1 & -\omega_{12}^2 \\ -\omega_{21}^2 & \omega_{22}^2 \end{bmatrix} \tag{7}$$

Furthermore, $b$ is the characteristic length, $\eta$ denotes a non-dimensional excitation frequency while $\mathbf{f}_0$ and $\mathbf{f}_a$ are the amplitude vectors of mean and alternating load, respectively. The element of the damping and stiffness matrices $\mathbf{Z}$ and $\mathbf{\Omega}$ are found to be

$$\zeta_{11} = \frac{c_1\left(\frac{1}{m_1} + \frac{1}{m_2}\right)}{2\omega_k}, \qquad \zeta_{22} = \frac{c_2\left(\frac{1}{m_2} + \frac{1}{m_3}\right)}{2\omega_k\,\omega_{22}}, \tag{8}$$

$$\zeta_{12} = \frac{c_2}{2m_2\omega_k\omega_{12}}, \qquad \zeta_{21} = \frac{c_1}{2m_2\omega_k\omega_{21}}, \tag{9}$$

$$\omega_{12}^2 = \frac{k_2}{m_2\omega_k^2}, \qquad \omega_{21}^2 = \frac{k_1}{m_2\omega_k^2}, \qquad \omega_{22}^2 = \frac{k_2\left(\frac{1}{m_2} + \frac{1}{m_3}\right)}{\omega_k^2} \tag{10}$$

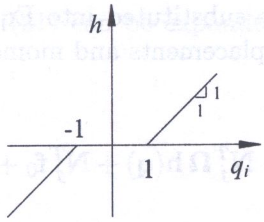


**Fig. 2.** The piecewise linear displacement function

The piecewise linear displacement function $h(q_i)$ for $i = 1, 2$ that describes the clearance of value $2b$ is defined as (Fig. 2)

$$h(q_i) = \begin{cases} q_i + 1 & q_i < -1, \\ 0 & -1 \le q_i \le 1, \\ q_i - 1 & q_i > 1. \end{cases} \tag{11}$$

## 3. Finite element in time method

The equation of motion (4) derived in an explicit form can be approximately solved in an implicit form by using a weak formulation based on weighted-integral statements. The Hamilton principle is one such weak formulation which describes the motion of the mechanical system between the two known non-dimensional times, $\tau_1$ and $\tau_2$,

$$\int_{\tau_1}^{\tau_2} (\delta L + \delta W)\,\mathrm{d}\tau = \delta\mathbf{q}^T\mathbf{p}\big|_{\tau_1}^{\tau_2}\ , \tag{12}$$

where $\delta L$ and $\delta W$ are the Lagrangian and the nonconservative virtual work, respectively, while $\mathbf{p} = \frac{\delta L}{\delta \mathbf{q}'}$ denotes the vector of the generalized momenta.

Employing Eq. (4), the Hamilton weak principle (12) can be expressed in the following form,

$$\int_{\tau_1}^{\tau_2} \left\{\delta\mathbf{q}'^T\mathbf{q}' + \delta\mathbf{q}^T\left[-\mathbf{Z}\mathbf{q}' - \boldsymbol{\Omega}\,\mathbf{h}(\mathbf{q}) + \mathbf{f}_0 + \mathbf{f}_a\cos(\eta\tau)\right]\right\}\mathrm{d}\tau = \delta\mathbf{q}^T\mathbf{p}\big|_{\tau_1}^{\tau_2}\ . \tag{13}$$

The basis of the finite element in time approximation is the division of the time interval $\tau_2 - \tau_1$ into a finite number $n_e$ of time elements, usually equally spaced for convenience. Similar to the standard finite element technique, for each degree of freedom $q_i$, a set of nodal displacements $\bar{\mathbf{q}}_j$ is defined for each time element $j$ $(j = 1(1)n_e)$ as a nodal variable. The displacement $\mathbf{q}_j(\tau)$ within each time element is interpolated among their respective nodal variables by using the standard shape functions, as

$$\begin{aligned} \mathbf{q}_j(\tau) &= \mathbf{N}_j(\tau)\,\bar{\mathbf{q}}_j\,, & j &= 1(1)n_e\,, \\ \mathbf{q}'_j(\tau) &= \mathbf{N}'_j(\tau)\,\bar{\mathbf{q}}_j\,, & j &= 1(1)n_e\,. \end{aligned} \tag{14}$$

where $\mathbf{N}_j$ is the matrix of shape functions. The Lagrange polynomials, for example, can be used as shape functions,

$$N_k(\tau) = \prod_{\substack{l=1\\ l\neq k}}^{r} \frac{\tau - \tau_l}{\tau_k - \tau_l}\,, \qquad k = 1(1)r, \tag{15}$$

where $r$ denotes the number of nodes at the time element $j$.

When the assumed solutions (14) are substituted into Eq. (13), the following variational form for time element $j$, in terms of nodal displacements and momenta of all spatial degrees-of-freedom, is obtained,

$$\int_{\tau_j}^{\tau_{j+1}} \delta\bar{\mathbf{q}}_j^T \left[\mathbf{N}_j'^T \mathbf{N}_j' \bar{\mathbf{q}}_j - \mathbf{N}_j^T \mathbf{Z} \mathbf{N}_j' \bar{\mathbf{q}}_j - \mathbf{N}_j^T \boldsymbol{\Omega}\, \mathbf{h}(\mathbf{q}) + \mathbf{N}_j^T \mathbf{f}_0 + \mathbf{N}_j^T \mathbf{f}_a \cos(\eta\tau)\right] \mathrm{d}\tau = \delta\bar{\mathbf{q}}_j^T \mathbf{p}_j\,, \tag{16}$$

where $\mathbf{p}_j^T = [\mathbf{p}(\tau_j), 0, \ldots, 0, \mathbf{p}(\tau_{j+1})]$ is the vector containing $r$ elements. Since $\delta\bar{\mathbf{q}}_j$ is completely arbitrary, the finite element approximation at the time element level is given by

$$\mathbf{A}_j\, \bar{\mathbf{q}}_j + \mathbf{g}_j(\bar{\mathbf{q}}_j) + \mathbf{f}_j = \mathbf{p}_j\,, \qquad j = 1(1)n_e\,, \tag{17}$$

where

$$\mathbf{A}_j = \int_{\tau_j}^{\tau_{j+1}} \left[\mathbf{N}_j'^T \mathbf{N}_j' - \mathbf{N}_j^T \mathbf{Z} \mathbf{N}_j'\right] \mathrm{d}\tau, \tag{18}$$

$$\mathbf{g}_j(\bar{\mathbf{q}}_j) = -\int_{\tau_j}^{\tau_{j+1}} \mathbf{N}_j^T \boldsymbol{\Omega}\, \mathbf{h}(\mathbf{q})\, \mathrm{d}\tau, \tag{19}$$

$$\mathbf{f}_j = \int_{\tau_j}^{\tau_{j+1}} \left[\mathbf{N}_j^T \mathbf{f}_0 + \mathbf{N}_j^T \mathbf{f}_a \cos(\eta\tau)\right] \mathrm{d}\tau. \tag{20}$$

The global finite element equation is determined by assembling the finite element equations at the time element level (17), as in the standard finite element modelling scheme, yielding

$$\mathbf{A}\bar{\mathbf{q}} + \mathbf{g}(\bar{\mathbf{q}}) + \mathbf{f} = \mathbf{0}, \tag{21}$$

where, with a summation notation representing the element assemblage operation

$$\bar{\mathbf{q}} = \sum_{j=1}^{n_e} \bar{\mathbf{q}}_j\,, \tag{22}$$

$$\mathbf{A} = \sum_{j=1}^{n_e} \mathbf{A}_j\,, \tag{23}$$

$$\mathbf{g}(\bar{\mathbf{q}}) = \sum_{j=1}^{n_e} \mathbf{g}_j(\bar{\mathbf{q}}_j), \tag{24}$$

$$\mathbf{f} = \sum_{j=1}^{n_e} \mathbf{f}_j\,. \tag{25}$$

For a periodic steady state response, the boundary terms on the right-side of Eq. (17) vanish, as the periodic boundary condition requires both displacements and velocities to be identical at $\tau_1$ and $\tau_2$,

$$\sum_{j=1}^{n_e} \mathbf{p}_j = \mathbf{0}. \tag{26}$$

Thus, the global finite element equation (21) results in a set of nonlinear algebraic equations in the unknown nodal displacement vector $\bar{\mathbf{q}}$. Using the Newton–Raphson numerical method, the nodal displacement $\bar{\mathbf{q}}$ may be expressed in an iteration procedure as

$$\bar{\mathbf{q}}^{(n+1)} = \bar{\mathbf{q}}^{(n)} + \Delta\bar{\mathbf{q}}^{(n)}. \tag{27}$$

For a small increment $\Delta\bar{\mathbf{q}}^{(n)}$, Eq. (21) can be expanded into a Taylor series retaining only the linear terms,

$$\mathbf{d}^{(n)} + \mathbf{K}^{(n)}\Delta\bar{\mathbf{q}}^{(n)} = \mathbf{0}, \tag{28}$$

where

$$\mathbf{d}^{(n)} = \mathbf{A}\,\bar{\mathbf{q}}^{(n)} + \mathbf{g}^{(n)}(\bar{\mathbf{q}}) + \mathbf{f}, \tag{29}$$

$$\mathbf{K}^{(n)} = \mathbf{A} + \left(\frac{\partial \mathbf{g}(\bar{\mathbf{q}})}{\partial \bar{\mathbf{q}}}\right)^{(n)}. \tag{30}$$

The iteration procedure requires evaluation of the integral components of the vector $\mathbf{d}^{(n)}$ and the tangent matrix $\mathbf{K}^{(n)}$ at each iteration step. The components of the matrix $\mathbf{A}$ and vector $\mathbf{f}$ are constants and they need to be calculated only once. The components of the vector $\mathbf{g}^{(n)}$ depend on the piecewise linear displacement functions (11), therefore, we have to return at time element level

$$\mathbf{g}_j(\bar{\mathbf{q}}_j) = -\int_{\tau_j}^{\tau_{j+1}} \mathbf{N}_j^T \boldsymbol{\Omega}\,\mathbf{h}(\mathbf{q})\,\mathrm{d}\tau, \tag{31}$$

$$\frac{\partial \mathbf{g}_j}{\partial \bar{\mathbf{q}}_j} = -\int_{\tau_j}^{\tau_{j+1}} \mathbf{N}_j^T \boldsymbol{\Omega}\,\frac{\partial \mathbf{h}}{\partial \mathbf{q}}\,\mathbf{N}_j\,\mathrm{d}\tau. \tag{32}$$

A calculation of the above integrals requires a knowing of the zeroes of the function $q_i(\tau) - 1$ within the interval $[\tau_j\,,\,\tau_{j+1}]$. Since $q_i(\tau)$ is approximated by polynomials, these zeroes can be easily determined.

The iteration procedure defined by Eqs. (22)–(23) will be terminated when the increment of nodal displacement vector $\Delta\bar{\mathbf{q}}^{(n)}$ converges towards zero.

## 4. Stability analysis

The stability of periodic solutions, based on the above iteration procedure, can be investigated by using the Floquet theory, because nonlinearity is restricted to brief intervals of maximum amplitudes. In order to determine the monodromy matrix which relates the final and initial perturbations of solutions, a standard procedure of static condensation has to be applied. This procedure is based on the elimination of the elemental degrees of freedom that are not involved in satisfying the inter-element compatibility before the time element is assembled into the global system equations.

The incremental equation (28) at the time element level is represented by

$$\mathbf{d}_j + \mathbf{K}_j\,\Delta\bar{\mathbf{q}}_j = \mathbf{p}_j\,, \qquad j = 1(1)n_e\,. \tag{33}$$

The partitioned form of Eq. (33) may be written as

$$\begin{bmatrix}\mathbf{d}_B\\ \mathbf{d}_I\end{bmatrix}_j + \begin{bmatrix}\mathbf{K}_{BB} & \mathbf{K}_{BI}\\ \mathbf{K}_{IB} & \mathbf{K}_{II}\end{bmatrix}_j \begin{bmatrix}\Delta\bar{\mathbf{q}}_B\\ \Delta\bar{\mathbf{q}}_I\end{bmatrix}_j = \begin{bmatrix}\mathbf{p}_B\\ \mathbf{0}\end{bmatrix}_j \tag{34}$$

where $B$ and $I$ correspond the temporal nodes at the ends and in the interior of time element $j$, respectively, while $\mathbf{p}_{Bj}^T = [-\mathbf{p}(\tau_j), \mathbf{p}(\tau_{j+1})]$ denotes the vector of momenta at the boundary nodes. Compare the lower part with the upper part of Eq. (34) to obtain

$$\left(\mathbf{d}_B - \mathbf{K}_{BI}\,\mathbf{K}_{II}^{-1}\,\mathbf{d}_I\right)_j + \left(\mathbf{K}_{BB} - \mathbf{K}_{BI}\,\mathbf{K}_{II}^{-1}\,\mathbf{K}_{IB}\right)_j \Delta\bar{\mathbf{q}}_{Bj} = \mathbf{p}_{Bj}\,. \tag{35}$$

Considering the assemblage procedure, Eq. (35) of all time elements are further condensed by imposing the compatibility conditions at the common node of the two elements, yielding the elimination of the common node. This procedure results into the global equation

$$\begin{bmatrix}\mathbf{d}_1\\ \mathbf{d}_2\end{bmatrix} + \begin{bmatrix}\mathbf{K}_{11} & \mathbf{K}_{12}\\ \mathbf{K}_{21} & \mathbf{K}_{22}\end{bmatrix} \begin{bmatrix}\Delta\bar{\mathbf{q}}_1\\ \Delta\bar{\mathbf{q}}_2\end{bmatrix} = \begin{bmatrix}-\mathbf{p}_1\\ \mathbf{p}_2\end{bmatrix}, \tag{36}$$

where 1 and 2 denote the initial and final time nodes

Let a periodic solution for the nodal displacement and the linear moment be denoted by $\bar{\mathbf{q}}_0$ and $\mathbf{p}_0$, respectively. Then, perturbations $\delta\bar{\mathbf{q}}$ and $\delta\mathbf{p}$ are superimposed on $\bar{\mathbf{q}}_0$ and $\mathbf{p}_0$, resulting

$$\begin{aligned}\bar{\mathbf{q}} &= \bar{\mathbf{q}}_0 + \delta\bar{\mathbf{q}},\\ \mathbf{p} &= \mathbf{p}_0 + \delta\mathbf{p}.\end{aligned} \tag{37}$$

Considering Eqs. (36) and (37), we obtain the perturbation equation

$$\begin{bmatrix}\mathbf{K}_{11} & \mathbf{K}_{12}\\ \mathbf{K}_{21} & \mathbf{K}_{22}\end{bmatrix}\begin{bmatrix}\delta\bar{\mathbf{q}}_1\\ \delta\bar{\mathbf{q}}_2\end{bmatrix} = \begin{bmatrix}-\delta\mathbf{p}_1\\ \delta\mathbf{p}_2\end{bmatrix}, \tag{38}$$

which can be modified into the form suitable for stability analysis

$$\begin{bmatrix}\delta\bar{\mathbf{q}}_2\\ \delta\mathbf{p}_2\end{bmatrix} = \mathbf{B}\begin{bmatrix}\delta\bar{\mathbf{q}}_1\\ \delta\mathbf{p}_1\end{bmatrix} \tag{39}$$

Hence, in view of Eq. (38), the monodromy matrix $\mathbf{B}$ takes the following form,

$$\mathbf{B} = \begin{bmatrix}-\mathbf{K}_{12}^{-1}\mathbf{K}_{11} & -\mathbf{K}_{12}^{-1}\\ \mathbf{K}_{21} - \mathbf{K}_{22}\mathbf{K}_{12}^{-1}\mathbf{K}_{11} & -\mathbf{K}_{22}\mathbf{K}_{12}^{-1}\end{bmatrix}. \tag{40}$$

According to the Floquet theory, the stability of periodic solutions is determined by the spectral radius of the monodromy matrix

$$\rho(\mathbf{B}) = \max|\lambda_i|, \tag{41}$$

where $\lambda_i$ are the eigenvalues of $\mathbf{B}$. The steady state response is classified through

$$\rho(\mathbf{B}) < 1 \quad \text{as asymptotically stable, or} \tag{42}$$

$$\rho(\mathbf{B}) > 1 \quad \text{as unstable.} \tag{43}$$

The limit case between stable and unstable solutions which defines the boundary of unstable region, corresponds to $\rho(\mathbf{B}) = 1$.

## 5. Numerical results

The above procedure of stability analysis is applied to investigate the responses of a three-degree-of-freedom semi-definite system with clearances under periodic excitation. The mechanical model is adopted from [10] where its dynamic behavior was studied by using the harmonic balance method. The model has the following non-dimensional parameters $\zeta_{11} = \zeta_{12} = \zeta_{21} = \zeta_{22} = 0.05$, $\omega_{12} = \omega_{21} = 0.6$ and $\omega_{22} = 1.1$ assuming that only $h_1(q_1)$ is piecewise linear (Eq. (11)) while $h_2(q_2) = q_2$.

The finite element in time method cannot solve problems with ideal clearance because the zero stiffness implies the singularity of the tangent matrix $\mathbf{K}^{(n)}$. In the finite element calculations, the clearance is approximated by the trilinear system at which the slope of the second stage is 1% of the slope of the first and third stages. Furthermore, the numerical procedure was performed with ten four nodes time elements and with the response of the linear system as the starting vector $\bar{\mathbf{q}}^{(0)}$. The third order interpolation polynomials are used as the shape functions.

As the results of the finite element incremental procedure and the stability analysis, the frequency responses for different values of the mean and alternating loads, in terms of the non-dimensional excitation frequency $\eta$ and the steady state amplitude $\hat{q}_1$, are given in Figs. 3–5. The converged solutions are obtained in maximum 100 iterations, usually only few steps.

The first loading case (Fig. 3) gives the unconverged or unstable solutions in the frequency range $\eta < 0.5$ and $1.18 < \eta < 1.5$. When the alternating load takes a half of previous value the range of unconverged or unstable solutions becomes smaller (Fig. 4), while for the double value of mean

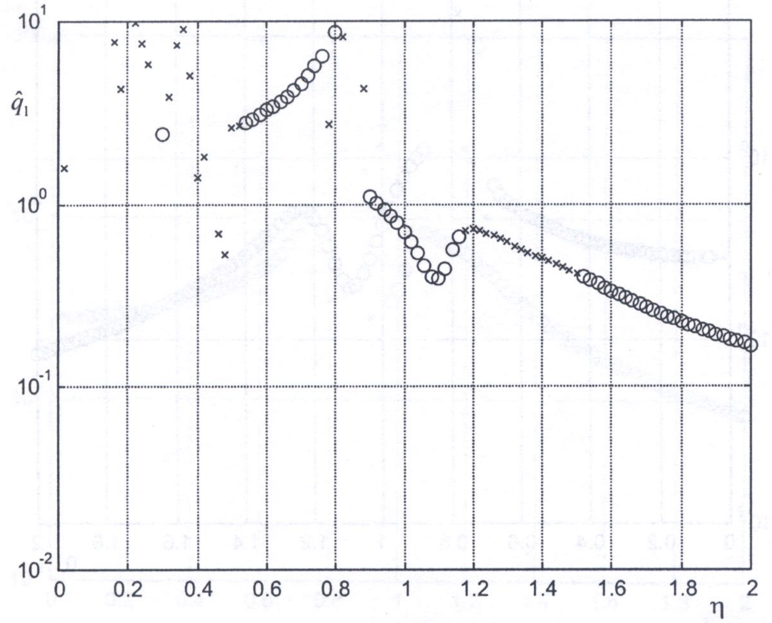


**Fig. 3.** Frequency response of the 3-DOF semi-definite system with clearance for $\mathbf{f}_0 = [0.25, 0.25]^T$ and $\mathbf{f}_a = [0.5, 0]^T$; ∘ – stable, × – unstable or achieved after maximum of 100 iterations

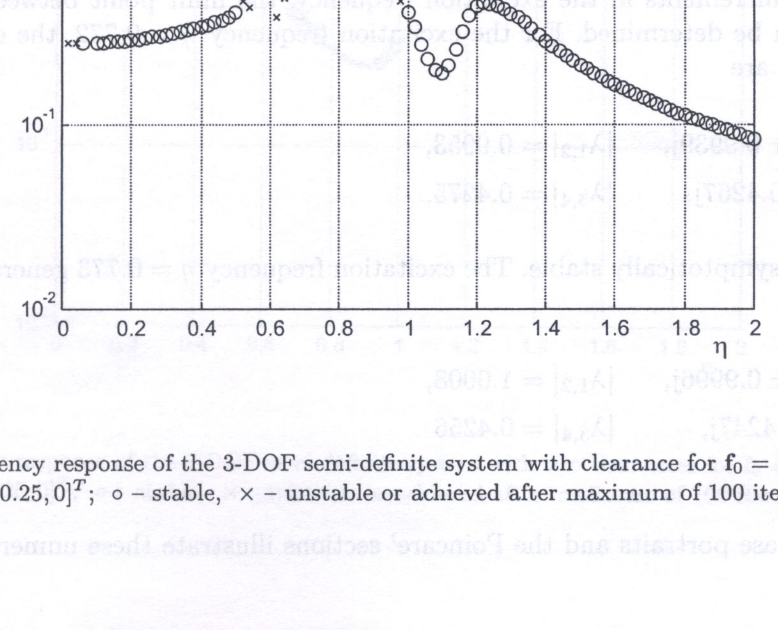


**Fig. 4.** Frequency response of the 3-DOF semi-definite system with clearance for $\mathbf{f}_0 = [0.25, 0.25]^T$ and $\mathbf{f}_a = [0.25, 0]^T$; ∘ – stable, × – unstable or achieved after maximum of 100 iterations

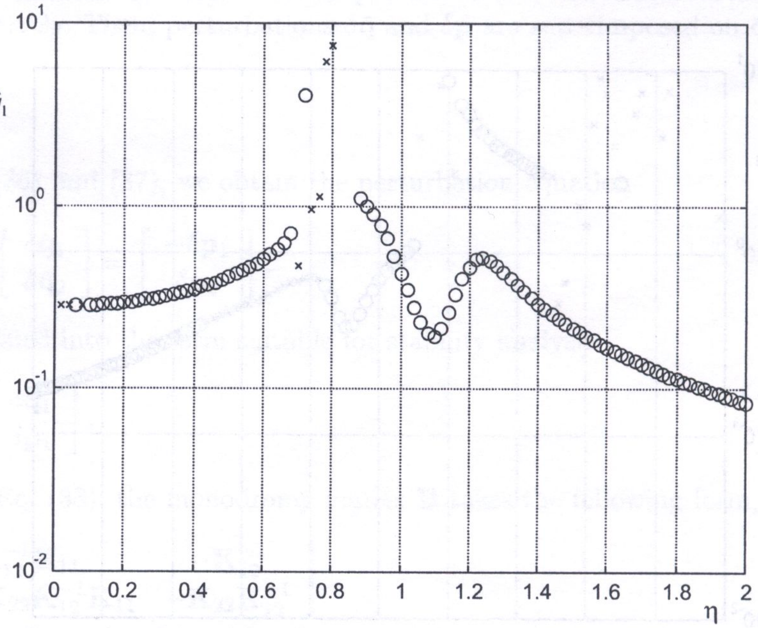


**Fig. 5.** Frequency response of the 3-DOF semi-definite system with clearance for $\mathbf{f}_0 = [0.5, 0.25]^T$ and $\mathbf{f}_a = [0.25, 0]^T$; ○ – stable, × – unstable or achieved after maximum of 100 iterations

load, only few such solutions are found (Fig. 5). This phenomenon is linked with the dominance of primary resonance in the solution.

The frequency responses from Figs. 3–5 are in agreement with those in reference [10] as well as with the solutions obtained by the piecewise full decoupling method [6, 7].

The finite element in time method is also applied to the same mechanical model but with two clearances. This problem was not solved in [10]. The frequency response is computed for the third loading case, only. Considering the response of linear system as the starting vector, the unconverged or unstable solutions are found in the interval of non-dimensional frequencies $0.7 < \eta < 0.96$ as shown in Fig. 6. Using the previous stable solution as the starting vector, the unconverged solutions are not found, any more. Only the unstable solutions are obtained in the range $0.77 < \eta < 0.94$ (Fig. 7). It proves that the convergence of FET solutions depends on initial conditions.

Imposing small increments in the excitation frequency, the limit point between stable and unstable solutions can be determined. For the excitation frequency $\eta = 0.772$, the eigenvalues of the monodromy matrix are

$$\lambda_{1,2} = -0.0527 \pm 0.9939\mathrm{j}, \qquad |\lambda_{1,2}| = 0.9953,$$
$$\lambda_{3,4} = 0.0266 \pm 0.4267\mathrm{j}, \qquad |\lambda_{3,4}| = 0.4275. \tag{44}$$

i.e. the solution is asymptotically stable. The excitation frequency $\eta = 0.773$ generates the following eigenvalues,

$$\lambda_{1,2} = -0.0496 \pm 0.9996\mathrm{j}, \qquad |\lambda_{1,2}| = 1.0008,$$
$$\lambda_{3,4} = 0.028 \pm 0.4247\mathrm{j}, \qquad |\lambda_{3,4}| = 0.4256 \tag{45}$$

(unstable solution).

In Fig. 8, the phase portraits and the Poincare' sections illustrate these numerical results.

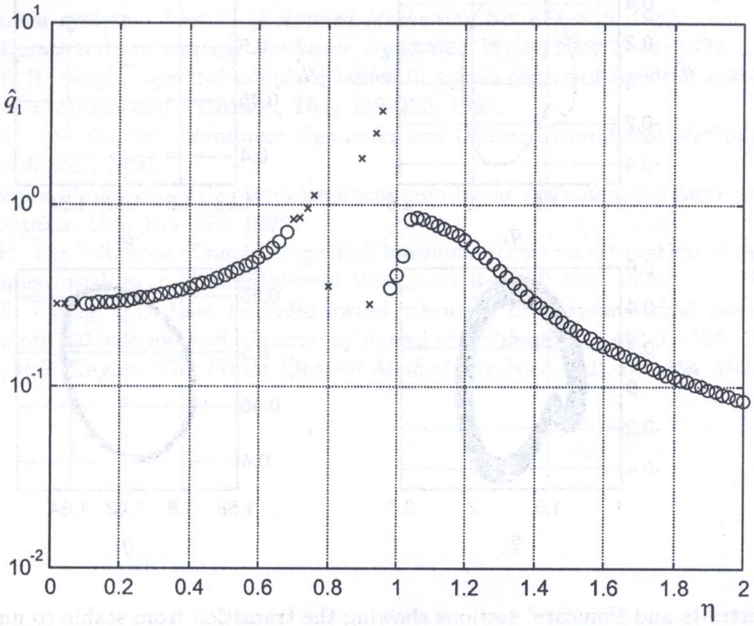


**Fig. 6.** Frequency response of the 3-DOF semi-definite system with two clearances for $\mathbf{f}_0 = [0.5, 0.25]^T$ and $\mathbf{f}_a = [0.25, 0]^T$; ∘ – stable, × – unstable or achieved after maximum of 100 iterations

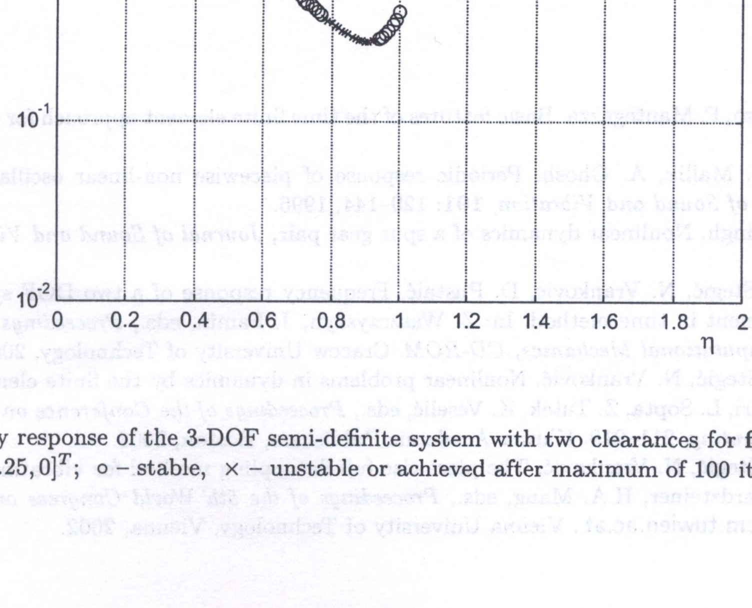


**Fig. 7.** Frequency response of the 3-DOF semi-definite system with two clearances for $\mathbf{f}_0 = [0.5, 0.25]^T$ and $\mathbf{f}_a = [0.25, 0]^T$; ∘ – stable, × – unstable or achieved after maximum of 100 iterations

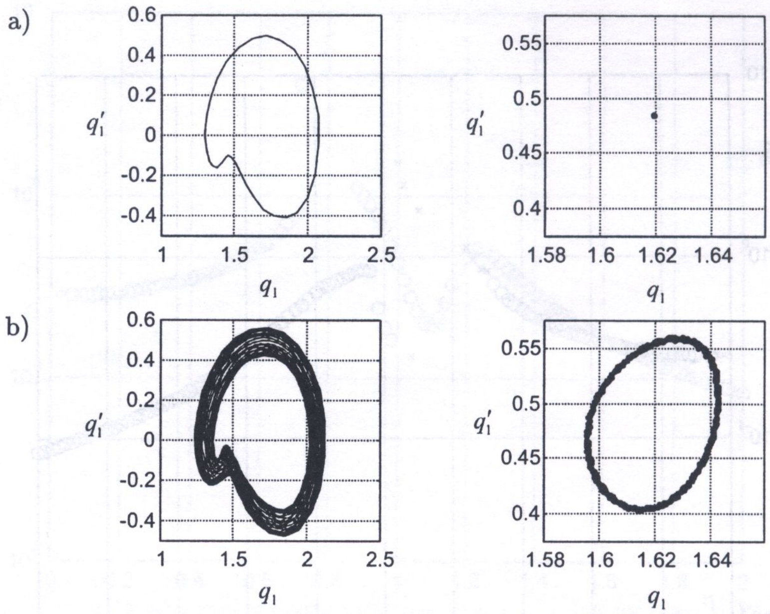


**Fig. 8.** Phase portraits and Poincare' sections showing the transition from stable to unstable response; a) stable response for $\eta = 0.772$, b) unstable response for $\eta = 0.773$

## 6. Concluding Remarks

The steady state motions of periodically forced two-degree-of-freedom systems with clearances are studied with the finite element in time method. Based on Hamilton weak principle, the finite element in time method gives accurate numerical results with only a few time elements and without significant computational effort. The solutions often tend to the first harmonic responses while the range of convergence for sub-periodic responses is very small. The convergence of frequency responses with dominance of subharmonics can be improved by using the previous stable solution as the starting vector.

The straightforward determination of the stability of periodic solution is the advantageous feature of this method. The stability of periodic solution can be investigated considering small perturbations of the solution. According to the Floquet theory, the stability of solutions is determined by the eigenvalues of the monodromy matrix that relates the initial and final perturbations.

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