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Large deflection analysis of moderately thick composite plates

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Large deflection analysis of laminated composite plates is considered. The Galerkin method along with Newton–Raphson method is applied to large deflection analysis of laminated composite plates with various edge conditions. First order shear deformation theory and von Kármán type nonlinearity are utilized and the governing differential equations are solved by choosing suitable polynomials as trial functions to approximate the plate displacement functions. The solutions are compared to that of Chebyshev polynomials and finite elements. A very close agreement has been observed with these approximating methods. In the solution process, analytical computation has been done wherever it is possible, and analytical-numerical type approach has been made for all problems.

Keywords: Galerkin method, large deflection, composite plates

1. Introduction

Different numerical techniques were employed to investigate the geometrically nonlinear behavior of thick plates. Pica *et al.* solved the large deflection problem of isotropic plates using finite element technique [8]. Reddy and Chao applied the finite element method (FEM) to large deflection and large amplitude free vibration problems of laminates [9]. Turvey and Osman performed the large deflection analysis of isotropic plates using Dynamic Relaxation technique [14]. Liu *et al.* solved the large deflection problem of elliptical plates using Galerkin method [6]. Shukla and Nath presented a Chebyshev polynomials (CP) solution for geometrically nonlinear problem of moderately thick laminates [12]. Liew *et al.* proposed a mesh-free kp-Ritz method for the large deflection flexural analysis of laminated composite plates [5]. First order shear deformation theory (FSDT) based on Mindlin's hypothesis and von Kármán type geometric nonlinearity were utilized in the aforementioned works.

Among the methods of weighted residuals, the Galerkin method (GM) is a powerful numerical solution technique to differential equations. The Galerkin technique has found a research area for a particular case of boundary conditions and trial functions for the large deflection analysis [4, 6, 11, 13].

This paper concerns the effect of shear deformations in the large deflection analysis of composite plates. The FSDT based on Mindlin's hypothesis is imposed using von Kármán type geometric nonlinearity. Governing nonlinear equations are solved by employing GM with the Newton–Raphson technique. In the solution process, computations have been carried out analytically wherever it is possible and analytical-numerical type approach has been made for all cases.

2. Governing equations

Consider a rectangular laminated plate with dimensions a, b and uniform thickness h. The origin of the coordinate system is chosen to coincide with the center of the midplane of the undeformed

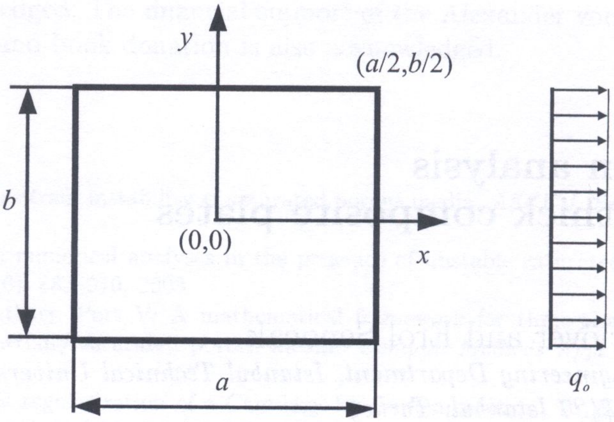

Fig. 1. Plate geometry and loading

plate (see Fig. 1). The plate is assumed to be subjected to a uniform transverse pressure q_o, and it is constructed of finite number of homogeneous orthotropic layers perfectly bonded together.

Under the assumptions of first order shear deformation theory based on Mindlin's hypothesis; let u, v, w denote the displacements at an arbitrary point of the plate in the x, y, z directions and $u^0(x, y)$, $v^0(x, y)$, $w^0(x, y)$ are the displacements at a corresponding point of the midplane of the plate in the x, y and z directions respectively. Then the displacement field of the first order theory is of the form [7],

$$\begin{aligned} u(x, y, z, t) &= u^0(x, y, t) + \phi_x(x, y, t)z, \\ v(x, y, z, t) &= v^0(x, y, t) + \phi_y(x, y, t)z, \\ w(x, y, z, t) &= w^0(x, y, t), \end{aligned} \tag{1}$$

where ϕ_x and ϕ_y are the rotations of a transverse normal about the y and x axes respectively. The corresponding total strains could be expressed as follows,

$$\begin{aligned} \varepsilon_x &= \varepsilon_x^0 + \kappa_x z, \\ \varepsilon_y &= \varepsilon_y^0 + \kappa_y z, \\ \gamma_{xy} &= \gamma_{xy}^0 + \kappa_{xy} z, \\ \gamma_{xz} &= \gamma_{xz}^0, \gamma_{yz} = \gamma_{yz}^0. \end{aligned} \tag{2}$$

Considering von Kármán type geometric nonlinearity [2], the strain displacement relations can be written as

$$\begin{aligned} \varepsilon_x^0 &= u_{,x}^0 + \frac{1}{2} w_{,x}^2 , \\ \varepsilon_y^0 &= v_{,y}^0 + \frac{1}{2} w_{,y}^2 , \\ \gamma_{xy}^0 &= u_{,y}^0 + v_{,x}^0 + w_{,x} w_{,y} , \\ \gamma_{xz}^0 &= w_{,x} + \phi_x , \\ \gamma_{yz}^0 &= w_{,y} + \phi_y , \end{aligned} \tag{3}$$

where differentiations are denoted by comma. Midplane curvatures and twist of the plate are the following,

$$\kappa_x = \phi_{x,x} , \qquad \kappa_y = \phi_{y,y} , \qquad \kappa_{xy} = \phi_{x,y} + \phi_{y,x} . \tag{4}$$

For a plate with an arbitrary number of layers, the constitutive relations are

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{Bmatrix}, \tag{5}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix}, \tag{6}$$

where $\mathbf{N}$ and $\mathbf{M}$ are the resultant forces and moments conjugate to $\boldsymbol{\varepsilon}^0$ and $\boldsymbol{\kappa}$, respectively. Q_x and Q_y are transverse forces and the parameter K is shear correction factor [15]. A_{ij} , B_{ij} and D_{ij} are symmetric matrices defined as follows,

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} (1, z, z^2)\, Q_{ij}\, \mathrm{d}z, \tag{7}$$

where Q_{ij} are the corresponding reduced stiffness coefficients.

Five governing equations of motion for the plate can be written as follows in the general form [10]:

$$\begin{aligned}
R_1 &= N_{x,x} + N_{xy,y} - I_0 u^0_{,tt} - I_1 \phi_{x,tt} = 0, \\
R_2 &= N_{xy,x} + N_{y,y} - I_0 v^0_{,tt} - I_1 \phi_{y,tt} = 0, \\
R_3 &= Q_{x,x} + Q_{y,y} + (w_{,x} N_x + w_{,y} N_{xy})_{,x} + (w_{,x} N_{xy} + w_{,y} N_y)_{,y} + q_o - I_0 w_{,tt} = 0, \\
R_4 &= M_{x,x} + M_{xy,y} - Q_x - I_2 \phi_{x,tt} - I_1 u^0_{,tt} = 0, \\
R_5 &= M_{xy,x} + M_{y,y} - Q_y - I_2 \phi_{y,tt} - I_1 v^0_{,tt} = 0,
\end{aligned} \tag{8}$$

where I_0, I_1 and I_2 are mass moments of inertia defined as

$$(I_0, I_1, I_2) = \int_{-h/2}^{h/2} \rho\,(1, z, z^2)\, \mathrm{d}z = \sum_i \int_{z_i}^{z_{i+1}} (1, z, z^2)\, \rho^{(i)}\, \mathrm{d}z, \tag{9}$$

$\rho^{(i)}$ being the material density of the i-th layer. The equations of motion (8) in terms of displacements can be obtained by making use of constitutive (5, 6), strain-displacement (3) and curvature-displacement (4) relations (these equations are not given here; see [10] for details).

For the Galerkin approach, the normalized displacements of the plate are approximated in the form shown below:

$$\begin{aligned}
u^0 &= \sum_{m=0}^{M} \sum_{n=0}^{N} a_{mn}(t)\, U_{mn}(x, y), \\
v^0 &= \sum_{m=0}^{M} \sum_{n=0}^{N} b_{mn}(t)\, V_{mn}(x, y), \\
w &= \sum_{m=0}^{M} \sum_{n=0}^{N} c_{mn}(t)\, W_{mn}(x, y), \\
\phi_x &= \sum_{m=0}^{M} \sum_{n=0}^{N} d_{mn}(t)\, S_{mn}(x, y), \\
\phi_y &= \sum_{m=0}^{M} \sum_{n=0}^{N} e_{mn}(t)\, T_{mn}(x, y),
\end{aligned} \tag{10}$$

where a_{mn} , b_{mn} , c_{mn} , d_{mn} and e_{mn} are unknown functions of time, U_{mn} , V_{mn} , W_{mn} , S_{mn} and T_{mn} are the trial functions, and M and N are the number of terms in x and y directions respectively. In general, M and N may take different values for each displacement function. Herein, polynomials are used as trial functions, which are chosen to satisfy the kinematic boundary conditions, where as natural boundary conditions are not satisfied. In this case, simultaneous approximation is made to the solutions of differential equations and to the boundary conditions. Since this is a static analysis a_{mn} , b_{mn} , c_{mn} , d_{mn} and e_{mn} are not time dependent functions (they are taken as unknown constants) and time derivative terms in the equations of motion are all cancelled.

Substituting Eq. (10) into nonlinear plate equilibrium equations and the boundary terms yields the residuals in the domain of the plate and at the boundaries of the plate. Forcing these residuals to be orthogonal to each member of a set of trial functions yields the following Galerkin equations,

$$\begin{aligned}
&\int_{-b/2}^{+b/2}\int_{-a/2}^{+a/2} U_{mn}R_1\,\mathrm{d}x\mathrm{d}y - \int_{-b/2}^{+b/2} U_{mn}N_x|_{x=\pm a/2}\,\mathrm{d}y - \int_{-a/2}^{+a/2} U_{mn}N_{xy}|_{y=\pm b/2}\,\mathrm{d}x = 0,\\
&\int_{-b/2}^{+b/2}\int_{-a/2}^{+a/2} V_{mn}R_2\,\mathrm{d}x\mathrm{d}y - \int_{-a/2}^{+a/2} V_{mn}N_y|_{y=\pm b/2}\,\mathrm{d}x - \int_{-b/2}^{+b/2} V_{mn}N_{xy}|_{x=\pm a/2}\,\mathrm{d}y = 0,\\
&\int_{-b/2}^{+b/2}\int_{-a/2}^{+a/2} W_{mn}R_3\,\mathrm{d}x\mathrm{d}y = 0,\\
&\int_{-b/2}^{+b/2}\int_{-a/2}^{+a/2} S_{mn}R_4\,\mathrm{d}x\mathrm{d}y - \int_{-b/2}^{+b/2} S_{mn}M_x|_{x=\pm a/2}\,\mathrm{d}y = 0,\\
&\int_{-b/2}^{+b/2}\int_{-a/2}^{+a/2} T_{mn}R_5\,\mathrm{d}x\mathrm{d}y - \int_{-a/2}^{+a/2} T_{mn}M_y|_{y=\pm b/2}\,\mathrm{d}x = 0.
\end{aligned} \tag{11}$$

3. Solution procedure

In the application of the Galerkin method the kinematic boundary conditions are satisfied by choosing appropriate trial functions. In the Galerkin method used here, the evaluations of integrals are symbolically computed by using a commercial computer math code *Mathematica*™ [16]. Trial functions are weighted polynomials given as follows,

$$\begin{aligned}
U_{mn} &= \Phi_1(x,y)\,x^m y^n,\\
V_{mn} &= \Phi_2(x,y)\,x^m y^n,\\
W_{mn} &= \Phi_3(x,y)\,x^m y^n,\\
S_{mn} &= \Phi_4(x,y)\,x^m y^n,\\
T_{mn} &= \Phi_5(x,y)\,x^m y^n,
\end{aligned} \tag{12}$$

where Φ_i, $(i = 1, \ldots, 5)$ denote the weight functions. Substituting Eq. (10) into Eqs. (11), nonlinear equations in terms of unknown coefficients a_{mn} , b_{mn} , c_{mn} , d_{mn} and e_{mn} are obtained. These equations are solved by employing the Newton–Raphson methodology.

4. Boundary conditions

Three different boundary conditions are considered and shown in Table 1. The Galerkin integrals in Eqs. (11) are in the general form and must be modified according to relevant boundary conditions. Note that whole plate models are analyzed in all cases presented here.

Table 1. Boundary conditions and corresponding weight functions

CC	$N_x = N_{xy} = w = \phi_x = \phi_y = 0$ at $x = \pm a/2$, $N_y = N_{xy} = w = \phi_x = \phi_y = 0$ at $y = \pm b/2$. $\Phi_i = 1\ (i = 1, 2)$, $\Phi_i = (x^2 - a^2/4)(y^2 - b^2/4)\ (i = 3, \ldots 5)$.
CS–1	$u^0 = v^0 = w = 0$ at $x = \pm a/2$ and $y = \pm b/2$, $\phi_x = 0$ at $x = -a/2$ and $y = \pm b/2$, $M_x = 0$ at $x = +a/2$, $M_y = 0$ at $y = \pm b/2$, $\phi_y = 0$ at $x = \pm a/2$. $\Phi_i = (x^2 - a^2/4)(y^2 - b^2/4)\ (i = 1, \ldots 3)$, $\Phi_4 = (x + a/2)(y^2 - b^2/4)$, $\Phi_5 = (x^2 - a^2/4)$.
CS–2	$u^0 = v^0 = w = \phi_x = 0$ at $x = \pm a/2$ and $y = \pm b/2$, $M_y = 0$ at $y = \pm b/2$, $\phi_y = 0$ at $x = \pm a/2$. $\Phi_i = (x^2 - a^2/4)(y^2 - b^2/4)\ (i = 1, \ldots 4)$, $\Phi_5 = (x^2 - a^2/4)$.

5. Results

Large deflection of a symmetric cross-ply laminate and unsymmetric angle-ply and cross-ply laminates are chosen as numerical examples for the application of the GM. Comparisons with results of the other solution techniques such as Chebyshev polynomials and finite elements are given.

In the application of the GM the displacement (trial) functions are approximated by polynomials expressed in Eq. (12). The proper choice of the trial functions considering symmetry of the problem can reduce the computation time [13]. The systematic choice of the trial functions is explained in detail for each case. The convergence study of the proposed solution technique is carried out and it has been determined that taking M and N as five is appropriate. Hence, for all the GM applications given here, M and N are taken as five.

5.1. Symmetric cross-ply laminate

The large deflection of a symmetric cross-ply [0°/90°/90°/0°] laminate under various boundary conditions is analyzed. The powers m and n are taken as in the following manner for CS–1,

$$\left.\begin{matrix} m = 0, 1, 2, \ldots, 5 \\ n = \text{even} \end{matrix}\right\} U_{mn}\,,\ W_{mn}\,,\ S_{mn}\,, \qquad \left.\begin{matrix} m = 0, 1, 2, \ldots, 5 \\ n = \text{odd} \end{matrix}\right\} V_{mn}\,,\ T_{mn}\,, \tag{13}$$

and for CS–2 type boundary condition,

$$\left.\begin{matrix} m = \text{odd} \\ n = \text{even} \end{matrix}\right\} U_{mn}\,,\ S_{mn}\,, \qquad \left.\begin{matrix} m = \text{even} \\ n = \text{odd} \end{matrix}\right\} V_{mn}\,,\ T_{mn}\,, \qquad \left.\begin{matrix} m = \text{even} \\ n = \text{even} \end{matrix}\right\} W_{mn}\,. \tag{14}$$

The material and geometry constants of the plate are taken from [12] and given as follows,

$$E_1 = 175.78\ \text{GPa}, \quad E_2 = E_1/25,$$
$$G_{12}/E_2 = G_{13}/E_2 = 0.5, \quad G_{23}/E_2 = 0.2,$$
$$\nu_{12} = 0.25, \quad a = b, \quad a/h = 10, \quad K = 5/6.$$

Normalized center deflection and center moment values of the plate under CS–1 and CS–2 boundary conditions for GM and CP (values are read from [12]) are given in Figs. 2 and 3.

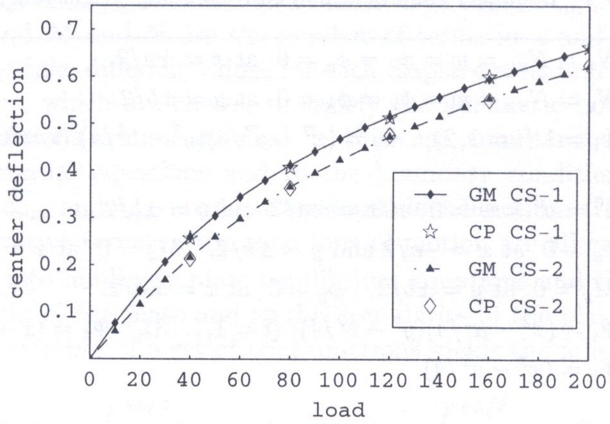

Fig. 2. Load ($q_o a^4/E_2h^4$) versus central deflection (w/h) of [0°/90°/90°/0°] laminate

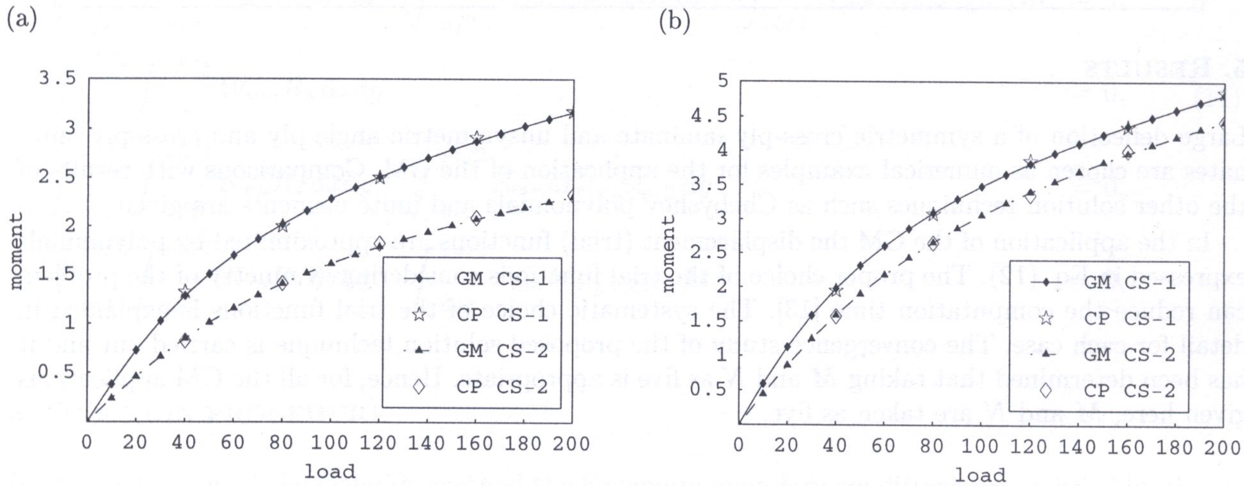

Fig. 3. Load ($q_o a^4/E_2h^4$) versus $M_x a^2/D_{11}h$ (a) and $M_y a^2/D_{22}h$ (b) at the center of [0°/90°/90°/0°] laminate

5.2. Unsymmetric laminates

Unsymmetric angle-ply and cross-ply laminates under CC type boundary condition are considered for the large deflection analysis. In the nonlinear analysis of the cross-ply laminate the powers m and n are taken as in the same manner given in Eq. (14). For the nonlinear analysis of angle-ply laminate the powers m and n are chosen in such a way that $(m+n)$ is odd for the in-plane displacement functions and rotations and even for the transverse displacement function.

Material and geometrical properties of the laminates are taken from [9] and given below.

$$E_2 = E_1/40, \quad G_{12}/E_2 = 0.6, \quad G_{13}/E_2 = 0.5, \quad G_{23} = G_{13},$$
$$\nu_{12} = 0.25, \quad a = b, \quad a/h = 40, \quad K = 5/6.$$

Dimensionless center deflection-load curves of these laminates are shown in Fig. 4. FEM [9] and perturbation technique [3] results (they are not shown here) for these laminates are found to be in good agreement with the present results. Results of the commercial FEM program ANSYS [1] can be seen in Fig. 4. Shell-91 type elements including shear deformations with a 10×10 mesh are used in ANSYS program.

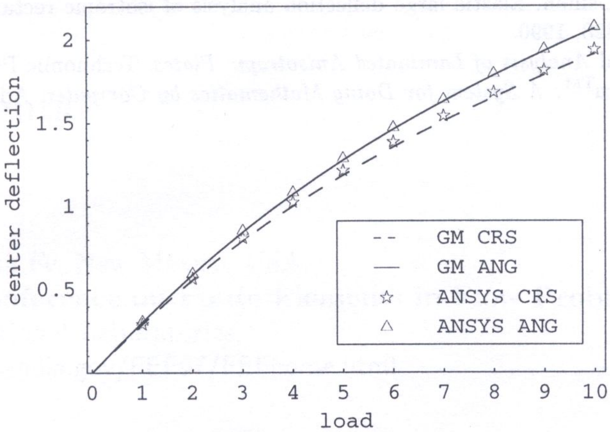

Fig. 4. Load ($10^{-2}q_o a^4/E_2 h^4$) versus central deflection (w/h) of cross-ply [0°/90°] (CRS) and angle-ply [45°/−45°] (ANG) laminates

6. Conclusions

Geometrically nonlinear analysis of thick composite plates based on FSDT is performed by using Galerkin approach along with Newton-Raphson technique. The choice of trial functions is crucial to approximate the two dimensional displacement field. The trial functions must be chosen in a way that essential boundary conditions are satisfied. The present solution methodology may be used to solve large deflection analysis of the moderately thick laminates in an easy and effective way with the help of a symbolic math package. The method is found to determine closely the displacements with a few number of terms. The results are compared to that of known other approximating methods (Chebyshev polynomials and finite elements), and commercial FEM code ANSYS. A very good agreement is observed. The convergence of the Galerkin method is found to be quite fast.

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