Laminar flow in trapezoidal grooves at finite Bond numbers with shear stress at the liquid-vapor interface by the method of fundamental solutions

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The problem of determination of Poiseuille number for a steady gravitational flow of liquid in an inclined open trapezoidal groove is addressed. The solution comprises of two parts. First, for a given groove's dimension, liquid-solid contact angle, and the Bond number, the shape of the free surface is determined starting from the Young–Laplace equation. The shooting method is used for solution of a two-point boundary value problem. Then, having determined the shape of the free surface and slope the groove, the fully developed laminar flow is determined. The boundary value problem is solved using the method of fundamental solutions. Given the distribution of liquid velocity, the Poiseuille number, as a function of the other parameters of the model is analysed.

1. INTRODUCTION

The problem of the laminar flow of viscous incompressible liquid in internally-grooved ducts are interesting in process equipment to improve heat transfer during evaporation (or condensation). In literature this problem is not new and has been considered by many authors. Ayyaswamy *et al.* [1] used the Galerkin boundary method to solve the Poiseuille flow in triangular grooves. They assumed that the free surface of liquid has a constant radius of curvature, which corresponds to a very low Bond number. Moreover, in that paper, interfacial shear stress was zero. The problem of the laminar flow of a viscous incompressible liquid in triangular and trapezoidal grooves due to gravity forces at very low Bond numbers with shear stress at the liquid-vapor interfaces numbers was considered in the following papers [3, 5–7, 11]. In the case of very low Bond number, the curvature of the free surface is assumed to be constant, so that the problem of determining the shape of this surface is omitted. In the paper [4], the flow in triangular grooves at finite a Bond number (not a constant radius of the free surface) but without interfacial shear stress was considered. The problem of the laminar flow of a viscous incompressible liquid in rectangular grooves by means of separation variables was considered in the papers [8, 9].

The purpose of this paper is to develop a numerical method which would be useful in analyzing the laminar flow in trapezoidal groove with the shear stress at the liquid-vapor interface for a wide distribution of the Bond number. The problem of determining the shape of the free surface is decoupled with the problem of determining fluid velocity. First, for a given groove's dimension, liquid-solid contact angle, and the Bond number, the shape of the free surface is determined starting from the Young–Laplace equation. Then, having determined the shape of the free surface and slope the groove, the fully developed laminar flow is determined by means of method of fundamental solutions.
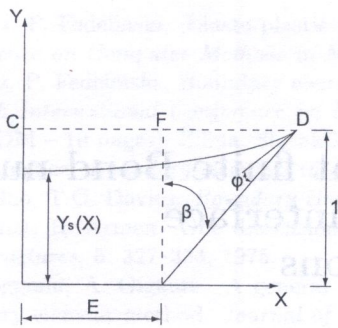
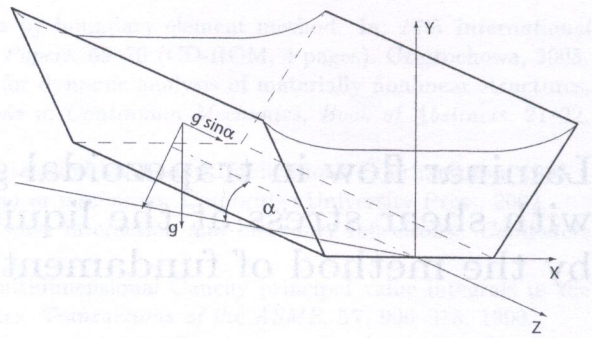Fig. 1. Dimensionless cross sections of the trapezoidal groove

Fig. 2. Three-dimensional view of channel

Geometry of the considered model is presented in Figs. 1 and 2, where Y_S is the distance between the free surface and the bottom of the groove, φ is the meniscus contact angle, β is the groove half-angle and α is the angle of the slope of the groove.

2. DETERMINING THE SHAPE OF THE FREE SURFACE

In order to determine the shape of the free surface, the Laplace–Young equation is used,

$$\Delta p = \sigma \frac{1}{R}.$$ (1)

The equation above, which is simplified by omitting one of its components because of the fact that one curvature radius is infinite, describes the relation between the surface tension σ, the curvature radius R and the pressure drop Δp at the surface.

On the other hand, the difference in pressure Δp can be expressed on the basis of hydrostatics,

$$\Delta p = p_1 - p_2 = (\gamma_2 - \gamma_1) y + C_1 + C_2.$$ (2)

Assuming that $\gamma_1 \ll \gamma_2 = \gamma$ and $C = C_1 + C_2$, Eq. (1) can be rewrite into the following form,

$$\frac{1}{R} = \frac{\gamma y + C}{\sigma}.$$ (3)

The curvature of the cylindrical surface is described by the following differential equation,

$$\frac{1}{R} = \frac{\dfrac{d^2 y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}.$$ (4)

Combining Eqs. (3) and (4) we obtain

$$\gamma y + C = \sigma \frac{\dfrac{d^2 y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}.$$ (5)

Next, the Bond number is introduced as a dimensionless parameter, which characterizes free surface tension,

$$Bo = \frac{\gamma h^2}{\sigma},$$ (6)

where h is the specific height of the groove.

Because of the fact that the Bond number value could be freely chosen, the problem cannot be simplified by assuming that the Bond number is very low and consequently the shape of the free surface cannot be approximate as a segment of circle arc. Subsequently, dimensionless variables are defined in the following form,

$$X = \frac{x}{h}, \qquad Y = \frac{y}{h}, \qquad E = \frac{a}{h}. \tag{7}$$

Finally, the equation in dimensionless form takes the following form,

$$(Bo\, Y_S + C)\left[1 + \left(\frac{dY_S}{dX}\right)^2\right]^{\frac{3}{2}} = \frac{d^2 Y_S}{dX^2}. \tag{8}$$

Boundary conditions are

$$Y_S(E + \tan(\beta)) = 1 \qquad \text{and} \qquad \left.\frac{dY_S}{dX}\right|_{X=E+\tan(\beta)} = -\frac{1}{\tan(\varphi + \beta)}. \tag{9}$$

The unknown parameter C is determined during solving Eqs. (8)–(9) in such a way that the condition

$$\left.\frac{dY_S}{dX}\right|_{X=0} = 0 \tag{10}$$

is satisfied. The two point boundary value problem formulated in Eq. (8) and the boundary conditions (9)–(10) by means of *Mathematica* [12] is solved using the function *NDSolve* with its default options set. First, the parameter C is freely chosen and than the equation is solved a few times modifying the value of the parameter C in such a way, that the condition (10) is satisfied (see the sample code in Section 5).

3. Determining fluid flow

Considering that the flow is laminar and it takes place in a straight groove of constant cross section, we can assume that there is only one velocity component different form zero, which in this case is w. The governing equations for the flow of viscous incompressible fluid, which are the Navier–Stokes equations, are as follows,

$$\frac{\partial w}{\partial z} = 0, \tag{11}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{12}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\rho g}{\mu}\cos(\alpha), \tag{13}$$

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) - \frac{\rho g}{\mu}\sin(\alpha), \tag{14}$$

where u, v, w are the components of velocity field, p – pressure, g – gravity, μ – dynamic viscosity, ρ – density of the fluid, α – the angle of the slope.

The consequence of these equations is the fact that the velocity component is not dependent on the z coordinate. Assuming that the flow is steady (time-independent), it is possible to consider the simplified form of the equation,

$$\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2} = -\frac{1}{\mu}\frac{\partial p(z)}{\partial z} - \frac{\rho g}{\mu}\sin(\alpha). \tag{15}$$

Because of the fact the pressure does not change along the z coordinate,

$$\frac{\partial p(z)}{\partial z} = 0.$$
(16)

Hence we obtain the following equation,

$$\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2} = -\frac{\rho g}{\mu} \sin(\alpha),$$
(17)

and then, introducing a dimensionless variable

$$W = \frac{w\mu}{h^2 \rho g \sin(\alpha)},$$
(18)

it's possible to rewrite the governing equation as follows,

$$\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} = -1.$$
(19)

The parameters which are very useful while analyzing mentioned fluid flow are: the Poiseuille number Po and the mean velocity $\bar{W}$. The expressions which allow us to determine the values of these parameters are as follows.

The Poiseuille number of the liquid in the groove is given by

$$Po = f\,Re = \frac{D_h^2}{2\bar{W}}$$
(20)

where f is friction factor and Re is the Reynolds number and the dimensionless hydraulic diameter D_h is defined by

$$D_h = \frac{4A}{P}.$$
(21)

The mean velocity $\bar{W}$ is defined as

$$\bar{W} = \frac{\int_A W(X,Y)\,\mathrm{d}A}{A},$$
(22)

where A is the dimensionless cross-section area and is the wetted perimeter defined by

$$P = E + \frac{1}{\cos(\beta)}.$$
(23)

4. DETERMINATION OF FLUID VELOCITY

The boundary value problem given in Eq. (19) is solved with the following boundary conditions,

$$\mathrm{n} \cdot \nabla W = \tau \qquad \text{– at the liquid-vapor interface,}$$
(24)

$$W = 0 \qquad \text{– at the boundary of the channel,}$$
(25)

where τ is the dimensionless shear stress.

For a given shape of free surface, velocity field is determined in the next step. A solution to the boundary value problem given by Eq. (19) and the boundary condition (24)–(25) by means of the fundamental solutions method [2] consists in assumed solution in the form

$$W(X,Y) = W_n(X,Y) + \sum_{j=1}^{N} c_j \varphi_j(X,Y)$$
(26)

where $W_n(X, Y)$ is a particular solution to Eq. (19), $\varphi_j(X, Y)$ are functions (fundamental solutions) which exactly satisfy the Laplace equation and c_j are unknown coefficients.

Particular solution is chosen in the form

$$W_n(X, Y) = -\frac{X^2 + Y^2}{4}. \tag{27}$$

A fundamental solution for a 2-D Laplace equation with a symmetrical solution in the domain considered (trapezoidal cross section channel) can be chosen in such a way that the symmetry condition is satisfied exactly by

$$\varphi_j = \frac{1}{2}\left[\ln\left((X - X_j)^2 + (Y - Y_j)^2\right) + \ln\left((X + X_j)^2 + (Y - Y_j)^2\right)\right]. \tag{28}$$

where (X_j, Y_j) are coordinates of source points placed outside the considered flow region.

Finally, the assumed solution has the form

$$W(X, Y) = -\frac{X^2 + Y^2}{4} + \frac{1}{2}\sum_{j=1}^{N} c_j\left[\ln\left((X - X_j)^2 + (Y - Y_j)^2\right) + \ln\left((X + X_j)^2 + (Y - Y_j)^2\right)\right]. \tag{29}$$

This solution satisfies the governing equation (19), boundary condition and symmetry condition. The unknown coefficients c_j are determined by satisfying appropriate boundary conditions on the free surface and rigid wall of the channel by means of boundary collocation in the least squares sense. For that reason $nbp1$ equidistant collocation points on the free surface, $nbp2$ equidistant collocation points on the boundary of the channel and equidistant source points are chosen (see Fig. 3). The source points are placed outside the region of the flow at the contour which is the scaled geometry of the channel.
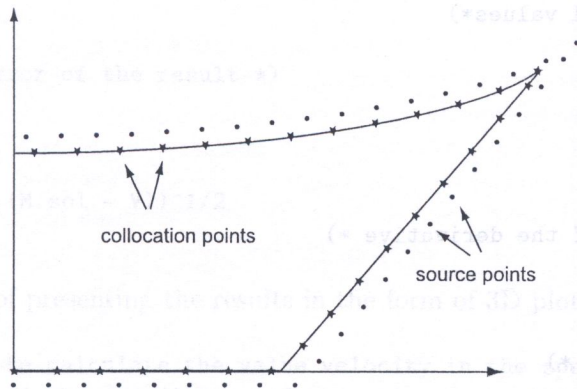

Fig. 3. Distribution of collocation and source points

Consequently, the problem is reduced to the system of linear equations for unknown coefficients c_j, $j = 1 \ldots N = nsp$. This overdetermined system ($N \geq nbp = nbp1 + nbp2$) is solved using a method based on QR decomposition of matrix. Putting the coefficients obtained c_j into Eq. (29), we have the formula for velocity field in the domain considered.

The integral appearing in Eq. (22) is calculated numerically using the Mathematica function *NIntegrate* with the Gauss–Kronrod-based algorithm [12]. This function usually uses an adaptive algorithm, which recursively subdivides the integration region as needed.

5. USE OF SYMBOLIC ALGEBRA SOFTWARE TO SOLVE THE NUMERICAL PROBLEM

Mathematica software [12] is a powerful tool which can be successfully used in wide range of numerical calculations. In this case Mathematica was used to solve the problem since the determination of the free surface curvature until the solution of the PDE and plotting the results as well.

For that reason we would like to present a sample code, which we used in our calculation in order to demonstrate the use of this software.

The first example is the code used in order to solve the differential equation and determining the unknown parameter C which is required in the free surface equation.

```
(*differential equation*)
eq[Bo_, cc_] := {y''[x] == (1 + (y'[x])^2)^(3/2)*(Bo*y[x] + cc)}

(*the expression of boundary condition*)
ic[fi_, beta_] := {y[0] == 1, y'[0] == -1/Tan[fi + beta]}

(*parameters *)
par := {Bo0 = 0.2, cc0 =-15, dc=1, fi0=6/5, beta0 = 2/3, ee = 1};

(*conversion to numerical values*)
par = N[par];

(* x - coordinate for specific beta angle parameter *)
xend = ee + Tan[beta0];

(* the values of the boundary conditions for specific parameters' values *)
ic1 = ic[fi0, beta0];

(* setting the initial values*)
cc1 = cc0;

(* setting the initial values*)
ydxendold = -1;

Do[
   (* step forward *)
   cc0 = cc1 - dc;

   (* initial value of the derivative *)
   ydxendold = 0;

   Do[
      (* step forward *)
      cc0 += dc;

      (* solving the equation *)
      sol1 = NDSolve[{
      eq[Bo0, cc0], ic1}, y, {x, 0, xend}][[1]];
      xe = sol1[[1, 2, 1, 1, 2]];
      yxend = y[x] /. sol1 /. {x -> xe};

      (* solving the derivative *)
      ydxend = D[y[x] /. sol1, x] /. {x -> xe};

      (*checking if the sign of the derivative has changed *)
      If[ydxend*ydxendold < 0, Break[]];

      (* overwriting the previous value of the derivative *)
      ydxendold = ydxend;
   , {step, 1, 30}
   ];
```
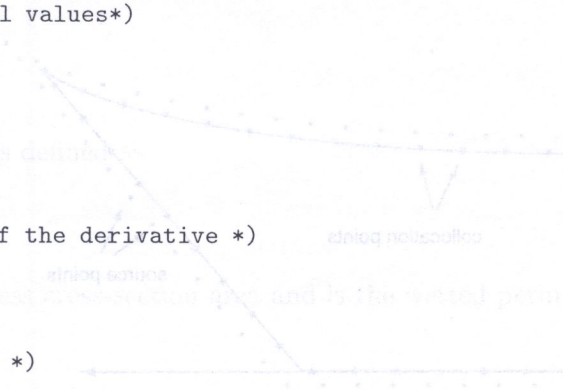
```
(* step backward if the sing of the derivative has changed *)
cc1 = cc0 - dc;

(* increasing the precision *)
dc *= 1/10;
,{1, 1, 15}
];

Print["ydx=", ydxend]
Print["c=", cc1, "=", N[cc1, 20]]
```

Another example is the solution of the overdetermined system of linear equations. The equation are solved using QR decomposition in order to find the least square solution of the system. Having determined the matrix of the system (M), the right-hand side vector (V) and the vector of unknown coefficients c_j, we solve the problem as follows:

```
(* M is the matrix of the system, Q, R are the matrices of decomposition *)
{Q, R} = QRDecomposition[M];

(* solving the system of linear equation *)
sol = LinearSolve[R, Q.V];

(* calculating the norm of the vector *)
Print["Norm of solution vector:"];
Norm[sol, 2]

(* calculating the error of the result *)
Print["Errors:"];
error1 = V -- M.sol;
Norm[error1, 2]
error = ((M.sol - V).(M.sol - V))^1/2
```

Finally – the example of presenting the results in the form of 3D plot of velocity field.

```
(* The function used to calculate the value velocity in the specified coordinate *)
(* vars_ is the list of the coordinates of the point: x, y *)
(* ps_ is thew list of the coordinates of the source points *)
(* c_ is the list of the coefficients (29) determined in the procedure above *)
(* nofps_ is the number of the source points *)

velocity[vars_, ps_, c_, nofps_] := Module[{w, rx1, ry1, rx2, x, y},
    x = vars[[1]];
    y = vars[[2]];
    w = -(x^2 + y^2)/4;
    Do[
        rx1 = (x - ps[[i, 1]])^2;
        ry1 = (y - ps[[i, 2]])^2;
        rx2 = (x + ps[[i, 1]])^2;
        w += c[[i]]*(Log[rx1 + ry1] + Log[rx2 + ry1])/2
    , {i, 1, nofps}
    ];
    Return[w]
];
```

```
velocityplot[vars_, ps_, c_, nofps_] := Module[{w, x, y},
    x = vars[[1]];
    y = vars[[2]];
    w = velocity[vars, ps, c, nofps];
    w = If[(yxplot /. x -> x) <= y, 0, w];
    w = If[((x - ee)/(xe - ee)) >= y, 0, w];
    Return[w]
];

Plot3D[velocityplot[{x, y}, ps, sol, nofps], {x, 0, xend}, {y, 0, 1},
    PlotPoints -> 80, Mesh -> False, FaceGrids -> All, AxesLabel -> {"X", "Y", "W"}]
```

The samples presented above shows only a few of the procedures which we used in our calculation whereas the complete Mathematica code one can find in [10].

6. RESULTS AND CONCLUSIONS

A numerical study was completed in which the flow in a trapezoidal groove was solved. Specifically, the values of the mean velocity and the Poiseuille number for various Bond numbers ($0.1 \leq Bo \leq 10$), contact angles ($\varphi = 0.4, 1.2$) and shear stress at liquid-vapor interface ($-0.2 \leq \tau \leq 7$) values are presented. Half-groove angle $0.1 \leq \beta \leq 1.7$.

The results of all examples were calculated using the method described above, in which the following numbers of collocation and source points: $nbp1 = 69$, $nbp2 = 138$, $nbp = 207$ and $nsp = 57$ were applied.

Figures 4a–h show the contour plots of the dimensionless mean velocity obtained for concurrent flow, no shear stress and countercurrent flow. The flow behavior changes significantly both when the shear stress is applied and when the Bond number is higher.

The smaller the value of the contact angle, the greater the influence of the Bond number. As shown in Figs. 5a–h the relation between the mean velocity and the shear stress is a linear function in a whole range of values of Bond numbers, whereas the Poiseuille number changes nonlinearly.

Sample three dimensional plot of velocity for the fallowing values of parameters: $Bo = 0.2$, $\varphi = 1.2$, $\beta = 0.6$, $\tau = 0$ is presented in Fig. 6.

ACKNOWLEDGMENT

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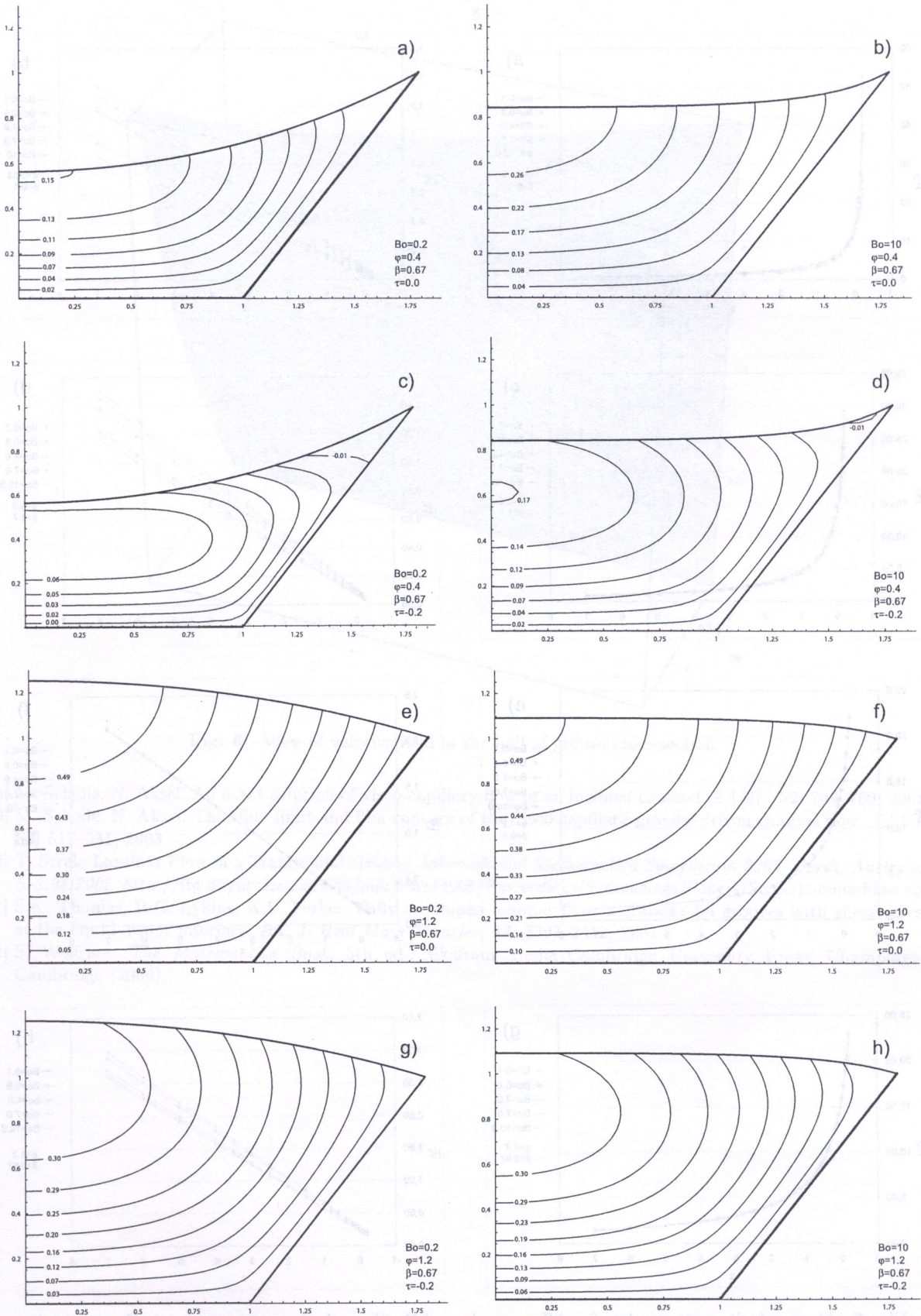
Fig. 4. Poiseuille number and mean velocity as a function of shear stress for different values of Bond numbers, contact angles and Half-grove angles
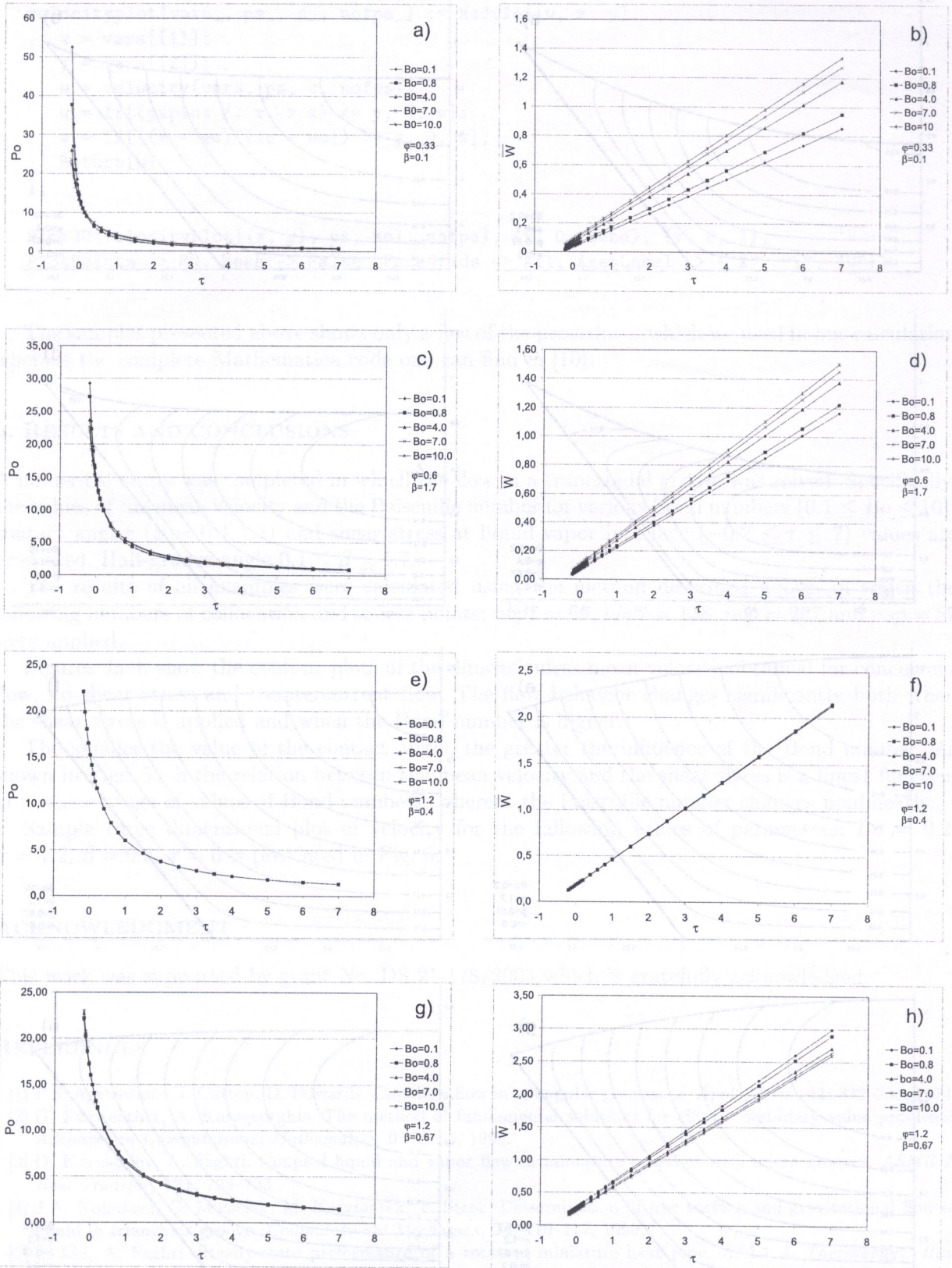
Fig. 5. Mean velocity contour plots for different values of Bond numbers, contact angles, half-grove angles and shear stress
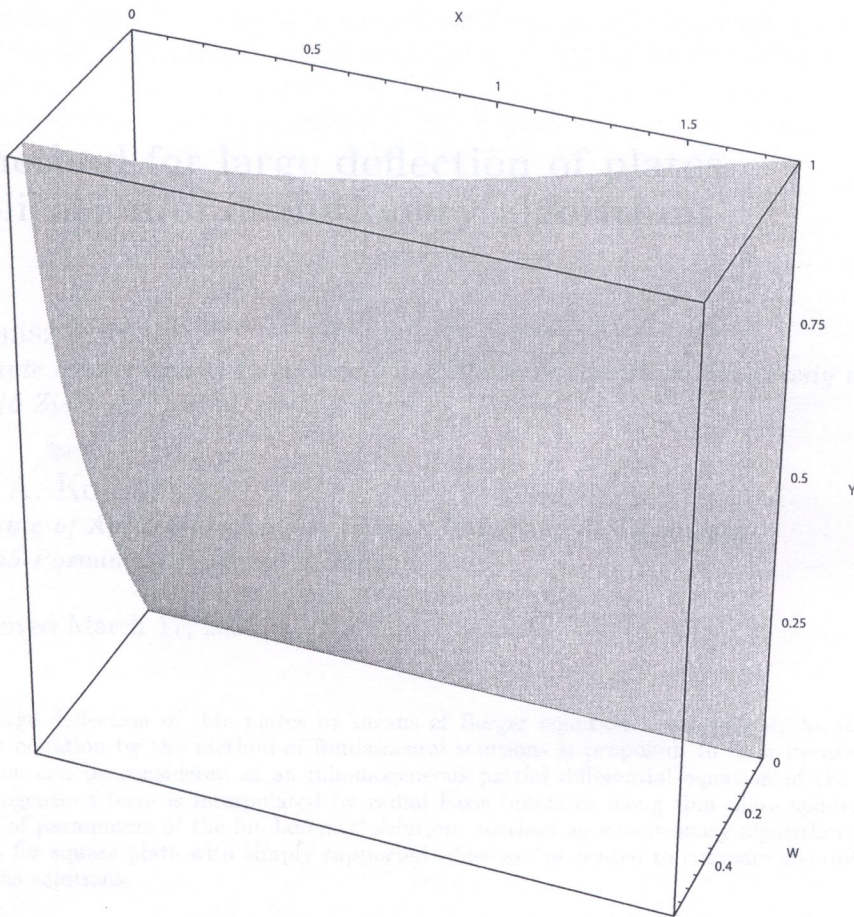
Fig. 6. View of velocity field in the half of groove cross-section

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