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Trefftz functions in FEM, BEM and meshless methods

Vladimír Kompiš, Mário Štiavnický
Academy of the Armed Forces of General Milan Rastislav Štefánik
Demänovská 393, 031 19 Lipt. Mikuláš, Slovakia

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The paper contains three different multi-domain formulations using Trefftz (T-) displacement approximation/interpolation, namely the hybrid-displacement FEM, reciprocity based FEM (multi-domain BEM) and the Boundary Meshless Method (BMM) for a single and multi-domain (MD) formulation. All three methods can lead to compatible formulation with the isoparametric FEM, when the displacements along the common boundaries are defined by same interpolation function. All three T-formulations enable to define more complicated elements/subdomains (the T-element can be also a multiply connected region) with integration along the element boundaries, only.

1. Introduction

There is 69 years since Trefftz published his paper [18] on the method which we call now the Trefftz method and there is 9 years since we started to organize the Workshops on Trefftz methods. The Workshops are organized every three years in different countries. Moreover, sessions on T-methods are organized in during many important conferences and congresses on Computational Mechanics. The T-functions improve efficiency of numerical methods and so, they are important tool of the numerical methods. The field of current application of the T-methods is large, but we want to show how they can be used in FEM, BEM and BMM for solids. The extension to other problems is straightforward.

The T-functions can have two roles in numerical models: 1) approximation and 2) interpolation of the field variables and the functions are approximated a) by the point collocation or b) in the weak (integral) form.

T-functions for solids may have different forms:

1. T-polynomials – obtained a) analytically (symbolically) [6, 7], or b) numerically [9]. In analytical models coefficients depend on the material properties and are obtained once for any material properties, but they are more complex and computationally not effective especially for 3D problems. In numerical models the coefficients are simply computed for each material and corresponding expressions are given in a matrix form for each order of the polynomial and for each material.

2. Kelvin (fundamental) solution [13] with source points outside the domain. The expressions have a weak singularity for displacements and strong singularity for strains, stresses and tractions in the source points. They correspond to action of a unit force in an infinite continuum.

3. Boussinesq–Cerruti [2, 3] solution is similar to the Kelvin solution and corresponds to action of a unit force on the boundary of the half space (or half plane in 2D).

4. Somigliana dislocation solution [17] with the source point outside the domain, too. The basic properties of these functions are similar as the previous two types, however, each of them has some advantages for particular purposes.

5. Special functions corresponding to the response of finite or infinite elastic continuum with holes, domains of special forms etc. to some special kinds of loads.

6. Also continuous distribution of Kelvin, Boussinesq–Cerruti and dislocation source functions as well as many other point and distributed source type functions can be included to this type of T-functions.

The polynomial and source type T-functions can be found in [1, 4, 8, 11] and some of them are given in Section 2.

Three kinds of basic displacement models (FEM, BEM and MM) are shown here and their MD presentation requires a weak (integral) representation of equilibrium in order to keep convergence. Then all three presentations are very similar and one can find some similar features in all of them.

First basic hybrid Trefftz displacement (HTD) formulations as defined by Jirousek [6] are presented. In these formulations two independent fields of displacements are used. The first one approximates the T-displacement field inside the element and has to satisfy the governing equations inside the element and the second field is defined on the element boundary only.

The second variant of the HTD FEM [6] uses the second (boundary) displacement field on the inter-element boundaries, only.

In the reciprocity based FEM [10] (or multi-domain BEM) formulation T-displacements and T-tractions are used as reciprocal states of the same body and are related to the known and unknown boundary conditions in the way as it is known from BEM [5]. For MD formulation the displacements are considered to be continuous on the sub-domain (element) boundaries and tractions have to satisfy the weak inter-element equilibrium.

The boundary meshless method (BMM) [11] is a point collocation method. A well known single domain BMM is the Method of Fundamental Solutions (MFS) [14], in which Kelvin type T-functions with source points outside the domain are used as interpolators of boundary conditions. No integration and no meshes are required. More general T-functions can be used in BMM in order to improve accuracy and numerical stability of the system of equations in the BPM. For more complicated problems and/or inhomogeneous materials a MD formulation can be used. In order to obtain a convergent solution for general problem, the weak form of equilibrium is necessary also for a homogeneous structure and the formulation is not totally meshless and requires some integration, too.

The singular value decomposition (SVD) [16] plays an important role in the formulation as matrices with more columns than rows, which arise in some cases of the MD formulations, are to be inverted in order to obtain more accurate solution. The SVD minimizes the solution vector.

All methods using T-elements enable to define large elements with more complicated form than classical FEM formulations and they are open to further development. The elements can be even multiply connected and can be simply extended to special use, e.g. they can contain special functions, which better approximate the field inside the element. This is similar to the technique used in extended finite elements (EXFEM).

The MD formulations of all three kinds of T-elements mentioned above are compatible with displacement FEM and can be combined with them.

2. Trefftz functions for linear elasticity

First we will show obtaining **T-polynomials** for a 3D isotropic elastic solid. The displacement field, which describes its behavior under static load condition have to satisfy the homogeneous equilibrium equations expressed in displacements, known as Lame–Navier equations,

$$(\lambda + \mu)\, u_{j,ij} + \mu\, u_{i,jj} = 0. \tag{1}$$

The components of the displacement field are expressed as

$$\left\{\begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array}\right\} = \left[\begin{array}{ccc} \mathbf{P(x)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P(x)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{P(x)} \end{array}\right] \left\{\begin{array}{c} \mathbf{C}^{(1)} \\ \mathbf{C}^{(2)} \\ \mathbf{C}^{(3)} \end{array}\right\} \tag{2}$$

where $\mathbf{P(x)}$ is the full polynomial of the n-th order,

$$\mathbf{P(x)} = \left[1\,,\, x_1\,,\, x_2\,,\, x_3\,, \dots\,,\, x_1^n\,,\, x_1^{n-1} x_2\,, \dots\,, x_2 x_3^{n-1}\,,\, x_3^n\right] \tag{3}$$

and $\mathbf{C}^{(j)}$ is the vector of unknown coefficients.

The equilibrium equation (1) contains the second derivatives of the displacements. Thus, the terms of the 0-th and 1-st order satisfy this equation automatically. However, the higher order polynomials cannot be chosen arbitrarily in order to satisfy the equilibrium equation (1). Equation (2) is split into the form

$$\left\{\begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array}\right\} = \left[\begin{array}{ccc} \mathbf{A}^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}^{(3)} \end{array}\right] \left\{\begin{array}{c} \mathbf{a}^{(1)} \\ \mathbf{a}^{(2)} \\ \mathbf{a}^{(3)} \end{array}\right\} + \left[\begin{array}{ccc} \mathbf{B}^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^{(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}^{(3)} \end{array}\right] \left\{\begin{array}{c} \mathbf{b}^{(1)} \\ \mathbf{b}^{(2)} \\ \mathbf{b}^{(3)} \end{array}\right\} \tag{4}$$

where the matrix $\mathbf{B}$ contains as many terms as many terms has the two order lower polynomial. The terms of each order of the polynomial are computed separately and they are functions of material properties only. As example the third order polynomial terms

$$\begin{array}{ccccccc} & & & x_1^3 & & & \\ & & x_1^2 x_2 & & x_1^2 x_3 & & \\ \hline & x_1 x_2^2 & & x_1 x_2 x_3 & & x_1 x_3^2 & \\ x_2^3 & & x_2^2 x_3 & & x_2 x_3^2 & & x_3^3 \end{array} \tag{5}$$

can be split so that the upper three terms will be included into $\mathbf{B}^{(1)}$ and the lower part terms into $\mathbf{A}^{(1)}$. The terms for $\mathbf{B}^{(j)}$ and $\mathbf{A}^{(j)}$ are obtained by changing cyclically the component indices and set into Eq. (1). In this way we can obtain the relation

$$[\mathbf{M}(x_i)]\{\mathbf{b}\} + [\mathbf{N}(x_i)]\{\mathbf{a}\} = \{\mathbf{0}\}. \tag{6}$$

Now the vector $\mathbf{b}$ contains dependent coefficients, which have to be expressed through the independent coefficients $\mathbf{a}$ in order to satisfy the equilibrium equations in each point.

The solution of this problem can be done symbolically or numerically. In the last case, we choose so many or more discrete points, as many dependent terms are in the corresponding order of the polynomial (in the given example of the third order polynomial there are at least three points). Note that if more points are chosen than is the number of the dependent terms, the system of equations will contain more equations as the number of unknown terms, but it will be not solved in the least square sense, as the redundant equations are a combination of the others.

We get Eq. (6) in the form

$$\left[\mathbf{M}\left({}^{(j)}x_i\right)\right]\{\mathbf{b}\} = -\left[\mathbf{N}\left({}^{(j)}x_i\right)\right]\{\mathbf{a}\} \tag{7}$$

from which we have

$$\{\mathbf{b}\} = -\left[\mathbf{M}^{-1}\right][\mathbf{N}]\{\mathbf{a}\}. \tag{8}$$

The upper left index corresponds to the nodal point for determination of coefficients of the T-polynomial displacement. With this, the T-polynomial displacement is

$$\{\mathbf{u}\} = \left([\mathbf{A}(x_i)] - [\mathbf{B}(x_i)]\left[\mathbf{M}^{-1}\right][\mathbf{N}]\right)\{\mathbf{a}\} = [\mathbf{U}(x_i)]\{\mathbf{a}\}. \tag{9}$$

Each column of the matrix **U** in Eq. (9) introduces a T-displacement function for each component defined by corresponding row. The T-polynomial contains then one independent term of the polynomial and the dependent terms, too. Note that in this way the matrix **U** contains polynomial terms of both matrices **A** and **B**.

The points for numerical evaluation of Eq. (4) have to be conveniently spaced so that the matrix **M** will not be singular or bad conditioned. We can conveniently choose the points lying on a quadrant of a sphere in order to overwhelm such problem. Note that the number of T-polynomial functions, which can be defined is $(2n+1)$ for 2D and $(n+1)^2$ for 3D problems, with n being the polynomial order (containing all the orders from 0 to n).

The **Kelvin functions** (fundamental solution for elasticity problems) are the basis for BEM and are well known from extensive BEM literature (see also nice paper on its history [4]). Displacement and boundary traction components in the point $\mathbf{x}$ corresponding to the unit point force acting in the point $\mathbf{y}$ in direction of j-axis are

$$U_{ij}(x,y) = -\frac{1}{16\pi(1-\nu)\mu r}\left[(3-4\nu)\,\delta_{ij} + r_{,i}r_{,j}\right] \tag{10}$$

and

$$T_{ij}(x,y) = -\frac{1}{8\pi(1-\nu)r^2}\left\{\left[(1-2\nu)\,\delta_{ij} + 3r_{,i}r_{,j}\right]\frac{\partial r}{\partial n} - (1-2\nu)\,(r_{,i}n_{,j} - r_{,j}n_{,i})\right\}, \tag{11}$$

respectively, for 3D problems. The first index, i, corresponds to the displacement, or traction component.

Similarly, for the plane strain problems,

$$U_{ij}(x,y) = -\frac{1}{8\pi(1-\nu)\mu}\left[(3-4\nu)\,\delta_{ij}\ln(r) - r_{,i}r_{,j}\right] \tag{12}$$

and

$$T_{ij}(x,y) = -\frac{1}{4\pi(1-\nu)r}\left\{\left[(1-2\nu)\,\delta_{ij} + 2r_{,i}r_{,j}\right]\frac{\partial r}{\partial n} - (1-2\nu)\,(r_{,i}n_{,j} - r_{,j}n_{,i})\right\} \tag{13}$$

with r being the distance between the points $\mathbf{x}$ and $\mathbf{y}$ and

$$r_{,i} = \frac{\partial r}{\partial x_i(y)} = \frac{r_i}{r} \qquad \text{with} \qquad r_i = x_i(y) - x_i(x) \tag{14}$$

and

$$\frac{\partial r}{\partial n} = \frac{r_i n_i}{r} = r_{,n}\,, \tag{15}$$

$\mathbf{n}$ is a unit vector in the outer normal direction to the surface S.

T-functions can be introduced by single forces acting in the continuum outside the domain of interest (d.o.i.), or they can be modeled by distributed field of tractions (forces). If they are presented by distributed forces $\mathbf{p}$ along some surface S (defined outside the d.o.i.), the resulting field of T-displacements is given by

$$u_j = \int_S U_{ji}p_i\,\mathrm{d}S. \tag{16}$$

In this case $\mathbf{p}$ denotes stress discontinuity along the surface S.

Similarly, the displacement field corresponding to a point dislocation [4, 17] in an unbounded 3D space

$$D_{ijk} = \frac{1}{8\pi(1-\nu)r^2}\left[(1-2\nu)(\delta_{kj}r_{,i} - \delta_{ij}r_{,k} - \delta_{ik}r_{,j}) - 3r_{,i}r_{,j}r_{,k}\right] \tag{17}$$

can represent a T-displacement. Corresponding distribution of dislocations with the density vector **d** along a surface S defines the displacement field

$$u_k = \int_S D_{kji} n_j d_i \, dS. \tag{18}$$

The density of dislocations is also known as the displacement discontinuity (along S).

Special T-functions are all solutions on some domain which satisfy the homogeneous equilibrium inside the domain. Many analytical solutions are known from the theory of elasticity. Examples can be the infinite 2D or 3D domain with a hole, loaded by pressure at the hole boundary, rigid inclusions in an elastic continuum, continuum with a crack inside, or on the domain boundary, etc. [1, 8]. If such functions enable to satisfy the boundary conditions not only at a point of the domain boundary, but also in some vicinity of the point, they can increase accuracy of the solution, and elements with holes, cracks, or some special forms can be defined as large elements without necessity to use very fine meshes in such parts of the structure.

Very convenient Trefftz-type functions are **Boussinesq and Cerruti functions** [2, 3] which give the displacements and stresses (and also corresponding tractions) for a half space or a half plane subjected to the unit force acting on their surface.

The displacements and tractions for a 3D problem of a concentrated normal force, P_3, acting at the origin on the half space, $x_3 \geq 0$, are given by

$$\begin{aligned} u_\alpha &= \frac{P_3}{4\pi\mu}\left[\frac{x_\alpha x_3}{r^3} - (1-2\nu)\frac{x_\alpha}{r(r+x_3)}\right], \qquad \text{with} \quad \alpha = 1,2, \\ u_3 &= \frac{P_3}{4\pi\mu}\left[\frac{x_3^2}{r^3} + \frac{2(1-\nu)}{r}\right], \end{aligned} \tag{19}$$

$$\begin{aligned} \sigma_{\alpha\beta} &= \frac{P_3}{2\pi}\left\{\frac{3x_\alpha x_\beta x_3}{r^5} - (1-2\nu)\left[\frac{x_3}{r^3}\delta_{\alpha\beta} - \frac{\delta_{\alpha\beta}}{r(r+x_3)} + \frac{x_\alpha x_\beta}{r^3}\frac{(2r+x_3)}{(r+x_3)^2}\right]\right\}, \\ \sigma_{3\alpha} &= -\frac{P_3}{2\pi}\frac{3x_\alpha x_3^2}{r^5}, \qquad \text{with} \quad \alpha,\beta = 1,2, \\ \sigma_{33} &= -\frac{P_3}{2\pi}\frac{3x_3^3}{r^5}. \end{aligned} \tag{20}$$

Similarly for the case of forces acting in tangential directions we have

$$\begin{aligned} u_\alpha &= \frac{P_\gamma}{2\pi\mu}\left\{\frac{\delta_{\alpha\gamma}}{r} + \frac{x_\alpha x_\gamma}{r^3} + (1-2\nu)\left[\frac{\delta_{\alpha\gamma}}{(r+x_3)^2} - \frac{x_\alpha x_\gamma}{r(r+x_3)^2}\right]\right\}, \qquad \text{with} \quad \alpha,\gamma = 1,2, \\ u_3 &= \frac{P_\gamma}{4\pi\mu}\left[\frac{x_\gamma x_3}{r^3} + (1-2\nu)\frac{x_\gamma}{r(r+x_3)}\right], \end{aligned} \tag{21}$$

$$\begin{aligned} \sigma_{\alpha\beta} &= -\frac{P_\gamma}{2\pi}\left\{\frac{3x_\alpha x_\beta x_\gamma}{r^5} - \frac{(1-2\nu)}{r(r+x_3)^2}\left[\frac{(3r+x_3)x_\alpha x_\beta x_\gamma}{r^2(r+x_3)} + \frac{(2r+x_3)}{r^2}x_\gamma x_3\delta_{\alpha\beta} - (x_j\delta_{i\gamma} + x_i\delta_{j\gamma})\right]\right\}, \\ \sigma_{3\alpha} &= -\frac{P_\gamma}{2\pi}\frac{3x_\alpha x_3 x_\gamma}{r^5}, \qquad \text{with} \quad \alpha,\beta,\gamma = 1,2, \\ \sigma_{33} &= -\frac{P_\gamma}{2\pi}\frac{3x_\gamma x_3^2}{r^5}. \end{aligned} \tag{22}$$

All source type of T-functions can be chosen with the source points on the domain boundaries, or outside the domain. If the source points is on the domain boundary and a discrete T-function is used in the solution, then all displacement and stresses are singular at such a point and the boundary conditions can simulate finite solution only in the sense of the Saint-Venant principle. However, distributed source functions can simulate very accurately even the boundary conditions with discontinuities, as it is in contact problems [11]. On the other side, using the T-source functions with the source point outside the domain simplifies the formulation. If a collocation of boundary conditions is used, the formulation of models can be very simple (such as MFS), however, if the source points are closer to the boundary then more collocation points are necessary for obtaining required accuracy. If the points are too far from the boundary the resulting system of equations can be ill-conditioned, which also results in the lost of accuracy.

3. HTD FEM

The basic formulation as defined by Jirousek [6] is obtained by first enforcing in a weak sense conformity of internal displacements, $\mathbf{u}$, and another independent field of displacements, $\tilde{\mathbf{u}}$, defined on the element boundaries, S_e, only

$$\int_{S_e} \delta\mathbf{t}(\mathbf{u}-\tilde{\mathbf{u}})\,\mathrm{d}S = 0, \tag{23}$$

where $\mathbf{t}$ are tractions corresponding to the displacements, $\mathbf{u}$, and then, using a weak equilibrium between tractions $\mathbf{t}$ and tractions $\bar{\mathbf{t}}$, prescribed on the domain boundaries as

$$\sum_e \int_{S_u+S_i} \delta\tilde{\mathbf{u}}\,\mathbf{t}\,\mathrm{d}S - \sum_e \int_{S_t} \delta\tilde{\mathbf{u}}\,(\mathbf{t}-\bar{\mathbf{t}})\,\mathrm{d}S = \sum_e \int_{S_e} \delta\tilde{\mathbf{u}}\,\mathbf{t}\,\mathrm{d}S - \sum_e \int_{S_t} \delta\tilde{\mathbf{u}}\,\bar{\mathbf{t}}\,\mathrm{d}S = 0. \tag{24}$$

The lower indices e, i, u and t belong to element, inter-element boundary and to boundary with prescribed displacement and tractions, respectively.

After discretisation of Eqs. (23) and (24) by replacing the boundary displacements, $\tilde{\mathbf{u}}$, by their nodal values and corresponding shape functions, $\mathbf{N}$, along the element boundaries and approximating the internal displacement and tractions by corresponding T-functions $\mathbf{U}$ and $\mathbf{T}$,

$$\begin{aligned} \mathbf{u} &= \mathbf{Uc}, \\ \mathbf{t} &= \mathbf{Tc}, \end{aligned} \tag{25}$$

where $\mathbf{c}$ denotes the vector of intensities of T-functions, we obtain the following system of equations (for more details see [6, 7])

$$\sum_e \mathbf{K}_e\mathbf{u}_e = \mathbf{b} \tag{26}$$

where

$$\begin{aligned} \mathbf{K}_e &= \mathbf{G}^T\mathbf{H}^{-1}\mathbf{G}, \\ \mathbf{H} &= \int_{S_e} \mathbf{T}^T\mathbf{U}\,\mathrm{d}S, \\ \mathbf{G} &= \int_{S_e} \mathbf{T}^T\mathbf{N}\,\mathrm{d}S. \end{aligned} \tag{27}$$

The right side vector $\mathbf{b}$ represents the static equivalent nodal loads defined exactly so, as it is well known from isoparametric FEM.

The second variant of the HTD FEM uses the second displacement field on the inter-element boundaries, only (see [6, 7] for more details). Compatibility of internal tractions with prescribed boundary tractions and compatibility of internal displacements with the displacements on inter-element boundaries and on the boundaries with prescribed displacements, S_u have to be satisfied as

$$\int_{S_t} \delta\mathbf{u}(\mathbf{t}-\bar{\mathbf{t}})\,\mathrm{d}S + \int_{S_u} \delta\mathbf{t}(\mathbf{u}-\bar{\mathbf{u}})\,\mathrm{d}S + \int_{S_i} \delta\mathbf{t}(\mathbf{u}-\tilde{\mathbf{u}})\,\mathrm{d}S = 0. \tag{28}$$

The equilibrium is expressed by compatibility of tractions with prescribed tractions on the domain boundaries (i.e. by the first term in Eq. (28)) and by additional inter-element weak form

$$\int_{S_i} \delta\tilde{\mathbf{u}}\,\mathbf{t}\,\mathrm{d}S = 0. \tag{29}$$

Following the same procedure as in the previous case, the stiffness matrix $\mathbf{K}_e$ in the resulting system of equations (26) and the matrix $\mathbf{G}$ are defined similarly as in Eq. (27), but

$$\mathbf{H} = \int_{S_u+S_i} \mathbf{T}^T\mathbf{U}\,\mathrm{d}S - \int_{S_t} \mathbf{U}^T\mathbf{T}\,\mathrm{d}S \tag{30}$$

and the right side vector is

$$\begin{aligned} \mathbf{b} &= \mathbf{G}^T\mathbf{H}^{-1}\mathbf{b}^*, \\ \mathbf{b}^* &= -\int_{S_u} \mathbf{T}^T\bar{\mathbf{u}}\,\mathrm{d}S + \int_{S_t} \mathbf{U}^T\bar{\mathbf{t}}\,\mathrm{d}S. \end{aligned} \tag{31}$$

Comparing both variants one can see that the second variant reduces the resulting system of equations and contains the nodal points on the inter-element boundaries, only and also the right side is modified according to this reduction.

4. Reciprocity based (RB) FEM

T-displacements, $\mathbf{U}$, and T-tractions, $\mathbf{T}$, are used as reciprocal states of the same body and are related to the known and unknown boundary conditions in tractions, $\mathbf{t}$, and displacements, $\mathbf{u}$ (as it is known from BEM; the body forces are omitted for simplicity), by

$$\int_{S_e} \mathbf{T}\mathbf{u}\,\mathrm{d}S = \int_{S_e} \mathbf{U}\mathbf{t}\,\mathrm{d}S. \tag{32}$$

In this case we consider non-singular T-functions on the element boundary. For MD formulation (FEM) the displacements are considered to be continuous on the sub-domain (element) boundaries and tractions have to satisfy the weak inter-element equilibrium

$$\sum_e \int_{S_i} \delta\mathbf{u}\,\mathbf{t}\,\mathrm{d}S = 0. \tag{33}$$

After discretisation, Eqs. (32) and (33) can be written in the sub-matrix form as

$$\begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} & -\mathbf{U}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} & -\mathbf{U}_{22} \\ \mathbf{0} & \mathbf{0} & \sum_e \mathbf{M} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_t \\ \mathbf{u}_i \\ \mathbf{t}_i \end{Bmatrix} = \begin{Bmatrix} \mathbf{U}_{11}\bar{\mathbf{t}} \\ \mathbf{U}_{21}\bar{\mathbf{t}} \\ \mathbf{0} \end{Bmatrix} = \begin{Bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{0} \end{Bmatrix} \tag{34}$$

with unknown displacements $\mathbf{u}_t$ at nodal points on boundaries with prescribed tractions, and displacements $\mathbf{u}_i$ and tractions $\mathbf{t}_i$ on the inter-element boundaries. The matrix terms are integral presentation of corresponding Eqs. (32) and (33). The first row corresponds to the points in the domain boundaries and the second to the inter-domain boundaries. All displacements on the domain boundaries and tractions on the inter-element boundaries can be eliminated on the element level and resulting system of equations is obtained in the form

$$\sum_e \mathbf{M}\mathbf{U}_{22}^{*-1}\mathbf{T}_{22}^{*}\mathbf{u}_i = \sum_e \mathbf{M}\mathbf{U}_{22}^{-1}\mathbf{p}_2^{*} \tag{35}$$

with

$$\begin{aligned} \mathbf{T}_{22}^* &= \mathbf{T}_{22} - \mathbf{T}_{21}\mathbf{T}_{11}^{-1}\mathbf{T}_{12}\,, \\ \mathbf{U}_{22}^* &= \mathbf{U}_{22} - \mathbf{T}_{21}\mathbf{T}_{11}^{-1}\mathbf{U}_{12}\,, \\ \mathbf{p}_2^* &= \mathbf{p}_2 - \mathbf{T}_{21}\mathbf{T}_{11}^{-1}\mathbf{p}_1\,. \end{aligned} \tag{36}$$

Equation (35) can be written as

$$\mathbf{K}\mathbf{u}_i = \mathbf{b}_i \tag{37}$$

where $\mathbf{K}$ is stiffness matrix obtained from the matrices of elements, $\mathbf{u}$ is the vector of nodal displacements on inter-element boundaries, and $\mathbf{b}$ is the vector of equivalent nodal loads.

5. BMM/BPM and its MD formulation

A well known BMM is the Method of Fundamental Solutions (MFS). The fundamental solutions with source points outside the domain are used as interpolators of boundary conditions as

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{U} \end{bmatrix} \{\mathbf{c}\} = \begin{Bmatrix} \bar{\mathbf{t}} \\ \bar{\mathbf{u}} \end{Bmatrix}, \tag{38}$$

with $\mathbf{T}, \mathbf{U}$ and $\mathbf{c}$ T-tractions, T-displacements by fundamental solutions (10)–(11) and intensities of the forces acting in infinite continuum outside of the investigated domain, respectively. $\bar{\mathbf{t}}$ and $\bar{\mathbf{u}}$ are prescribed boundary conditions (tractions and displacements) at collocation points. No integration and no meshes are required. More general T-functions can be used in order to improve accuracy and numerical stability of the system of equations in more general point collocation method, the Boundary Point Method (BPM).

For more complicated problems and/or inhomogeneous materials the MD formulation can be used. In order to obtain a convergent solution for general problem, the weak form of equilibrium (24) is necessary also for a single domain formulation and the formulation is not totally meshless and requires some integration, too.

The following system of equations is obtained,

$$\begin{bmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} & -\mathbf{I}_u & \mathbf{0} \\ \mathbf{T}_{21} & \mathbf{T}_{22} & \mathbf{0} & -\mathbf{I}_t \\ \mathbf{T}_{31} & \mathbf{T}_{32} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \sum_e \mathbf{M} \end{bmatrix} \begin{Bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{u}_i \\ \mathbf{t}_i \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \bar{\mathbf{t}} \\ \mathbf{0} \end{Bmatrix} \tag{39}$$

The matrices $\mathbf{I}$ are identity matrices and the index denotes the dimension corresponding to the vectors of nodal displacements and tractions and $\mathbf{c}_i$ are sub-vectors of the vector of unknown coefficients of T-functions chosen so that the inverse matrices in Eq. (43) below will not be underdetermined.

The first row of this sub-matrix form equation expresses the intensities of the T-functions by the displacements at discrete points of the whole sub-domain (at both the domain and at the inter-domain boundaries with common points at the neighbor sub-domains). The second row expresses the unknown intensities by tractions at discrete points of the inter-domain boundaries, only. The third row gives the intensities by prescribed boundary conditions at the discrete points of the domain boundaries. Only prescribed tractions are denoted for simplicity, but the prescribed displacements can be included to this row, too. The fourth row of Eq. (39) has to be evaluated in the integral form

$$\int_{S_i} (\mathbf{t}^A - \mathbf{t}^B)\,\mathrm{d}S = \mathbf{0}, \tag{40}$$

which is numerically computed as

$$\sum_e \int_{S_i} \mathbf{t}\,\mathrm{d}S = \sum_e \sum_i t_i w_i = 0, \tag{41}$$

if the nodal displacements are the Gaussian integration points on the inter-element boundaries. The upper indices A and B correspond to neighbor elements. The nodes on the inter-domain boundaries have to be Gauss integration points. The other way is to write

$$\sum_i \int_{S_i} \delta \mathbf{u}\, \mathbf{t}\, \mathrm{d}S = 0. \tag{42}$$

Instead of having tractions in the Gauss integration points at the inter-domain boundaries, there are displacements at nodes on the inter-domain boundaries and the displacements $\mathbf{u}$ are expressed by their nodal values and by corresponding shape functions, $\mathbf{N}$, so as it is known from FEM and BEM formulations.

Eliminating all unknowns except the displacements in the nodal points on the inter-domain boundaries at the sub-domain level, the following system of equations is obtained,

$$\sum_e \mathbf{M}\left(\mathbf{T}_{21} + \mathbf{T}_{21}\mathbf{U}_{11}^{-1}\mathbf{U}_{12}\mathbf{T}_{32}^{*-1}\mathbf{T}_{31} - \mathbf{T}_{22}\mathbf{T}_{32}^{*-1}\mathbf{T}_{31}\right)\mathbf{U}_{11}^{-1}\mathbf{u}_i$$

$$= \sum_e \mathbf{M}\left(\mathbf{T}_{21}\mathbf{U}_{11}^{-1}\mathbf{U}_{12}\mathbf{T}_{32}^{-1} - \mathbf{T}_{22}\mathbf{T}_{32}^{-1}\right)\bar{\mathbf{t}} \tag{43}$$

or

$$\mathbf{K}\mathbf{u}_i = \mathbf{b}_i \tag{44}$$

where

$$\mathbf{T}_{32}^{*} = \mathbf{T}_{32} - \mathbf{T}_{31}\mathbf{U}_{11}^{-1}\mathbf{U}_{12} \tag{45}$$

and the unknown T-intensities and tractions are given by

$$\begin{aligned}
\mathbf{c}_1 &= \left(\mathbf{U}_{11}^{-1} + \mathbf{U}_{11}^{-1}\mathbf{U}_{12}\mathbf{T}_{32}^{*-1}\mathbf{T}_{31}\mathbf{U}_{11}^{-1}\right)\mathbf{u}_i - \mathbf{U}_{11}^{-1}\mathbf{U}_{12}\mathbf{T}_{32}^{*-1}\bar{\mathbf{t}},\\
\mathbf{c}_2 &= \mathbf{T}_{32}^{*-1}\left(\bar{\mathbf{t}} - \mathbf{T}_{31}\mathbf{U}_{11}^{-1}\mathbf{u}_i\right),\\
\mathbf{t}_i &= \mathbf{T}_{21}\mathbf{c}_1 + \mathbf{T}_{22}\mathbf{c}_2.
\end{aligned} \tag{46}$$

The singular value decomposition plays an important role in the formulation when any inversion matrix in then expressions above has more columns than rows. It enables to increase the accuracy and has similar effect as the higher order integration schemes in the weak forms.

6. Conclusions and remarks

Use of Trefftz functions for hybrid FEM, reciprocity based FEM and meshless method has been shown for linear elastic problems. The Trefftz functions enable to obtain stiffness matrices by integration over the element boundaries only. The elements can be large also for complicated problem, they can be even multiply connected. The Trefftz functions can be conveniently chosen so that the integrals are non-singular and neither special computational models, nor high order quadratures are necessary.

Special T-functions enable to increase the computational efficiency. Functions which model local effects (holes, cracks) in infinite continuum are very convenient also for finite domains as the interpolation or approximation functions. For example the Bussinesq–Cerruti functions can be used to model local loads, or local contact; they can improve the local approximation of displacements and tractions near crack tips, or corners. The asymptotic solutions of the infinite problems are good candidates for this purpose.

The meshless methods using T-functions are very simple to obtain and are fully meshless and integration-free, but problems with stability require using a multidomain formulation with a weak inter-domain equilibrium definition. The formulation is then not fully meshless and boundary elements and numerical integration are necessary on the inter-domain boundaries and the meshless form is kept on the domain boundaries only.

All, the second variant HTD FEM, RB FEM and MD BMM, lead to similar structure of resulting matrix defined by boundary elements on the inter-domain boundaries, while the domain boundaries and boundary conditions are treated in different ways.

Only basic displacement formulations of T-elements were presented here. Many other possibilities were shown in the review on T-elements [7].

Important application of T-functions is in the post-processing phase by computing stresses from displacements obtained e.g. by classical FEM. They enable increase the accuracy of stresses and to obtain smoothed stresses over the mesh of elements [15].

The application of T-functions to non-linear problems [12] is also an interesting field and can contribute to increased rate of convergence. As the equilibrium equations expressed in Cauchy stress and deformed coordinates are same for small and large displacements, T-stress polynomials are useful in post-processing from stresses in Gaussian integration points and boundary tractions, which is one of challenges for the next research.

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