Computer Assisted Mechanics and Engineering Sciences, 13: 209–233, 2006.

A survey of various FE models with conceptual diagrams for linear analysis

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(Received October 1, 2004)

In the paper a general survey of existing finite element (FE) models is presented using a conceptual description in diagram form, which was initiated in paper [1]. The analysis is focused on the description of FE models, which is uniform in its concept but specific for each FE type. All FE models are associated with certain variational principles (and their stationarity conditions) in their original or modified versions. The diagrams are used to visualize the equations which are satisfied inside an individual FE and on interelement boundaries. The use of conceptual diagrams is very convenient in the presentation of finite element method (FEM), it simplifies the understanding and teaching of this method.

1. Introduction

In paper [1] a conceptual description of a set of equations of the boundary value problem (BVP) of elastic equilibrium was presented in diagram form. The structure of basic matrix transformations for the FEM was given, but only for displacement and some mixed FE formulations.

The present paper is a survey of variational principles which provide a basis for different types of FEs, in particular for compatible displacement, equilibrium, mixed (two or three fields), hybrid displacement, hybrid stress and mixed/hybrid models. The paper is devoted to a systematic presentation of energy functionals with appriopriate variational principles, for the derivation of which we must assume suitable equations to hold.

After the choice of FE approximation has been made all variational principles are transformed into FE formulation and in the next step matrix equations describing an individual finite element are presented.

The employment of conceptual diagrams is a very convenient metodology in presentation of FEM. For brevity the relations between all variables are presented in a matrix or differential operator form. On the other hand, the diagrams show expressively the similarities or differences between various types of FEs. The first section covers the local formulation of a boundary value problem of elasticity. The seven subsections in the second section describe the main types of FEs in a similar style. The specifications of the models are limited to corresponding equations and diagrams with only most important remarks about the structure of FE equations and their interpretation.

It is stressed that in the teaching and application of FEM the deep knowledge and understanding of different types of locally or globally formulated boundary value problems and their FEM solutions are very important.

1.1. Locally formulated boundary-value problem

The strong form of BVP is defined by the following set of differential and algebraic equations (ralated to domain Ω and two virtual subdomains Ω_1, Ω_2 – Fig. 1):

– constitutive equations (in stiffness and compliance form) (1), kinematic relations (2), equilibrium conditions (3):

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}, \qquad \boldsymbol{\varepsilon} = \mathbf{C}^{-1}\boldsymbol{\sigma}, \tag{1}$$

$$\boldsymbol{\varepsilon} = \partial \mathbf{u}, \tag{2}$$

$$\partial^{\mathrm{T}}\boldsymbol{\sigma} + \hat{\mathbf{p}} = \mathbf{0}, \tag{3}$$

for $\mathbf{x} \in \Omega$;

– geometric (kinematic) and static boundary conditions, given on parts of boundary $\partial\Omega_u$ and $\partial\Omega_\sigma$ with prescribed boundary displacements and/or loads:

$$\mathbf{u}_b = \mathbf{n}^u\mathbf{u} \quad \rightarrow \quad \check{\mathbf{u}} = \hat{\mathbf{u}}, \quad \text{for} \quad \mathbf{x} \in \partial\Omega_u, \tag{4}$$

$$\mathbf{t}_b = \mathbf{n}^\sigma\boldsymbol{\sigma}, \quad \rightarrow \quad \check{\mathbf{t}} = \hat{\mathbf{t}}, \quad \text{for} \quad \mathbf{x} \in \partial\Omega_\sigma; \tag{5}$$

in some cases the boundary conditions are related to selected or modified variables $\check{\mathbf{u}}$, $\check{\mathbf{t}}$, using matrices $\mathbf{R}_u$ and $\mathbf{R}_\sigma$ respectively:

$$\check{\mathbf{u}} = \mathbf{R}_u\mathbf{u}_b, \qquad \check{\mathbf{t}} = \mathbf{R}_t\mathbf{t}_b$$

and the sign "$\check{-}$" above a symbol denotes selected and/or modified components of vectors $\mathbf{u}_b$ and $\mathbf{t}_b$;

– continuity requirements (compatibility conditions) for displacements and equilibrium conditions for tractions along interdomain line $\partial\Omega_{(1,2)}$:

$$\mathbf{u}_b^{(1)} = \mathbf{n}^u\mathbf{u}^{(1)}, \quad \mathbf{u}_b^{(2)} = \mathbf{n}^u\mathbf{u}^{(2)} \quad \rightarrow \quad \check{\mathbf{u}}^{(1)} = \check{\mathbf{u}}^{(2)} \quad \text{for} \quad \mathbf{x} \in \partial\Omega_{(1,2)}, \tag{6}$$

$$\mathbf{t}_b^{(1)} = \mathbf{n}^\sigma\boldsymbol{\sigma}^{(1)}, \quad \mathbf{t}_b^{(2)} = \mathbf{n}^\sigma\boldsymbol{\sigma}^{(2)} \quad \rightarrow \quad \check{\mathbf{t}}^{(1)} + \check{\mathbf{t}}^{(2)} = \mathbf{0} \quad \text{for} \quad \mathbf{x} \in \partial\Omega_{(1,2)}. \tag{7}$$

The set of equations is presented in the first diagram (Fig. 1). Equations (1)–(3) must be satisfied at point P, boundary conditions at P_{ub}, $P_{\sigma b}$. The fields $\hat{\mathbf{p}}(\mathbf{x})$, $\boldsymbol{\sigma}(\mathbf{x})$, $\boldsymbol{\varepsilon}(\mathbf{x})$, $\mathbf{u}(\mathbf{x})$ are related to each other and equilibrium equations described by displacements

$$\partial^{\mathrm{T}}\mathbf{C}\partial\mathbf{u} + \hat{\mathbf{p}} = \mathbf{0} \tag{8}$$

can be introduced. In Fig. 1 continuity conditions (6) and (7) on line $\partial\Omega_{(1,2)}$ between two areas Ω_1 and Ω_2 are presented among other objects and relations.

In Fig. 1 and in the following diagrams we distinguish by different kinds of arrows: i) static boundary load or traction between subdomains are marked by arrows with full heads and ii) prescribed boundary displacements or displacements on the contact surface are marked by arrows with empty heads.

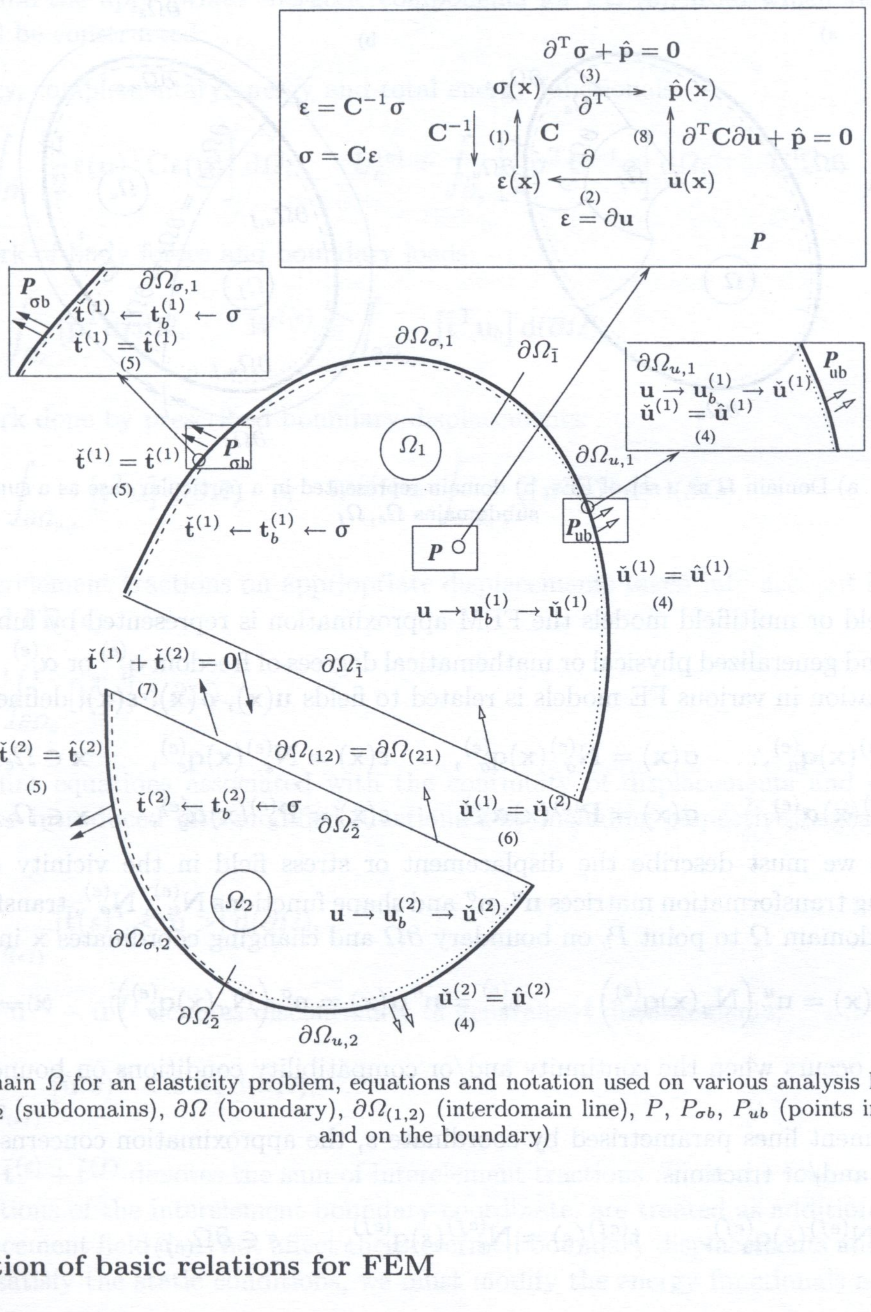

Fig. 1. Domain Ω for an elasticity problem, equations and notation used on various analysis levels: Ω (domain), Ω_1, Ω_2 (subdomains), $\partial\Omega$ (boundary), $\partial\Omega_{(1,2)}$ (interdomain line), P, $P_{\sigma b}$, P_{ub} (points in the domain and on the boundary)

1.2. Description of basic relations for FEM

In FEM we represent the problem domain Ω as a sum of a finite number of elements domains Ω_e (Fig. 2a). To abridge the paper the designation of main objects is presented in this subsection. In Fig. 2a the division of region Ω into the set of FEs Ω_e, with $e = 1, ..., E$, is shown

$$\Omega = \sum_{e=1}^{E} \Omega_e,$$

but in Fig. 2b we limit our interest to two FEs Ω_e and Ω_f. In general, a boundary of FE can be a sum of external and internal parts, i.e. a sum of external boundary parts with described loads and displacements and of interelement lines

$$\partial\Omega_e = \partial\Omega_{\bar{e}} \cup \partial\Omega_{\check{e}} = (\partial\Omega_{\sigma,e} \cup \partial\Omega_{u,e}) \cup \sum_{f=1}^{E(e)} \partial\Omega_{(ef)}.$$

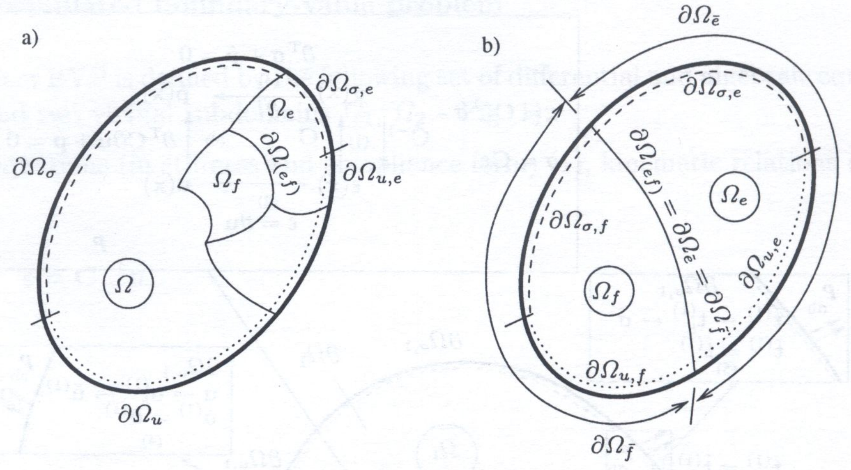

Fig. 2. a) Domain Ω as a set of FEs, b) domain represented in a particular case as a sum of two subdomains Ω_e, Ω_f

In single-field or multifield models the FEM approximation is represented by functions $\mathbf{N}_i^{(e)}(\mathbf{x})$ and $\mathbf{P}_i^{(e)}(\mathbf{x})$, and generalized physical or mathematical degrees of freedom $\mathbf{q}_i^{(e)}$ or $\boldsymbol{\alpha}_i^{(e)}$, for $i = u, \sigma, \varepsilon$. The approximation in various FE models is related to fields $\mathbf{u}(\mathbf{x})$, $\boldsymbol{\sigma}(\mathbf{x})$, $\boldsymbol{\varepsilon}(\mathbf{x})$, defined in Ω_e

$$\mathbf{u}(\mathbf{x}) = \mathbf{N}_u^{(e)}(\mathbf{x})\mathbf{q}_u^{(e)}, \qquad \boldsymbol{\sigma}(\mathbf{x}) = \mathbf{N}_\sigma^{(e)}(\mathbf{x})\mathbf{q}_\sigma^{(e)}, \qquad \boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{N}_\varepsilon^{(e)}(\mathbf{x})\mathbf{q}_\varepsilon^{(e)}, \qquad \mathbf{x} \in \Omega_e,$$

$$\mathbf{u}(\mathbf{x}) = \mathbf{P}_u^{(e)}(\mathbf{x})\boldsymbol{\alpha}_u^{(e)}, \qquad \boldsymbol{\sigma}(\mathbf{x}) = \mathbf{P}_\sigma^{(e)}(\mathbf{x})\boldsymbol{\alpha}_\sigma^{(e)}, \qquad \boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{P}_\varepsilon^{(e)}(\mathbf{x})\boldsymbol{\alpha}_\varepsilon^{(e)}, \qquad \mathbf{x} \in \Omega_e.$$

In some cases we must describe the displacement or stress field in the vicinity of the domain boundary, using transformation matrices $\mathbf{n}^u$, $\mathbf{n}^\sigma$ and shape functions $\mathbf{N}_u^{(e)}, \mathbf{N}_\sigma^{(e)}$, transferring suitable point P from domain Ω to point P_b on boundary $\partial\Omega$ and changing coordinates $\mathbf{x}$ into $s \in \partial\Omega$

$$\mathbf{u}_b^{(e)} = \mathbf{n}^u\mathbf{u}(\mathbf{x}) = \mathbf{n}^u\left(\mathbf{N}_u(\mathbf{x})\mathbf{q}_u^{(e)}\right), \qquad \mathbf{t}_b^{(e)} = \mathbf{n}^\sigma\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{n}^\sigma\left(\mathbf{N}_\sigma(\mathbf{x})\mathbf{q}_\sigma^{(e)}\right) \qquad \mathbf{x} \to s, \quad s \in \partial\Omega_e.$$

This situation occurs when the continuity and/or compatibility conditions on boundaries are postulated.

On interelement lines parametrised by coordinate s, the approximation concerns the boundary displacements and/or tractions:

$$\mathbf{u}^{(ef)}(s) = \mathbf{N}_u^{(ef)}(s)\mathbf{q}_u^{(ef)}, \qquad \mathbf{t}^{(ef)}(s) = \mathbf{N}_t^{(ef)}(s)\mathbf{q}_t^{(ef)}, \qquad s \in \partial\Omega_{(ef)}.$$

1.3. Energy functionals

Variational problems related to mechanics may be classified according to the following aspects:

- number of independently approximated fields which are subjected to variation,
- type of constraints imposed on particular functions,
- enforcement of boundary conditions in an explicit or implicit manner.

In the next section the well-known primary and modified energy principles are presented, assuming the sum of energy for a set of FEs:

$$I = \sum_{e=1}^{E} I^{(e)}.$$

Now, we describe the appropriate energetic components for FE (e), from which various types of functionals will be constructed:

- strain energy, complementary energy and total energy functionals:

$$U_p^{(e)} = \int_{\Omega_e} \left[\frac{1}{2}\boldsymbol{\varepsilon}(\mathbf{u})^{\mathrm{T}}\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u})\right] \mathrm{d}\Omega, \qquad U_c^{(e)} = \int_{\Omega_e} \left[\frac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{C}^{-1}\boldsymbol{\sigma}\right] \mathrm{d}\Omega, \qquad U^{(e)} = \int_{\Omega_e} \left[\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\sigma}\right] \mathrm{d}\Omega,$$

- external work of body forces and boundary loads:

$$W^{\hat{p}(e)} = \int_{\Omega_e} [\hat{\mathbf{p}}^{\mathrm{T}}\mathbf{u}] \, \mathrm{d}\Omega, \qquad W^{\hat{t}(e)} = \int_{\partial\Omega_{\sigma,e}} [\hat{\mathbf{t}}^{\mathrm{T}}\mathbf{u}_b] \, \mathrm{d}(\partial\Omega),$$

- external work done by prescribed boundary displacements:

$$W^{\hat{u}(e)} = \int_{\partial\Omega_{u,e}} [\mathbf{t}^{\mathrm{T}}\hat{\mathbf{u}}] \, \mathrm{d}(\partial\Omega) \quad \text{or} \quad W^{\hat{u}(e)} = \int_{\partial\Omega_{u,e}} [(\mathbf{n}^{\sigma}\boldsymbol{\sigma})^{\mathrm{T}}\hat{\mathbf{u}}] \, \mathrm{d}(\partial\Omega),$$

- work of interelement tractions on appriopriate displacements taken into account in the analysis of individual FE (e):

$$W^{t(\bar{e})} = \int_{\partial\Omega_{\bar{e}}} [\mathbf{t}^{\mathrm{T}}\mathbf{u}] \, \mathrm{d}(\partial\Omega).$$

The constraint equations associated with the continuity of displacements and equilibrium of tractions can be introduced into modified functionals by including respective Lagrange multiplier terms:

$$H^{(ef)} = \int_{\partial\Omega_{(ef)}} [\check{\mathbf{t}}(s)^{\mathrm{T}} < \check{\mathbf{u}} >] \, \mathrm{d}(\partial\Omega),$$

where $< \check{\mathbf{u}} > = \check{\mathbf{u}}^{(e)} - \check{\mathbf{u}}^{(f)}$ denotes discontinuity of generalized displacements,

$$G^{(ef)} = \int_{\partial\Omega_{(ef)}} [\check{\mathbf{u}}(s)^{\mathrm{T}} < \check{\mathbf{t}} >] \, \mathrm{d}(\partial\Omega),$$

where $< \check{\mathbf{t}} > = \check{\mathbf{t}}^{(e)} + \check{\mathbf{t}}^{(f)}$ denotes the sum of interelement tractions. Then the Lagrange multipliers, which are functions of the interelement boundary coordinate, are treated as additional variables.

If the displacement field does not affect the prescribed boundary displacements and/or the stress field does not satisfy the static conditions, we must modify the energy functionals adding suitable components:

$$H_b^{(e)} = \int_{\partial\Omega_{u,e}} \left[\check{\mathbf{t}}^{(e)\mathrm{T}}(\check{\mathbf{u}} - \hat{\mathbf{u}})\right] \mathrm{d}(\partial\Omega) \quad \text{and/or} \quad G_b^{(e)} = \int_{\partial\Omega_{\sigma,e}} \left[\check{\mathbf{u}}^{(e)\mathrm{T}}(\check{\mathbf{t}} - \hat{\mathbf{t}})\right] \mathrm{d}(\partial\Omega).$$

In the above equations the Lagrange functions are applied according to mechanical interpretation. They are treated as additional functions, approximated on the boundary, but in certain cases they can be inferred from the inside of FE on the grounds of domain approximation, using two following expressions:

$$H_b^{(e)} = \int_{\partial\Omega_{u,e}} \left[(\mathbf{n}^{\sigma}\boldsymbol{\sigma}^{(e)})^{\mathrm{T}}(\check{\mathbf{u}} - \hat{\mathbf{u}})\right] \mathrm{d}(\partial\Omega) \quad \text{and/or} \quad G_b^{(e)} = \int_{\partial\Omega_{\sigma,e}} \left[(\mathbf{n}^{\sigma}\mathbf{u}^{(e)})^{\mathrm{T}}(\check{\mathbf{t}} - \hat{\mathbf{t}})\right] \mathrm{d}(\partial\Omega).$$

Finally it is worth emphasizing that integration is performed over the FE domain Ω_e, on external boundaries $\partial\Omega_{\sigma,e}$, $\partial\Omega_{u,e}$ and along interelement lines $\partial\Omega_{(ef)}$.

1.4. Martices and vectors for various FE models

In order to obtain a more compact form of the description of FE models we will introduce the following abbreviations of designations:

- CD-FE – a Compatible Displacement FE model,
- ES-FE – an Equilibrium Stress FE model,
- DS-FE – a Displacement-Stress (mixed/two-field) FE model,
- DE-FE – a Displacement-strain(E) (mixed/two-field) FE model,
- DSE-FE – a Displacement-Stress-strain(E) (mixed/three-field) FE model,
- HD-FE – a Hybrid Displacement FE model,
- HS-FE – a Hybrid Stress FE model,
- HDSC-FE – a Hybrid (mixed) Displacement-Stress model with the satisfaction of displacement Continuity in a variational way on interelement lines,
- HDSE-FEM – a Hybrid (mixed) Displacement-Stress model with the assurance of Equilibrium of interelement tractions in a variational manner.

In the next sections we introduce many functionals as products of suitable FE matrices and vectors of unknown degrees of freedom for adequate fields. In integral expressions we find: matrices from physical and kinematic relations $\mathbf{C}$, ∂, respectively; shape function matrices $\mathbf{N}_i$, $\mathbf{P}_i$, $i = u, \sigma, \varepsilon$; matrices of boundary transformations $\mathbf{n}^u$, $\mathbf{n}^\sigma$ and prescribed loads $\hat{\mathbf{p}}$, $\hat{\mathbf{t}}$ and displacements $\hat{\mathbf{u}}$. The integration is carried out over domain Ω_e and lines $\partial\Omega_{\sigma,e}$, $\partial\Omega_{u,e}$, $\partial\Omega_{(ef)}$.

Now, we define the matrices used for single-field or multifield models:

$$\mathbf{K}_{uu}^{(e)} = \int_{\Omega_e} \left[(\partial\mathbf{N}_u^{(e)})^{\mathrm{T}} \mathbf{C} (\partial\mathbf{N}_u^{(e)}) \right] \mathrm{d}\Omega,$$

$$\mathbf{A}_{\sigma\sigma}^{(e)} = -\int_{\Omega_e} \left[\mathbf{N}_\sigma^{(e)\mathrm{T}} \mathbf{C}^{-1} \mathbf{N}_\sigma^{(e)} \right] \mathrm{d}\Omega,$$

$$\mathbf{F}_{\varepsilon\varepsilon}^{(e)} = \int_{\Omega_e} \left[\mathbf{N}_\varepsilon^{(e)\mathrm{T}} \mathbf{C} \mathbf{N}_\varepsilon^{(e)} \right] \mathrm{d}\Omega,$$

$$\mathbf{F}_{\alpha\alpha}^{(e)} = \int_{\Omega_e} \left[\mathbf{P}_\alpha^{(e)\mathrm{T}} \mathbf{C} \mathbf{P}_\alpha^{(e)} \right] \mathrm{d}\Omega,$$

$$\mathbf{G}_{\sigma u}^{(e)} = \int_{\Omega_e} \left[\mathbf{N}_\sigma^{(e)\mathrm{T}} (\partial\mathbf{N}_u^{(e)}) \right] \mathrm{d}\Omega,$$

$$\tilde{\mathbf{G}}_{\sigma u}^{(e)} = -\int_{\Omega_e} \left[(\partial^{\mathrm{T}}\mathbf{N}_\sigma^{(e)})^{\mathrm{T}} \mathbf{N}_u^{(e)} \right] \mathrm{d}\Omega,$$

$$\mathbf{E}_{\sigma\varepsilon}^{(e)} = -\int_{\Omega_e} \mathbf{N}_\sigma^{(e)\mathrm{T}} \mathbf{N}_\varepsilon^{(e)} \mathrm{d}\Omega,$$

$$\mathbf{R}_{\varepsilon u}^{(e)} = \int_{\Omega_e} \left[\mathbf{N}_\varepsilon^{(e)\mathrm{T}} \mathbf{C} (\partial\mathbf{N}_u^{(e)}) \right] \mathrm{d}\Omega,$$

$$\mathbf{R}_{\alpha u}^{(e)} = \int_{\Omega_e} \left[\mathbf{P}_\alpha^{(e)\mathrm{T}} \mathbf{C} (\partial\mathbf{N}_u^{(e)}) \right] \mathrm{d}\Omega.$$

To continue the definitions the following matrices for hybrid models are introduced:

$$\mathbf{L}_{ut}^{(e)} = \sum_{f=1}^{E_{(e)}} \int_{\partial\Omega_{(ef)}} \left[(\mathbf{n}^u \mathbf{N}_u^{(e)})^{\mathrm{T}} \mathbf{N}_t^{(ef)} \right] \mathrm{d}(\partial\Omega),$$

$$\mathbf{L}_{ut}^{(f)} = -\sum_{e=1}^{F_{(f)}} \int_{\partial\Omega_{(ef)}} \left[(\mathbf{n}^u \mathbf{N}_u^{(f)})^{\mathrm{T}} \mathbf{N}_t^{(ef)} \right] \mathrm{d}(\partial\Omega),$$

$$\mathbf{M}_{\sigma u}^{(e)} = \sum_{f=1}^{E_{(e)}} \int_{\partial\Omega_{(ef)}} \left[(\mathbf{n}^\sigma \mathbf{N}_\sigma^{(e)})^{\mathrm{T}} \mathbf{N}_u^{(ef)} \right] \mathrm{d}(\partial\Omega).$$

The following vectors will be used as the nodal representation of static and kinematic loads:

$$\mathbf{f}_u^{\hat{p}(e)} = \int_{\Omega_e} \left[\mathbf{N}_u^{(e)\mathrm{T}} \hat{\mathbf{p}} \right] \mathrm{d}\Omega,$$

$$\mathbf{f}_u^{\hat{t}(e)} = \int_{\partial\Omega_{\sigma,e}} \left[(\mathbf{n}^u \mathbf{N}_u^{(e)})^{\mathrm{T}} \hat{\mathbf{t}} \right] \mathrm{d}(\partial\Omega),$$

$$\mathbf{f}_\sigma^{\hat{u}(e)} = \int_{\partial\Omega_{u,e}} \left[(\mathbf{n}^\sigma \mathbf{N}_\sigma^{(e)})^{\mathrm{T}} \hat{\mathbf{u}} \right] \mathrm{d}(\partial\Omega).$$

In some cases we will analyse an isolated FE, involving an interelement nodal action vector $\mathbf{r}_u^{(\tilde{e})}$, which appears in the discrete work:

$$W^{t(\tilde{e})} = \mathbf{r}_u^{(\tilde{e})\mathrm{T}} \mathbf{q}_u^{(\tilde{e})}.$$

1.5. Concept of description of FEs in diagram form

The presentation of each FE model will contain a short description and a diagram. To represent a particular FE type the following issues will be pointed out in the same order:

I) appropriate variational principle with the specification of equations which are satisfied in the formulation *a priori* or are used to eliminate a particular field during the definition of a functional,

II) applied finite element approximation,

III) interpretation of matrix equations describing a certain FE model,

IV) conceptual diagram and its short description.

In each figure three levels are introduced: structure, FE and point. For a single-, two- and three-field models one isolated FE Ω_e is shown in the diagram. In hybrid models we exhibit two neighbouring elements Ω_e and Ω_f, and additionally we consider the boundary displacement and/or traction components over the respective sides of the common boundary $\partial\Omega_{(ef)}$.

At point P inside FE the relations between $\mathbf{u}$, $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$ fields are marked in special boxes (using numbers of equations introduced in Subsec. 1.1). From nodal vectors appriopriate arrows are plotted in the direction of approximated fields and of matrix equations defined for each FE model. The structure of FE equations results from the links of particular fields, represented by vectors of degrees of freedom of several types.

2. Formulation of equations for various FE models

2.1. Compatible displacement model – CD-FE

I) The compatible displacement model is derived from the principle of minimum potential energy, which depends only on displacement field $\mathbf{u}$ and loads $\hat{\mathbf{p}}$, $\hat{\mathbf{t}}$

$$I_p[\mathbf{u}] = \int_\Omega \left[\frac{1}{2}(\partial\mathbf{u})^{\mathrm{T}}\mathbf{C}(\partial\mathbf{u}) - \hat{\mathbf{p}}^{\mathrm{T}}\mathbf{u}\right] d\Omega - \int_{\partial\Omega_\sigma} [\hat{\mathbf{t}}^{\mathrm{T}}\mathbf{u}]\, d(\partial\Omega).$$

To construct this functional equation (1) and (2) as subsidiary ones are used.

II) In CD-FE displacements $\mathbf{u}$ occur as the only independent field and they are represented by interpolation functions $\mathbf{N}_u^{(e)}$ and generalized nodal displacements $\mathbf{q}_u^{(e)}$ for each element

$$\mathbf{u}(\mathbf{x}) = \mathbf{N}_u^{(e)}(\mathbf{x}) \cdot \mathbf{q}_u^{(e)}, \quad \mathbf{x} \in \Omega_e.$$

For the variational functional $I_p[\mathbf{u}]$ to be definable, the interpolation functions should satisfy: i) displacement continuity conditions in each element, ii) displacement conformability conditions on interelement boundaries, iii) kinematic compatibility on external boundary $\partial\Omega_{u,e}$.
For a set of FEs and for an individual element the potential energy functional can be written as

$$I_p[\mathbf{u}] = \sum_{e=1}^{E} I_p^{(e)}[\mathbf{u}] = \sum_{e=1}^{E} \left\{ \int_{\Omega_e} \left[\frac{1}{2}(\partial\mathbf{u})^{\mathrm{T}}\mathbf{C}(\partial\mathbf{u}) - \hat{\mathbf{p}}^{\mathrm{T}}\mathbf{u}\right] d\Omega - \int_{\partial\Omega_{\sigma,e}} [\hat{\mathbf{t}}^{\mathrm{T}}\mathbf{u}]\, d(\partial\Omega) \right\}.$$

After applying FEM, the potential energy is represented by generalized nodal displacements and suitable matrices and vectors, defined for CD-FE

$$I_p[\mathbf{q}_u] = \sum_{e=1}^{E} I_p^{(e)}[\mathbf{q}_u^{(e)}],$$

$$I_p^{(e)}[\mathbf{q}_u] = U_p^{(e)} - W^{\hat{p}(e)} - W^{\hat{t}(e)} - W^{t(\tilde{e})} = \frac{1}{2}\mathbf{q}_u^{(e)T}\mathbf{K}_{uu}^{(e)}\mathbf{q}_u^{(e)} - \mathbf{q}_u^{(e)T}\left(\mathbf{f}_u^{\hat{p}(e)} + \mathbf{f}_u^{\hat{t}(e)}\right) - \mathbf{q}_u^{(\tilde{e})T}\mathbf{r}_u^{(\tilde{e})}.$$

From the principle of minimum potential energy (stationarity conditions with respect to variations of $\mathbf{q}_u^{(e)}$) the following equations for individual CD-FE are derived

$$\delta I_p^{(e)} = \frac{\partial I_p^{(e)}}{\partial \mathbf{q}_u^{(e)}} \cdot \delta\mathbf{q}_u^{(e)} = 0 \quad\rightarrow\quad \frac{\partial I_p^{(e)}}{\partial \mathbf{q}_u^{(e)}} = 0 \quad\rightarrow\quad \mathbf{K}_{uu}^{(e)}\mathbf{q}_u^{(e)} = \mathbf{f}_u^{\hat{p}(e)} + \mathbf{f}_u^{\hat{t}(e)} + \mathbf{r}_u^{(\tilde{e})}.$$

III) The last equation above and

$$\mathbf{K}_{uu}\mathbf{Q}_u = \mathbf{F}_u^{\hat{p}} + \mathbf{F}_u^{\hat{t}} + \hat{\mathbf{P}}_u$$

can be interpreted as equilibrium equations of an individual FE and of an assembled structure. It is worth noticing that the vector of nodal interelement action $\mathbf{r}_u^{(\tilde{e})}$ disappears on the whole structure level. On the other hand, the vector of external nodal loads $\hat{\mathbf{P}}_u$ is taken into account for the whole structure.

IV) The diagram for CD-FE model is presented in Fig. 3. For every internal point in FE three fields $\mathbf{u}$, $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$ are connected by Eqs. (1),(2) in the grey box in Fig. 3. At nodal points only generalized displacements are introduced, which are unknowns in the final equations. In additional boxes the continuity displacement requirement along interelement lines and kinematic admissibility of the displacement field are pointed out.

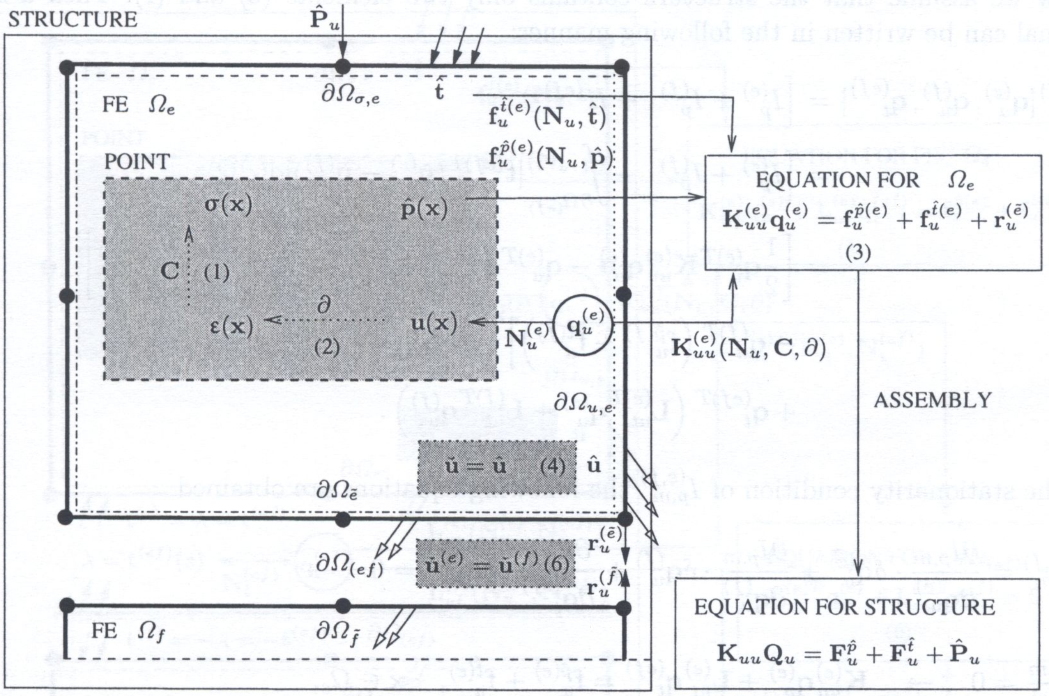

Fig. 3. Diagram for compatible displacement model – CD-FE

2.2. Hybrid displacement model – HD-FE

I) Relaxing the requirement of displacement continuity to be satisfied by trial functions along interelement boundaries

$$\check{\mathbf{u}}^{(e)}(s) = \check{\mathbf{u}}^{(f)}(s), \quad s \in \partial\Omega_{(ef)},$$

the hybrid displacement model can be formulated on the base of modified potential energy functional

$$I_{p,m}[\mathbf{u},\mathbf{t}] = I_p[\mathbf{u}] + H[\mathbf{u},\mathbf{t}].$$

II) The displacement field and boundary tractions are independently interpolated, using as unknowns nodal displacements $\mathbf{q}_u^{(e)}$ and forces $\mathbf{q}_t^{(ef)}$:

$$\mathbf{u}(\mathbf{x}) = \mathbf{N}_u^{(e)}(\mathbf{x}) \cdot \mathbf{q}_u^{(e)}, \quad \mathbf{x} \in \Omega_e,$$

$$\boldsymbol{\lambda}(s) = \mathbf{t}^{(ef)}(s) = \mathbf{N}_t^{(ef)}(s) \cdot \mathbf{q}_t^{(ef)}, \quad s \in \partial\Omega_{(ef)}, \quad \text{where} \quad \mathbf{t}^{(e)} = -\mathbf{t}^{(f)} = \mathbf{t}^{(ef)}.$$

Integration over interelement lines must be performed with care, according to the following formula:

$$\begin{aligned} I_{p,m}[\mathbf{u},\mathbf{t}] &= \sum_{e=1}^{E} I_p^{(e)}[\mathbf{u}^{(e)}] + \sum_{e=1}^{E} H^{(ef)}[\mathbf{u}^{(e)},\mathbf{t}^{(ef)}] \\ &= \sum_{e=1}^{E} I_p^{(e)}[\mathbf{u}^{(e)}] + \sum_{e=1}^{E} \left(\sum_{f=1}^{E(e)} \int_{\partial\Omega_{(ef)}} \left[\check{\mathbf{t}}^{(ef)T} \check{\mathbf{u}}^{(e)} \right] \mathrm{d}(\partial\Omega) \right) \\ &= \sum_{e=1}^{E} I_p^{(e)}[\mathbf{u}^{(e)}] + \sum_{e=1}^{E} \int_{\partial\Omega_{(ef)}} \left[\check{\mathbf{t}}^{(ef)T} \left(\check{\mathbf{u}}^{(e)} - \check{\mathbf{u}}^{(f)} \right) \right] \mathrm{d}(\partial\Omega). \end{aligned}$$

For now we assume that the structure contains only two elements (e) and (f). Then a suitable functional can be written in the following manner:

$$
\begin{aligned}
I_{p,m}^{(e,f)}[\mathbf{q}_u^{(e)}, \mathbf{q}_u^{(f)}, \mathbf{q}_t^{(ef)}] &= \left[I_p^{(e)} + I_p^{(f)}\right] + H^{(ef)} \\
&= \left[I_p^{(e)} + I_p^{(f)}\right] + \int_{\partial\Omega_{(ef)}} [\check{\mathbf{t}}^{(ef)T}(\check{\mathbf{u}}^{(e)} - \check{\mathbf{u}}^{(f)})] \mathrm{d}(\partial\Omega) \\
&= \left[\frac{1}{2}\mathbf{q}_u^{(e)T}\mathbf{K}_{uu}^{(e)}\mathbf{q}_u^{(e)} - \mathbf{q}_u^{(e)T}\left(\mathbf{f}_u^{\hat{p}(e)} + \mathbf{f}_u^{\hat{t}(e)}\right) + \frac{1}{2}\mathbf{q}_u^{(f)T}\mathbf{K}_{uu}^{(f)}\mathbf{q}_u^{(f)}\right. \\
&\quad \left. - \mathbf{q}_u^{(f)T}\left(\mathbf{f}_u^{\hat{p}(f)} + \mathbf{f}_u^{\hat{t}(f)}\right)\right] \\
&\quad + \mathbf{q}_t^{(ef)T}\left(\mathbf{L}_{ut}^{(e)T}\mathbf{q}_u^{(e)} + \mathbf{L}_{ut}^{(f)T}\mathbf{q}_u^{(f)}\right).
\end{aligned}
$$

From the stationarity condition of $I_{p,m}^{(e,f)}$ the following equations are obtained

$$
\delta I_{p,m}^{(e,f)} = \frac{\partial I_{p,m}}{\partial \mathbf{q}_u^{(e)}} \cdot \delta \mathbf{q}_u^{(e)} + \frac{\partial I_{p,m}}{\partial \mathbf{q}_u^{(f)}} \cdot \delta \mathbf{q}_u^{(f)} + \frac{\partial I_{p,m}}{\partial \mathbf{q}_t^{(ef)}} \cdot \delta \mathbf{q}_t^{(ef)} = 0 \quad \rightarrow
$$

$$
\frac{\partial I_{p,m}}{\partial \mathbf{q}_u^{(e)}} = 0 \quad \rightarrow \quad \mathbf{K}_{uu}^{(e)}\mathbf{q}_u^{(e)} + \mathbf{L}_{ut}^{(e)}\mathbf{q}_t^{(ef)} = \mathbf{f}_u^{\hat{p}(e)} + \mathbf{f}_u^{\hat{t}(e)}, \quad \mathbf{x} \in \Omega_e,
$$

$$
\frac{\partial I_{p,m}}{\partial \mathbf{q}_u^{(f)}} = 0 \quad \rightarrow \quad \mathbf{K}_{uu}^{(f)}\mathbf{q}_u^{(f)} + \mathbf{L}_{ut}^{(f)}\mathbf{q}_t^{(ef)} = \mathbf{f}_u^{\hat{p}(f)} + \mathbf{f}_u^{\hat{t}(f)}, \quad \mathbf{x} \in \Omega_f.
$$

$$
\frac{\partial I_{p,m}}{\partial \mathbf{q}_t^{(ef)}} = 0 \quad \rightarrow \quad \mathbf{L}_{ut}^{(e)T}\mathbf{q}_u^{(e)} + \mathbf{L}_{ut}^{(f)T}\mathbf{q}_u^{(f)} = \mathbf{0}, \quad s \in \partial\Omega_{(ef)},
$$

$$
\begin{bmatrix} \mathbf{K}_{uu}^{(e)} & \mathbf{L}_{ut}^{(e)} & \mathbf{0} \\ \mathbf{L}_{ut}^{(e)T} & \mathbf{0} & \mathbf{L}_{ut}^{(f)T} \\ \mathbf{0} & \mathbf{L}_{ut}^{(f)} & \mathbf{K}_{uu}^{(f)} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{q}_u^{(e)} \\ \mathbf{q}_t^{(ef)} \\ \mathbf{q}_u^{(f)} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_u^{\hat{p}(e)} + \mathbf{f}_u^{\hat{t}(e)} \\ \mathbf{0} \\ \mathbf{f}_u^{\hat{p}(f)} + \mathbf{f}_u^{\hat{t}(f)} \end{bmatrix}.
$$

III) The set of three equations contains: two equilibrium conditions for FE (e) and (f)

$$
\mathbf{K}_{uu}^{(e)}\mathbf{q}_u^{(e)} + \mathbf{L}_{ut}^{(e)}\mathbf{q}_t^{(ef)} = \mathbf{f}_u^{\hat{p}(e)} + \mathbf{f}_u^{\hat{t}(e)}, \qquad \mathbf{K}_{uu}^{(f)}\mathbf{q}_u^{(f)} + \mathbf{L}_{ut}^{(f)}\mathbf{q}_t^{(ef)} = \mathbf{f}_u^{\hat{p}(f)} + \mathbf{f}_u^{\hat{t}(f)},
$$

represented by nodal displacement vectors $\mathbf{q}_u$, taking into consideration nodal interelement actions $\mathbf{q}_t^{(ef)}$, and the additional equation

$$
\mathbf{L}_{ut}^{(e)T}\mathbf{q}_u^{(e)} + \mathbf{L}_{ut}^{(f)T}\mathbf{q}_u^{(f)} = \mathbf{0},
$$

which denotes a variational satisfaction of the requirement of displacement continuity.

IV) In the diagram for HD-FE (Fig. 4) the interelement line is distinguished and additional unknowns $\mathbf{q}_t^{(ef)}$ are connected with it. Two equilibrium equations for FEs (e) and (f) and the equation for boundary $\partial\Omega_{(ef)}$ are exposed. On the interelement line we introduce the approximation of traction $\mathbf{t}^{(ef)}(s)$ (shown as full arrows). The requirement of displacement continuity (6) includes parts of vectors $\mathbf{q}_u^{(e)}$, $\mathbf{q}_u^{(f)}$ and matrices $\mathbf{L}_{ut}^{(e)}$, $\mathbf{L}_{ut}^{(f)}$, which are related with $\partial\Omega_{(ef)}$. The contents of FE (e) box in the diagrams for CD-FE and HD-FE models is identical, but now the equilibrium equation (3) for FE (e) gives the connection of $\mathbf{q}_u^{(e)}$ with $\mathbf{q}_t^{(ef)}$. It is worth emphasising that in the HD-FE model two types of unknowns exist: displacements and interelement actions.

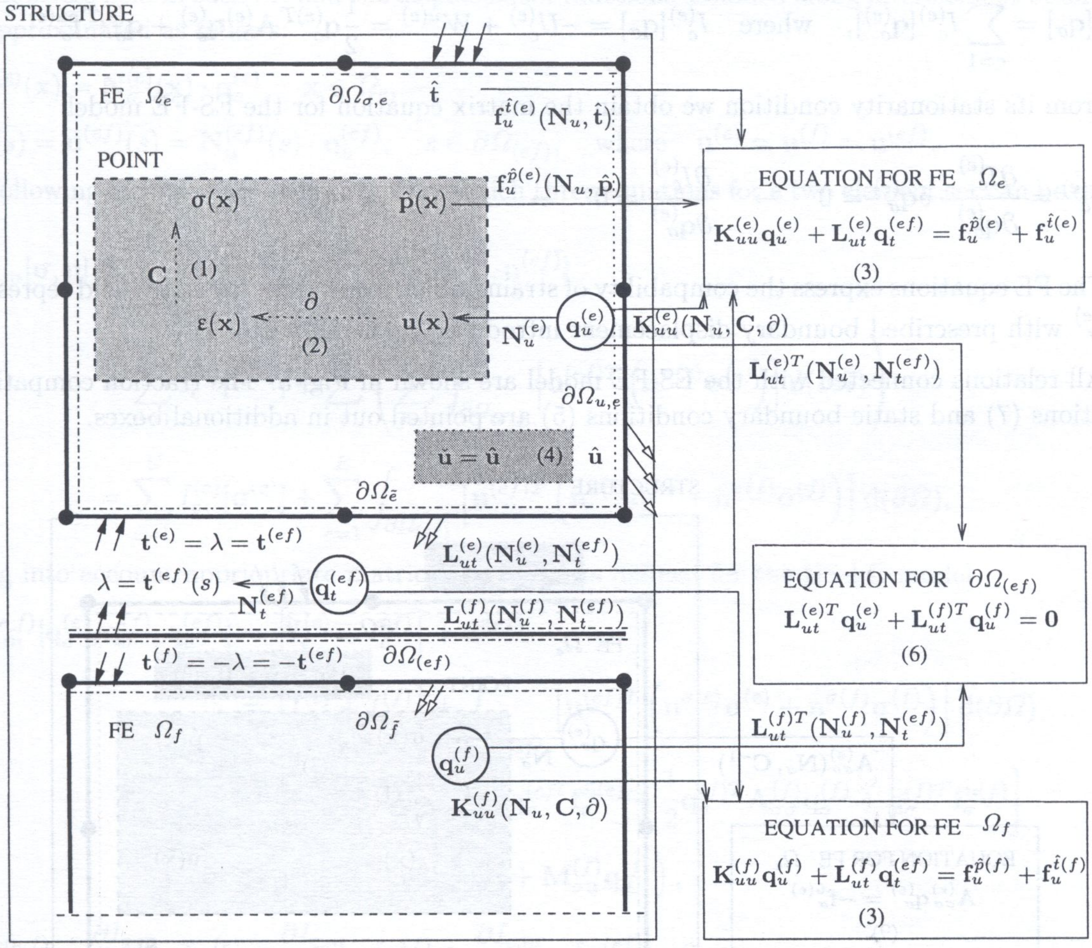

Fig. 4. Diagram for hybrid displacement model - HD-FE

2.3. An equilibrium stress model – ES-FE

I) The equilibrium stress model is based on the principle of minimum of complementary energy

$$I_c[\sigma] = -\int_{\Omega}\Big[\frac{1}{2}\sigma^{\rm T}\mathbf{C}^{-1}\sigma\Big]\,d\Omega + \int_{\partial\Omega_u}[(\mathbf{n}^{\sigma}\sigma)^{\rm T}\hat{\mathbf{u}}]\,d(\partial\Omega).$$

In this case the equilibrium equation and static boundary conditions are satisfied *a priori*. The constitutive relation is used in the construction of complementary energy.

II) The set of stress functions

$$\sigma(\mathbf{x}) = \mathbf{N}_{\sigma}^{(e)}(\mathbf{x})\cdot\mathbf{q}_{\sigma}^{(e)}, \qquad \mathbf{x}\in\Omega_e$$

can be taken as admissible functions if they satisfy the following requirements: i) they are continuous, single-valued and satisfy the equation of equilibrium in each FE, ii) they satisfy equilibrium conditions on interelement boundaries, iii) they ensure the compliance with static boundary conditions. The complementary energy for a set of FEs can be written in the following manner

$$I_c[\sigma] = \sum_{e=1}^{E}\Big\{-\int_{\Omega_e}\Big[\frac{1}{2}\sigma^{\rm T}\mathbf{C}^{-1}\sigma\Big]\,d\Omega + \int_{\partial\Omega_{u,e}}[(\mathbf{n}^{\sigma}\sigma)^{\rm T}\hat{\mathbf{u}}]\,d(\partial\Omega)\Big\},$$

$$I_c[\mathbf{q}_\sigma] = \sum_{e=1}^{E} I_c^{(e)}[\mathbf{q}_\sigma^{(e)}], \quad \text{where} \quad I_c^{(e)}[\mathbf{q}_\sigma] = -U_c^{(e)} + W^{\hat{u}(e)} = \frac{1}{2}\mathbf{q}_\sigma^{(e)T}\mathbf{A}_{\sigma\sigma}^{(e)}\mathbf{q}_\sigma^{(e)} + \mathbf{q}_\sigma^{(e)T}\mathbf{f}_\sigma^{\hat{u}(e)}$$

and from its stationarity condition we obtain the matrix equation for the ES-FE model:

$$\delta I_c^{(e)} = \frac{\partial I_c^{(e)}}{\partial \mathbf{q}_\sigma^{(e)}} \cdot \delta \mathbf{q}_\sigma^{(e)} = 0 \quad \rightarrow \quad \frac{\partial I_c^{(e)}}{\partial \mathbf{q}_\sigma^{(e)}} = 0 \quad \rightarrow \quad \mathbf{A}_{\sigma\sigma}^{(e)}\mathbf{q}_\sigma^{(e)} = -\mathbf{f}_\sigma^{\hat{u}(e)}.$$

III) The FE equations express the compability of strains which result from the stress field represented by $\mathbf{q}_\sigma^{(e)}$ with prescribed boundary displacement introduced by nodal vector $\mathbf{f}_\sigma^{\hat{u}(e)}$.

IV) All relations connected with the ES-FE model are shown in Fig. 5. The traction compatibility conditions (7) and static boundary conditions (5) are pointed out in additional boxes.

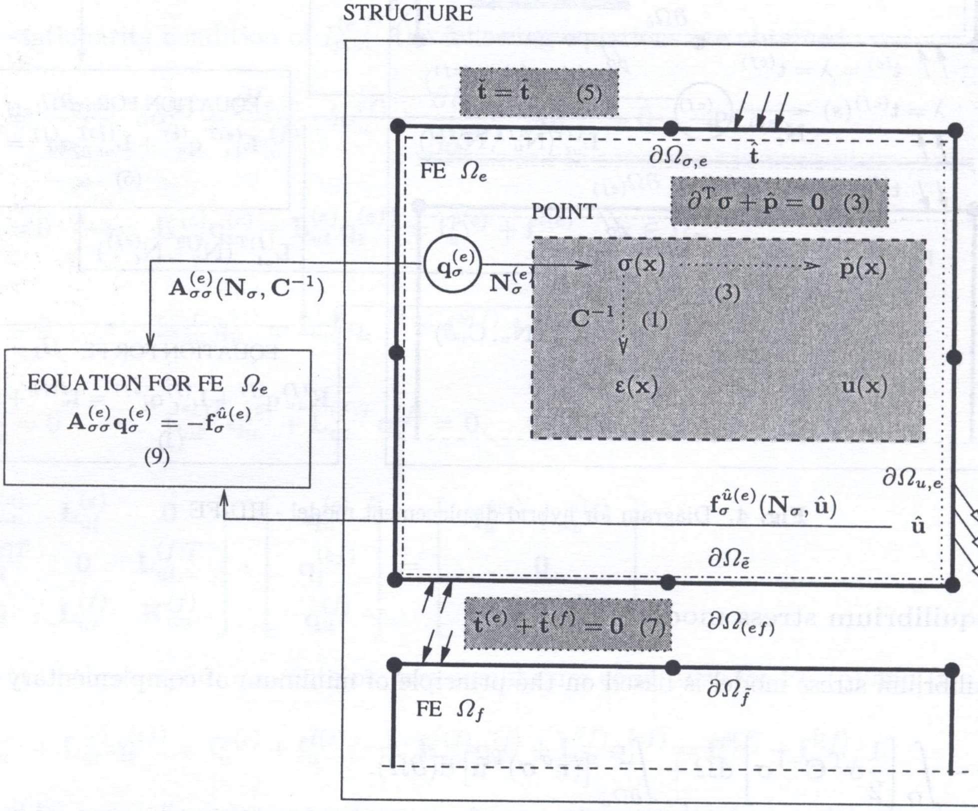

Fig. 5. Diagram for equilibrium stress model ES-FE

2.4. Hybrid stress model – HS-FE

I) The traction compatibility condition

$$\mathbf{t}^{(e)}(s) + \mathbf{t}^{(f)}(s) = \mathbf{0}, \qquad s \in \partial\Omega_{(ef)}$$

is enforced through a modification of complementary energy, using as Lagrange functions the displacements appearing along interelement lines

$$I_{c,m}[\boldsymbol{\sigma}, \mathbf{u}] = I_c[\boldsymbol{\sigma}] + G[\boldsymbol{\sigma}, \mathbf{u}].$$

II) The stress field in each FE and the displacement functions assumed along interelement boundaries are approximated as follows:

$$\boldsymbol{\sigma}^{(e)}(\mathbf{x}) = \mathbf{N}_\sigma^{(e)}(\mathbf{x}) \cdot \mathbf{q}_\sigma^{(e)}, \quad \mathbf{x} \in \Omega_e,$$

$$\boldsymbol{\lambda}(s) = \mathbf{u}^{(ef)}(s) = \mathbf{N}_u^{(ef)}(s) \cdot \mathbf{q}_u^{(ef)}, \quad s \in \partial\Omega_{(ef)}, \quad \text{where} \quad \mathbf{u}^{(e)} = \mathbf{u}^{(f)} = \mathbf{u}^{(ef)}.$$

The following formulae are obtained, from which three equations for a two element set can be derived

$$\begin{aligned} I_{c,m}[\boldsymbol{\sigma}, \mathbf{u}] &= \sum_{e=1}^{E} I_c^{(e)}[\boldsymbol{\sigma}^{(e)}] + \sum_{e=1}^{E} G^{(ef)}[\boldsymbol{\sigma}^{(e)}, \mathbf{u}^{(ef)}] \\ &= \sum_{e=1}^{E} I_c^{(e)}[\boldsymbol{\sigma}^{(e)}] + \sum_{e=1}^{E} \left(\sum_{f=1}^{E(e)} \int_{\partial\Omega_{(ef)}} \left[\mathbf{u}^{(ef)T} \left(\mathbf{n}^{\sigma(e)} \boldsymbol{\sigma}^{(e)} \right) \right] d(\partial\Omega) \right) \\ &= \sum_{e=1}^{E} I_c^{(e)}[\boldsymbol{\sigma}^{(e)}] + \sum_{e=1}^{E} \int_{\partial\Omega_{(ef)}} \left[\mathbf{u}^{(ef)T} \left(\mathbf{n}^{\sigma(e)} \boldsymbol{\sigma}^{(e)} + \mathbf{n}^{\sigma(f)} \boldsymbol{\sigma}^{(f)} \right) \right] d(\partial\Omega), \end{aligned}$$

taking into account appriopriate matrices and vectors defined for the HS-FE model

$$\begin{aligned} I_{c,m}^{(e,f)}[\mathbf{q}_\sigma^{(e)}, \mathbf{q}_\sigma^{(f)}, \mathbf{q}_u^{(ef)}] &= \left[I_c^{(e)} + I_c^{(f)} \right] + G^{(ef)} \\ &= \left[I_c^{(e)} + I_c^{(f)} \right] + \int_{\partial\Omega_{(ef)}} \left[\mathbf{u}^{(ef)T} \left(\mathbf{n}^{\sigma(e)} \boldsymbol{\sigma}^{(e)} + \mathbf{n}^{\sigma(f)} \boldsymbol{\sigma}^{(f)} \right) \right] d(\partial\Omega) \\ &= \left[\frac{1}{2} \mathbf{q}_\sigma^{(e)T} \mathbf{A}_{\sigma\sigma}^{(e)} \mathbf{q}_\sigma^{(e)} + \mathbf{q}_\sigma^{(e)T} \mathbf{f}_\sigma^{\hat{u}(e)} + \frac{1}{2} \mathbf{q}_\sigma^{(f)T} \mathbf{A}_{\sigma\sigma}^{(f)} \mathbf{q}_\sigma^{(f)} + \mathbf{q}_\sigma^{(f)T} \mathbf{f}_\sigma^{\hat{u}(f)} \right] \\ &\quad + \mathbf{q}_u^{(ef)T} \left(\mathbf{M}_{\sigma u}^{(e)} \mathbf{q}_\sigma^{(e)} + \mathbf{M}_{\sigma u}^{(f)} \mathbf{q}_\sigma^{(f)} \right), \end{aligned}$$

$$\delta I_{c,m}^{(e,f)} = \frac{\partial I_{c,m}}{\partial \mathbf{q}_\sigma^{(e)}} \cdot \delta \mathbf{q}_\sigma^{(e)} + \frac{\partial I_{c,m}}{\partial \mathbf{q}_\sigma^{(f)}} \cdot \delta \mathbf{q}_\sigma^{(f)} + \frac{\partial I_{c,m}}{\partial \mathbf{q}_u^{(ef)}} \cdot \delta \mathbf{q}_u^{(ef)} = 0 \quad \rightarrow$$

$$\frac{\partial I_{c,m}}{\partial \mathbf{q}_\sigma^{(e)}} = \mathbf{0} \quad \rightarrow \quad \mathbf{A}_{\sigma\sigma}^{(e)} \mathbf{q}_\sigma^{(e)} + \mathbf{M}_{\sigma u}^{(e)} \mathbf{q}_u^{(ef)} = -\mathbf{f}_\sigma^{\hat{u}(e)}, \quad \mathbf{x} \in \Omega_e,$$

$$\frac{\partial I_{c,m}}{\partial \mathbf{q}_\sigma^{(f)}} = 0 \quad \rightarrow \quad \mathbf{A}_{\sigma\sigma}^{(f)} \mathbf{q}_\sigma^{(f)} + \mathbf{M}_{\sigma u}^{(f)} \mathbf{q}_u^{(ef)} = -\mathbf{f}_\sigma^{\hat{u}(f)}, \quad \mathbf{x} \in \Omega_f,$$

$$\frac{\partial I_{c,m}}{\partial \mathbf{q}_u^{(ef)}} = 0 \quad \rightarrow \quad \mathbf{M}_{\sigma u}^{(e)T} \mathbf{q}_\sigma^{(e)} + \mathbf{M}_{\sigma u}^{(f)T} \mathbf{q}_\sigma^{(f)} = \mathbf{0}, \quad s \in \partial\Omega_{(ef)},$$

$$\begin{bmatrix} \mathbf{A}_{\sigma\sigma}^{(e)} & \mathbf{M}_{\sigma u}^{(e)} & \mathbf{0} \\ \mathbf{M}_{\sigma u}^{(e)T} & \mathbf{0} & \mathbf{M}_{\sigma u}^{(f)T} \\ \mathbf{0} & \mathbf{M}_{\sigma u}^{(f)} & \mathbf{A}_{\sigma\sigma}^{(f)} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{q}_\sigma^{(e)} \\ \mathbf{q}_u^{(ef)} \\ \mathbf{q}_\sigma^{(f)} \end{bmatrix} = \begin{bmatrix} -\mathbf{f}_\sigma^{\hat{u}(e)} \\ \mathbf{0} \\ -\mathbf{f}_\sigma^{\hat{u}(f)} \end{bmatrix}.$$

III) The three equations above can be written in the following order

$$\mathbf{A}_{\sigma\sigma}^{(e)} \mathbf{q}_\sigma^{(e)} + \mathbf{M}_{\sigma u}^{(e)} \mathbf{q}_u^{(ef)} = -\mathbf{f}_\sigma^{\hat{u}(e)}, \qquad \mathbf{A}_{\sigma\sigma}^{(f)} \mathbf{q}_\sigma^{(f)} + \mathbf{M}_{\sigma u}^{(f)} \mathbf{q}_u^{(ef)} = -\mathbf{f}_\sigma^{\hat{u}(f)},$$
$$\mathbf{M}_{\sigma u}^{(e)T} \mathbf{q}_\sigma^{(e)} + \mathbf{M}_{\sigma u}^{(f)T} \mathbf{q}_\sigma^{(f)} = \mathbf{0}.$$

Unlike the ES-FE model, matrices $\mathbf{M}_{\sigma u}^{(e)}$, $\mathbf{M}_{\sigma u}^{(f)}$ now appear. They link vectors $\mathbf{q}_\sigma^{(e)}$ and $\mathbf{q}_\sigma^{(f)}$, since the third equation ensures in a variational way the interelement action compatibility. The two other equations related to FEs (e) and (f) connect the stress degrees of freedom $\mathbf{q}_\sigma^{(e)}$, $\mathbf{q}_\sigma^{(f)}$ and generalized displacements $\mathbf{q}_u^{(ef)}$ at a finite number of boundary nodes and vectors $\mathbf{f}_\sigma^{\hat{u}(e)}$, $\mathbf{f}_\sigma^{\hat{u}(f)}$, ensuring compatibility of strains.

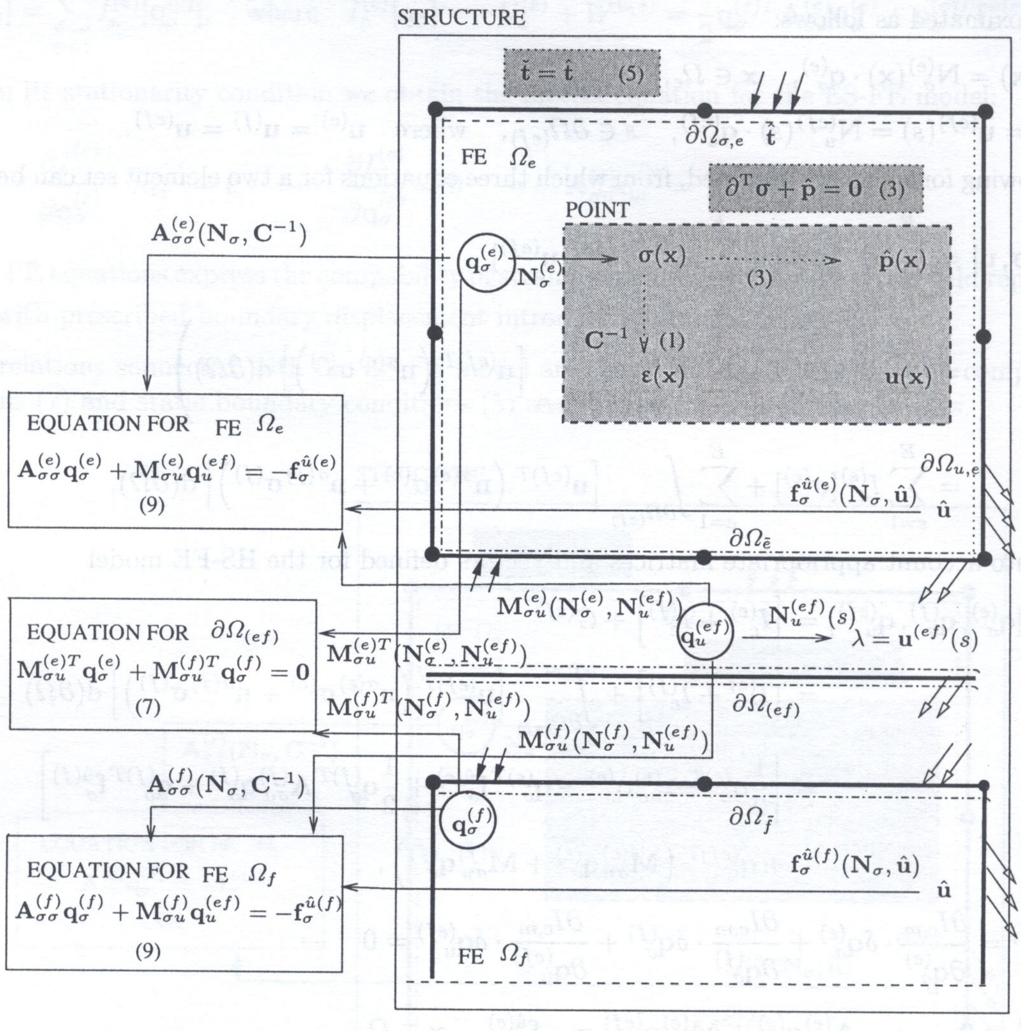

Fig. 6. Diagram for hybrid stress model – HS-FE

IV) The diagram for HS-FE model (Fig. 6) should be analysed in comparison with the ES-FE diagram in Fig. 5.

2.5. Displacement-stress (mixed) model – DS-FE

I) By enforcing the stress-strain relation in a strong sense we pass from the Hu-Washizu functional to the two-field Hellinger–Reissner one

$$I_{HR}[\mathbf{u},\boldsymbol{\sigma}] = \int_\Omega \left[-\frac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{C}^{-1}\boldsymbol{\sigma} + \boldsymbol{\sigma}^{\mathrm{T}}(\partial\mathbf{u}) - \hat{\mathbf{p}}^{\mathrm{T}}\mathbf{u} \right] \mathrm{d}\Omega - \int_{\partial\Omega_\sigma} [\hat{\mathbf{t}}^{\mathrm{T}}\mathbf{u}]\, \mathrm{d}(\partial\Omega) - \int_{\partial\Omega_u} [(\mathbf{n}^\sigma\boldsymbol{\sigma})^{\mathrm{T}}(\mathbf{u}-\hat{\mathbf{u}})]\, \mathrm{d}(\partial\Omega).$$

The integration by parts leads to the alternative expression for the Hellinger–Reissner functional

$$\tilde{I}_{HR}[\mathbf{u},\sigma] = -\int_\Omega \left[\frac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{C}^{-1}\boldsymbol{\sigma}\right] \mathrm{d}\Omega - \int_\Omega [(\partial^{\mathrm{T}}\boldsymbol{\sigma}+\hat{\mathbf{p}})^{\mathrm{T}}\mathbf{u}]\, \mathrm{d}\Omega + \int_{\partial\Omega_\sigma} [(\mathbf{n}^\sigma\boldsymbol{\sigma}-\hat{\mathbf{t}})^{\mathrm{T}}\mathbf{u}]\, \mathrm{d}(\partial\Omega) + \int_{\partial\Omega_u} [(\mathbf{n}^\sigma\boldsymbol{\sigma})^{\mathrm{T}}\hat{\mathbf{u}}]\, \mathrm{d}(\partial\Omega),$$

with assumes the continuity of displacements $\mathbf{u}$ and equilibrium of tractions $\mathbf{t}$ on interelement boundaries.

II) The displacement and stress fields are assumed as independent variables inside each individual FE

$$\mathbf{u}(\mathbf{x}) = \mathbf{N}_u^{(e)}(\mathbf{x}) \cdot \mathbf{q}_u^{(e)}, \qquad \mathbf{x} \in \Omega_e,$$

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{N}_\sigma^{(e)}(\mathbf{x}) \cdot \mathbf{q}_\sigma^{(e)}, \qquad \mathbf{x} \in \Omega_e.$$

Based on the stationarity condition of $\tilde{I}_{HR}$ we obtain the following equations for the DS-FE model:

$$\tilde{I}_{HR}[\mathbf{q}_u, \mathbf{q}_\sigma] = \sum_{e=1}^{E} \tilde{I}_{HR}^{(e)},$$

$$\tilde{I}_{HR}^{(e)} = \frac{1}{2}\mathbf{q}_\sigma^{(e)\mathrm{T}} \mathbf{A}_{\sigma\sigma}^{(e)} \mathbf{q}_\sigma^{(e)} + \mathbf{q}_\sigma^{(e)\mathrm{T}} \tilde{\mathbf{G}}_{\sigma u}^{(e)} \mathbf{q}_u^{(e)} + \mathbf{q}_\sigma^{(e)\mathrm{T}} \mathbf{S}_{\sigma u}^{\hat{t}(e)} \mathbf{q}_u^{(e)} - \mathbf{q}_u^{(e)\mathrm{T}} \left(\mathbf{f}_u^{\hat{p}(e)} + \mathbf{f}_u^{\hat{t}(e)} \right) + \mathbf{q}_\sigma^{(e)\mathrm{T}} \mathbf{f}_\sigma^{\hat{u}(e)}$$

$$\delta \tilde{I}_{HR}^{(e)} = \frac{\partial \tilde{I}_{HR}^{(e)}}{\partial \mathbf{q}_\sigma^{(e)}} \cdot \delta \mathbf{q}_\sigma^{(e)} + \frac{\partial \tilde{I}_{HR}^{(e)}}{\partial \mathbf{q}_u^{(e)}} \cdot \delta \mathbf{q}_u^{(e)} = 0 \quad \rightarrow$$

$$\frac{\partial \tilde{I}_{HR}^{(e)}}{\partial \mathbf{q}_\sigma^{(e)}} = 0 \quad \rightarrow \quad \mathbf{A}_{\sigma\sigma}^{(e)} \mathbf{q}_\sigma^{(e)} + \tilde{\mathbf{G}}_{\sigma u}^{(e)} \mathbf{q}_u^{(e)} + \mathbf{S}_{\sigma u}^{\hat{t}(e)} \mathbf{q}_u^{(e)} = -\mathbf{f}_\sigma^{\hat{u}(e)},$$

$$\frac{\partial \tilde{I}_{HR}^{(e)}}{\partial \mathbf{q}_u^{(e)}} = 0 \quad \rightarrow \quad \tilde{\mathbf{G}}_{\sigma u}^{(e)\mathrm{T}} \mathbf{q}_\sigma^{(e)} + \mathbf{S}_{\sigma u}^{\hat{t}(e)\mathrm{T}} \mathbf{q}_\sigma^{(e)} = \mathbf{f}_u^{\hat{p}(e)} + \mathbf{f}_u^{\hat{t}(e)},$$

$$\begin{bmatrix} \mathbf{A}_{\sigma\sigma}^{(e)} & (\tilde{\mathbf{G}}_{\sigma u}^{(e)} + \mathbf{S}_{\sigma u}^{\hat{t}(e)}) \\ (\tilde{\mathbf{G}}_{\sigma u}^{(e)\mathrm{T}} + \mathbf{S}_{\sigma u}^{\hat{t}(e)\mathrm{T}}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{q}_\sigma^{(e)} \\ \mathbf{q}_u^{(e)} \end{bmatrix} = \begin{bmatrix} -\mathbf{f}_\sigma^{\hat{u}(e)} \\ \mathbf{f}_u^{\hat{p}(e)} + \mathbf{f}_u^{\hat{t}(e)} \end{bmatrix}.$$

III) The connection of stress and displacement fields, represented by $\mathbf{q}_\sigma^{(e)}$, $\mathbf{q}_u^{(e)}$ in the first equation

$$\mathbf{A}_{\sigma\sigma}^{(e)} \mathbf{q}_\sigma^{(e)} + (\tilde{\mathbf{G}}_{\sigma u}^{(e)} + \mathbf{S}_{\sigma u}^{\hat{t}(e)}) \mathbf{q}_u^{(e)} = -\mathbf{f}_\sigma^{\hat{u}(e)}$$

means the consistence of the strains, obtained from the stresses as well as from the displacements, additionally taking into consideration prescribed boundary displacements $\hat{\mathbf{u}}$ and loads $\hat{\mathbf{t}}$. The second equality

$$(\tilde{\mathbf{G}}_{\sigma u}^{(e)\mathrm{T}} + \mathbf{S}_{\sigma u}^{\hat{t}(e)\mathrm{T}}) \mathbf{q}_\sigma^{(e)} = \mathbf{f}_u^{\hat{p}(e)} + \mathbf{f}_u^{\hat{t}(e)}$$

has the interpretation of equilibrium conditions formulated by $\mathbf{q}_\sigma^{(e)}$ and the nodal representation of loads $\hat{\mathbf{p}}$, $\hat{\mathbf{t}}$.

The employment of the second form of Hellinger–Reissner functional $\tilde{I}_{HR}[\mathbf{q}_u, \mathbf{q}_\sigma]$ (ensuing from integration by parts), as a starting point has consequences in the appearance of matrix $\mathbf{S}_{\sigma u}^{\hat{t}(e)}$ apart from vector $\mathbf{f}_u^{\hat{t}(e)}$.

IV) The connection of all fields and nodal representatives in diagram (Fig. 7) is clarified for the case when boundary loads $\hat{\mathbf{t}}$ and displacements $\hat{\mathbf{u}}$ disappear.

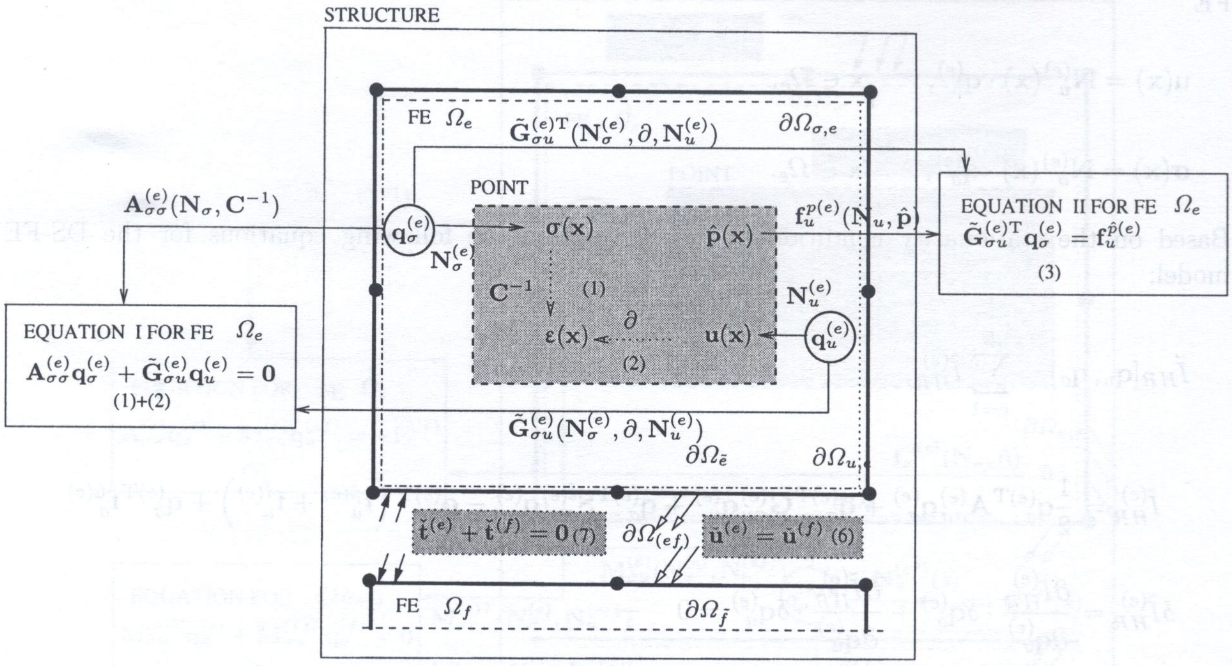

Fig. 7. Diagram for displacement-stress (mixed) model – DS-FE

2.6. Hybrid/mixed displacement–stress models

Various continuity requirements for the displacements and the stresses at interelement boundaries are considered and there is a great number of versions of the Hellinger–Riessner modified functional. In this section a boundary static and kinematic loads are omited.

2.6.1. Hybrid/mixed displacement-stress model with satisfaction displacement continuity in variational way – HDSC-FE

I) If the conditions of kinematic compatibility

$$\check{\mathbf{u}}^{(e)}(s) = \check{\mathbf{u}}^{(f)}(s), \qquad s \in \partial\Omega_{(ef)}$$

are introduced as constraints through Lagrange multipliers (which turn out to be interelement tractions) a respectively modified Hellinger-Reisner functional is used

$$I_{HR,mH}[\mathbf{u}, \boldsymbol{\sigma}, \mathbf{t}] = I_{HR}[\mathbf{u}, \boldsymbol{\sigma}] + H[\mathbf{u}, \mathbf{t}].$$

II) The three fields (displacements and stresses in FE, and interelement action) are approximated

$$\mathbf{u}(\mathbf{x}) = \mathbf{N}^{(e)}_{u}(\mathbf{x}) \cdot \mathbf{q}^{(e)}_{u}, \qquad \mathbf{x} \in \Omega_e,$$

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{N}^{(e)}_{\sigma}(\mathbf{x}) \cdot \mathbf{q}^{(e)}_{\sigma}, \qquad \mathbf{x} \in \Omega_e,$$

$$\boldsymbol{\lambda}(s) = \mathbf{t}^{(ef)}(s) = \mathbf{N}^{(ef)}_{t}(s) \cdot \mathbf{q}^{(ef)}_{t}, \qquad s \in \partial\Omega_{(ef)}.$$

As previously, a set of two FEs is taken into consideration. The energetic term $H^{(ef)}$ defined in Subsec. 1.3 is added to functional I_{HR}

$$I^{(e,f)}_{HR,mH}\left[\mathbf{q}^{(e)}_u,\mathbf{q}^{(e)}_\sigma,\mathbf{q}^{(f)}_u,\mathbf{q}^{(f)}_\sigma,\mathbf{q}^{(ef)}_t\right]=\left[I^{(e)}_{HR}+I^{(f)}_{HR}\right]+H^{(ef)}$$

$$=\frac{1}{2}\mathbf{q}^{(e)\mathrm{T}}_\sigma\mathbf{A}^{(e)}_{\sigma\sigma}\mathbf{q}^{(e)}_\sigma+\mathbf{q}^{(e)\mathrm{T}}_\sigma\mathbf{G}^{(e)}_{\sigma u}\mathbf{q}^{(e)}_u-\mathbf{q}^{(e)\mathrm{T}}_u\mathbf{f}^{\hat{p}(e)}_u$$

$$+\frac{1}{2}\mathbf{q}^{(f)\mathrm{T}}_\sigma\mathbf{A}^{(f)}_{\sigma\sigma}\mathbf{q}^{(f)}_\sigma+\mathbf{q}^{(f)\mathrm{T}}_\sigma\mathbf{G}^{(f)}_{\sigma u}\mathbf{q}^{(f)}_u-\mathbf{q}^{(f)\mathrm{T}}_u\mathbf{f}^{\hat{p}(f)}_u$$

$$+\mathbf{q}^{(ef)\mathrm{T}}_t\left(\mathbf{L}^{(e)\mathrm{T}}_{ut}\mathbf{q}^{(e)}_u+\mathbf{L}^{(f)\mathrm{T}}_{ut}\mathbf{q}^{(f)}_u\right)$$

and from the stationarity condition the following equations for the HDSE-FE model are obtained

$$\delta I^{(e,f)}_{HR,mH}=\frac{\partial I_{HR,mH}}{\partial\mathbf{q}^{(e)}_\sigma}\cdot\delta\mathbf{q}^{(e)}_\sigma+\frac{\partial I_{HR,mH}}{\partial\mathbf{q}^{(f)}_\sigma}\cdot\delta\mathbf{q}^{(f)}_\sigma+\frac{\partial I_{HR,mH}}{\partial\mathbf{q}^{(e)}_u}\cdot\delta\mathbf{q}^{(e)}_u+\frac{\partial I_{HR,mH}}{\partial\mathbf{q}^{(f)}_u}\cdot\delta\mathbf{q}^{(f)}_u$$

$$+\frac{\partial I_{HR,mH}}{\partial\mathbf{q}^{(ef)}_t}\cdot\delta\mathbf{q}^{(ef)}_t=0\quad\rightarrow$$

$$\frac{\partial I_{HR,mH}}{\partial\mathbf{q}^{(e)}_\sigma}=0\quad\rightarrow\quad\mathbf{A}^{(e)}_{\sigma\sigma}\mathbf{q}^{(e)}_\sigma+\mathbf{G}^{(e)}_{\sigma u}\mathbf{q}^{(e)}_u=\mathbf{0},\qquad\mathbf{x}\in\Omega_e,$$

$$\frac{\partial I_{HR,mH}}{\partial\mathbf{q}^{(f)}_\sigma}=0\quad\rightarrow\quad\mathbf{A}^{(f)}_{\sigma\sigma}\mathbf{q}^{(f)}_\sigma+\mathbf{G}^{(f)}_{\sigma u}\mathbf{q}^{(f)}_u=\mathbf{0},\qquad\mathbf{x}\in\Omega_f,$$

$$\frac{\partial I_{HR,mH}}{\partial\mathbf{q}^{(e)}_u}=0\quad\rightarrow\quad\mathbf{G}^{(e)\mathrm{T}}_{\sigma u}\mathbf{q}^{(e)}_\sigma+\mathbf{L}^{(e)}_{ut}\mathbf{q}^{(ef)}_t=\mathbf{f}^{\hat{p}(e)}_u,\qquad\mathbf{x}\in\Omega_e,$$

$$\frac{\partial I_{HR,mH}}{\partial\mathbf{q}^{(f)}_u}=0\quad\rightarrow\quad\mathbf{G}^{(f)\mathrm{T}}_{\sigma u}\mathbf{q}^{(f)}_\sigma+\mathbf{L}^{(f)}_{ut}\mathbf{q}^{(ef)}_t=\mathbf{f}^{\hat{p}(f)}_u,\qquad\mathbf{x}\in\Omega_f,$$

$$\frac{\partial I_{HR,mH}}{\partial\mathbf{q}^{(ef)}_t}=0\quad\rightarrow\quad\mathbf{L}^{(e)\mathrm{T}}_{ut}\mathbf{q}^{(e)}_u+\mathbf{L}^{(f)\mathrm{T}}_{ut}\mathbf{q}^{(f)}_u=\mathbf{0},\qquad s\in\partial\Omega_{(ef)},$$

$$\begin{bmatrix}\mathbf{A}^{(e)}_{\sigma\sigma}&\mathbf{G}^{(e)}_{\sigma u}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{G}^{(e)\mathrm{T}}_{\sigma u}&\mathbf{0}&\mathbf{L}^{(e)}_{ut}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{L}^{(e)\mathrm{T}}_{ut}&\mathbf{0}&\mathbf{L}^{(f)\mathrm{T}}_{ut}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{L}^{(f)}_{ut}&\mathbf{0}&\mathbf{G}^{(f)\mathrm{T}}_{\sigma u}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{G}^{(f)}_{\sigma u}&\mathbf{A}^{(f)}_{\sigma\sigma}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{q}^{(e)}_\sigma\\\mathbf{q}^{(e)}_u\\\mathbf{q}^{(ef)}_t\\\mathbf{q}^{(f)}_u\\\mathbf{q}^{(f)}_\sigma\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\\mathbf{f}^{\hat{p}(e)}_u\\\mathbf{0}\\\mathbf{f}^{\hat{p}(f)}_u\\\mathbf{0}\end{bmatrix}.$$

III) Analysing the similarities and differences between the DS-FE and HDSE-FE models, in the latter case we become aware of the appearance of matrices $\mathbf{L}^{(e)}_{ut}$, $\mathbf{L}^{(f)}_{ut}$ in equilibrium conditions written for FE (e) and (f), and an additional equation which has the interpretation of kinematic compatibility condition.

IV) In Fig. 8 in five boxes the appriopriate equations are demonstrated (two equations for each FE (e) and (f) and an equation connected with interelement line $\partial\Omega_{(ef)}$).

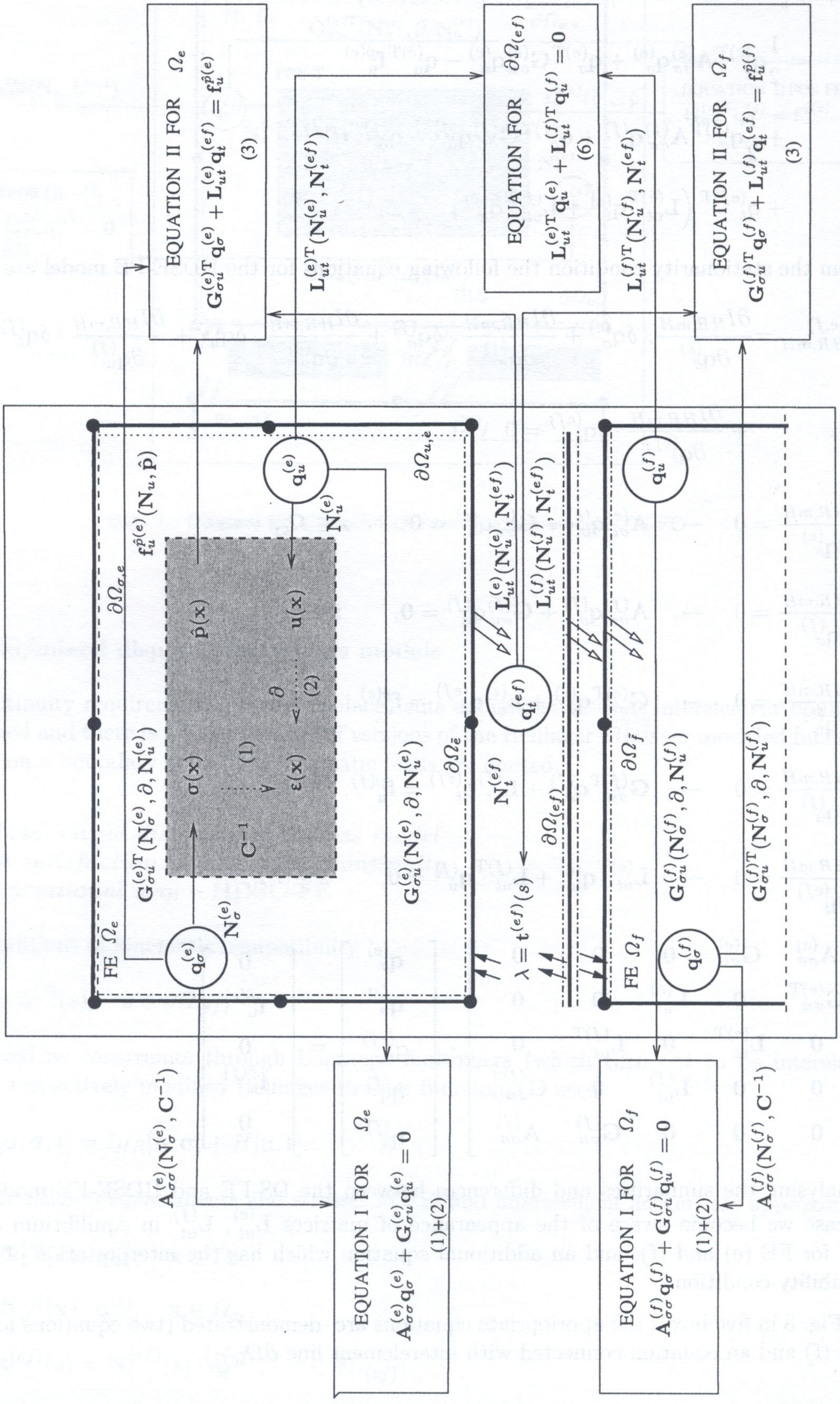

Fig. 8. Diagram for HDSE-FE model

2.6.2. Hybrid (mixed) displacement-stress model with assurance equilibrium of interelement traction in a variational way – HDSE-FE

I) If the condition of equilibrium of interelement tractions

$$\check{\mathbf{t}}^{(e)}(s) + \check{\mathbf{t}}^{(f)}(s) = \mathbf{0}, \quad s \in \partial\Omega_{(ef)}$$

is introduced as a constraint condition through a Lagrange multiplier field (which turns out to be displacement), another modification of the Hellinger–Reissner functional is introduced

$$I_{HR,mG}[\mathbf{u}, \boldsymbol{\sigma}] = I_{HR}[\mathbf{u}, \boldsymbol{\sigma}] + G[\boldsymbol{\sigma}, \mathbf{u}].$$

II) Besides the approximation of the displacement and stress fields we now introduce a discretization of the displacements on interelement boundaries

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \mathbf{N}_u^{(e)}(\mathbf{x}) \cdot \mathbf{q}_u^{(e)}, \quad \mathbf{x} \in \Omega_e, \\ \boldsymbol{\sigma}(\mathbf{x}) &= \mathbf{N}_\sigma^{(e)}(\mathbf{x}) \cdot \mathbf{q}_\sigma^{(e)}, \quad \mathbf{x} \in \Omega_e, \\ \lambda(s) &= \mathbf{u}^{(ef)}(s) = \mathbf{N}_u^{(ef)}(s) \cdot \mathbf{q}_u^{(ef)}, \quad s \in \partial\Omega_{(ef)}. \end{aligned}$$

Adding $G^{(ef)}$ (term defined in Subsec. 1.3) to I_{HR} we consider another modified Hellinger–Reissner functional, different from that in Subsec. 2.6.1

$$\begin{aligned} I_{HR,mG}^{(e,f)}\left[\mathbf{q}_u^{(e)}, \mathbf{q}_\sigma^{(e)}, \mathbf{q}_u^{(f)}, \mathbf{q}_\sigma^{(f)}, \mathbf{q}_u^{(ef)}\right] &= \left[I_{HR}^{(e)} + I_{HR}^{(f)}\right] + G^{(ef)} \\ &= \frac{1}{2}\mathbf{q}_\sigma^{(e)\mathrm{T}}\mathbf{A}_{\sigma\sigma}^{(e)}\mathbf{q}_\sigma^{(e)} + \mathbf{q}_\sigma^{(e)\mathrm{T}}\mathbf{G}_{\sigma u}^{(e)}\mathbf{q}_u^{(e)} - \mathbf{q}_u^{(e)\mathrm{T}}\mathbf{f}_u^{\hat{p}(e)} \\ &\quad + \frac{1}{2}\mathbf{q}_\sigma^{(f)\mathrm{T}}\mathbf{A}_{\sigma\sigma}^{(f)}\mathbf{q}_\sigma^{(f)} + \mathbf{q}_\sigma^{(f)\mathrm{T}}\mathbf{G}_{\sigma u}^{(f)}\mathbf{q}_u^{(f)} - \mathbf{q}_u^{(f)\mathrm{T}}\mathbf{f}_u^{\hat{p}(f)} \\ &\quad + \mathbf{q}_u^{(ef)\mathrm{T}}\left(\mathbf{M}_{\sigma u}^{(e)\mathrm{T}}\mathbf{q}_\sigma^{(e)} + \mathbf{M}_{\sigma u}^{(f)\mathrm{T}}\mathbf{q}_\sigma^{(f)}\right). \end{aligned}$$

From the stationarity condition

$$\begin{aligned} \delta I_{HR,mG}^{(e,f)} = \frac{\partial I_{HR,mG}}{\partial \mathbf{q}_\sigma^{(e)}} \cdot \delta\mathbf{q}_\sigma^{(e)} + \frac{\partial I_{HR,mG}}{\partial \mathbf{q}_\sigma^{(f)}} \cdot \delta\mathbf{q}_\sigma^{(f)} + \frac{\partial I_{HR,mG}}{\partial \mathbf{q}_u^{(e)}} \cdot \delta\mathbf{q}_u^{(e)} + \frac{\partial I_{HR,mG}}{\partial \mathbf{q}_u^{(f)}} \cdot \delta\mathbf{q}_u^{(f)} \\ + \frac{\partial I_{HR,mG}}{\partial \mathbf{q}_u^{(ef)}} \cdot \delta\mathbf{q}_u^{(ef)} = 0 \quad \rightarrow \end{aligned}$$

$$\frac{\partial I_{HR,mG}}{\partial \mathbf{q}_\sigma^{(e)}} = 0 \quad \rightarrow \quad \mathbf{A}_{\sigma\sigma}^{(e)}\mathbf{q}_\sigma^{(e)} + \mathbf{G}_{\sigma u}^{(e)}\mathbf{q}_u^{(e)} + \mathbf{M}_{\sigma u}^{(e)}\mathbf{q}_u^{(ef)} = \mathbf{0}, \quad \mathbf{x} \in \Omega_e,$$

$$\frac{\partial I_{HR,mG}}{\partial \mathbf{q}_\sigma^{(f)}} = 0 \quad \rightarrow \quad \mathbf{A}_{\sigma\sigma}^{(f)}\mathbf{q}_\sigma^{(f)} + \mathbf{G}_{\sigma u}^{(f)\mathrm{T}}\mathbf{q}_u^{(f)} + \mathbf{M}_{\sigma u}^{(f)}\mathbf{q}_u^{(ef)} = \mathbf{0}, \quad \mathbf{x} \in \Omega_f,$$

$$\frac{\partial I_{HR,mG}}{\partial \mathbf{q}_u^{(e)}} = 0 \quad \rightarrow \quad \mathbf{G}_{\sigma u}^{(e)\mathrm{T}}\mathbf{q}_\sigma^{(e)} = \mathbf{f}_u^{\hat{p}(e)}, \quad \mathbf{x} \in \Omega_e,$$

$$\frac{\partial I_{HR,mG}}{\partial \mathbf{q}_u^{(f)}} = 0 \quad \rightarrow \quad \mathbf{G}_{\sigma u}^{(f)\mathrm{T}}\mathbf{q}_\sigma^{(f)} = \mathbf{f}_u^{\hat{p}(f)}, \quad \mathbf{x} \in \Omega_f,$$

$$\frac{\partial I_{HR,mG}}{\partial \mathbf{q}_u^{(ef)}} = 0 \quad \rightarrow \quad \mathbf{M}_{\sigma u}^{(e)\mathrm{T}}\mathbf{q}_\sigma^{(e)} + \mathbf{M}_{\sigma u}^{(f)\mathrm{T}}\mathbf{q}_\sigma^{(f)} = \mathbf{0}, \quad s \in \partial\Omega_{(ef)}$$

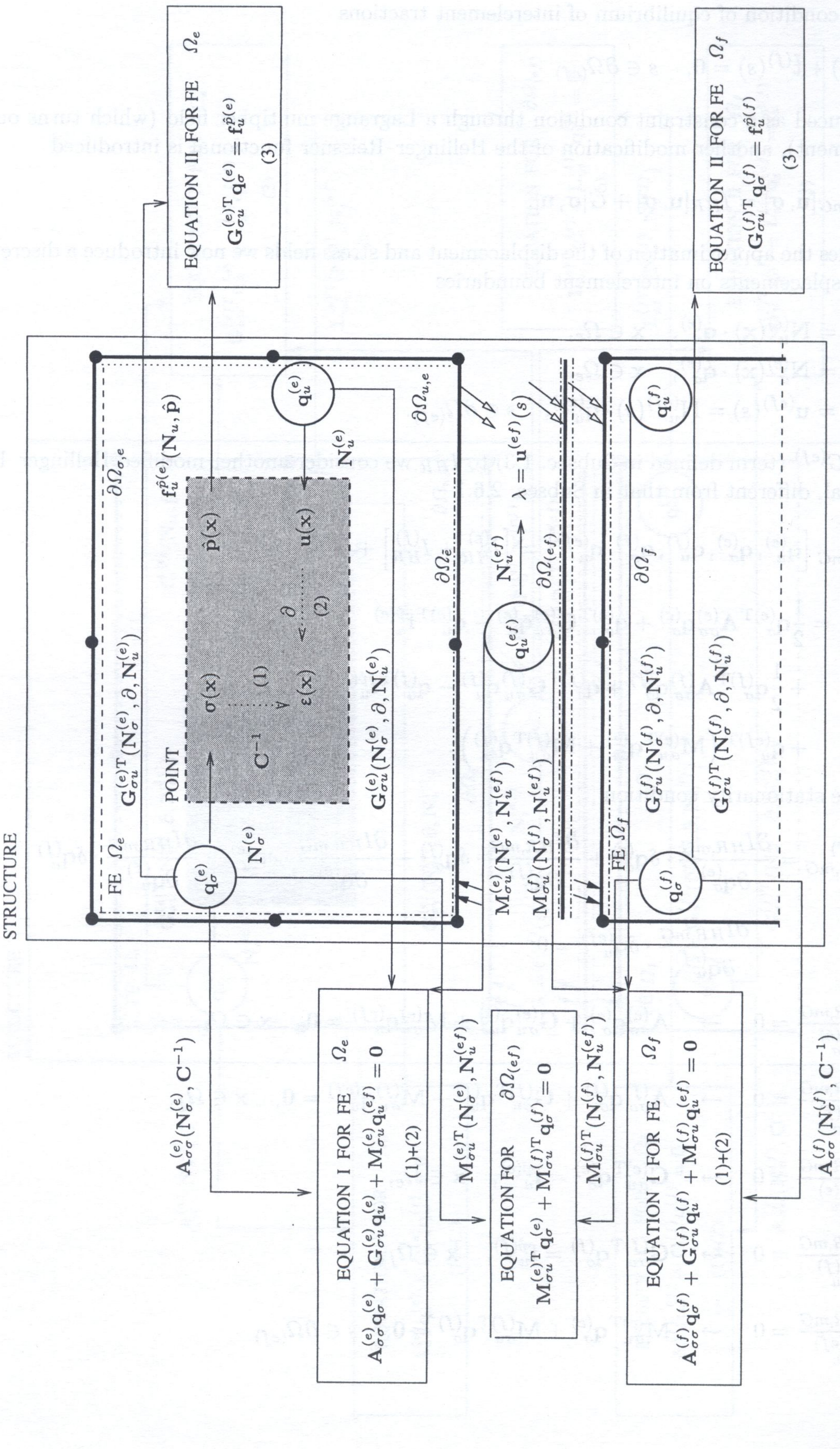

Fig. 9. Diagram for HDSE-FE model

the set of five equations is obtained

$$\begin{bmatrix} \mathbf{A}^{(e)}_{\sigma\sigma} & \mathbf{G}^{(e)}_{\sigma u} & \mathbf{M}^{(e)}_{\sigma u} & \mathbf{0} & \mathbf{0} \\ \mathbf{G}^{(e)\mathrm{T}}_{\sigma u} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{M}^{(e)\mathrm{T}}_{\sigma u} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}^{(f)\mathrm{T}}_{\sigma u} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}^{(f)\mathrm{T}}_{\sigma u} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}^{(f)}_{\sigma u} & \mathbf{G}^{(f)}_{\sigma u} & \mathbf{A}^{(f)}_{\sigma\sigma} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{q}^{(e)}_{\sigma} \\ \mathbf{q}^{(e)}_{u} \\ \mathbf{q}^{(ef)}_{u} \\ \mathbf{q}^{(f)}_{u} \\ \mathbf{q}^{(f)}_{\sigma} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}^{\hat{p}(e)}_{u} \\ \mathbf{0} \\ \mathbf{f}^{\hat{p}(f)}_{u} \\ \mathbf{0} \end{bmatrix}.$$

III) In this case we introduced matrices $\mathbf{M}^{(e)}_{\sigma u}$, $\mathbf{M}^{(f)}_{\sigma u}$. They must be taken into account in equations which represent the strain consistence conditions, written for two isolated FE (e) and (f). Additionally, by matrices $\mathbf{M}^{(e)}_{\sigma u}$, $\mathbf{M}^{(f)}_{\sigma u}$ we express the condition of equilibrium of interelement tractions on line $\partial\Omega_{(ef)}$.

IV) In Fig. 9 we can see five boxes which contain the above equations related to the set of FEs (e) and (f) connectes with line $\partial\Omega_{(ef)}$.

2.7. Displacement-strain (mixed) model – DE-FE (enhanced assumed strain EAS)

I) Starting from the three-field Hu-Washizu functional a another version of Hellinger–Reissner functional can be considered

$$I_{HR}[\mathbf{u}, \boldsymbol{\varepsilon}] = \int_{\Omega} \left[-\frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{C}\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{C}(\partial\mathbf{u}) - \hat{\mathbf{p}}^{\mathrm{T}}\mathbf{u} \right] \mathrm{d}\Omega - \int_{\partial\Omega_\sigma} [\hat{\mathbf{t}}^{\mathrm{T}}\mathbf{u}]\, \mathrm{d}(\partial\Omega).$$

According to the assumed strain concept two independent fields are introduced: displacement field $\mathbf{u}(\mathbf{x})$ (notice that strain field $\boldsymbol{\varepsilon}^u(\mathbf{u})$ is related with it) and an additional strain field $\boldsymbol{\varepsilon}^\alpha$

$$I_{HR,mS}[\mathbf{u}, \boldsymbol{\varepsilon}^u(\mathbf{u}), \boldsymbol{\varepsilon}^\alpha] = \int_{\Omega} \left[-\frac{1}{2}\boldsymbol{\varepsilon}^{\alpha\mathrm{T}}\mathbf{C}\boldsymbol{\varepsilon}^\alpha + \boldsymbol{\varepsilon}^{\alpha\mathrm{T}}\mathbf{C}\boldsymbol{\varepsilon}^u - \hat{\mathbf{p}}^{\mathrm{T}}\mathbf{u} \right] \mathrm{d}\Omega - \int_{\partial\Omega_\sigma} [\hat{\mathbf{t}}^{\mathrm{T}}\mathbf{u}]\, \mathrm{d}(\partial\Omega).$$

II) According to the above statement the following FE aproximation is used

$$\mathbf{u}(\mathbf{x}) = \mathbf{N}^{(e)}_u(\mathbf{x}) \cdot \mathbf{q}^{(e)}_u, \quad \mathbf{x} \in \Omega_e,$$
$$\boldsymbol{\varepsilon}^\alpha(\mathbf{x}) = \mathbf{P}^{(e)}_\alpha \cdot \boldsymbol{\alpha}^{(e)}_\varepsilon, \quad \mathbf{x} \in \Omega_e,$$

with the additional relation

$$\boldsymbol{\varepsilon}^u(\mathbf{u}) = \partial\mathbf{u} = \mathbf{B} \cdot \mathbf{q}^{(e)}_u.$$

III) Now, we can write the modified Hellinger-Reissner functional using the matrices defined for DE-FE

$$I_{HR,mS}[\mathbf{q}^{(e)}_u, \boldsymbol{\alpha}^{(e)}_\varepsilon] = -\boldsymbol{\alpha}^{(e)\mathrm{T}}_\varepsilon \mathbf{F}^{(e)}_{\alpha\alpha} \boldsymbol{\alpha}^{(e)}_\varepsilon + \boldsymbol{\alpha}^{(e)\mathrm{T}}_\varepsilon \mathbf{R}^{(e)}_{\alpha u} \mathbf{q}^{(e)}_u - \mathbf{q}^{(e)\mathrm{T}}_u \mathbf{f}^{\hat{p}(e)}_u - \mathbf{q}^{(e)\mathrm{T}}_u \mathbf{f}^{\hat{t}(e)}_u$$

and the following set of equations results

$$\begin{bmatrix} -\mathbf{F}^{(e)}_{\alpha\alpha} & \mathbf{R}^{(e)}_{\alpha u} \\ \mathbf{R}^{(e)\mathrm{T}}_{\alpha u} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{\alpha}^{(e)}_\varepsilon \\ \mathbf{q}^{(e)}_u \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}^{\hat{p}(e)}_u + \mathbf{f}^{\hat{t}(e)}_u \end{bmatrix}.$$

A reduction of the number of DOFs at the element level is possible by relating nodal displacements $\mathbf{q}_u^{(e)}$ with strain degrees of freedom $\boldsymbol{\alpha}_\varepsilon^{(e)}$

$$\boldsymbol{\alpha}_\varepsilon^{(e)} = (\mathbf{F}_{\alpha\alpha}^{(e)})^{-1} \cdot \mathbf{R}_{\alpha u}^{(e)} \cdot \mathbf{q}_u^{(e)}$$

and performing a condensation. We then obtain the equilibrium equation with an unknown nodal displacement vector

$$\bar{\mathbf{K}}_{uu}\mathbf{q}_u^{(e)} = \mathbf{f}_u^{\hat{p}(e)} + \mathbf{f}_u^{\hat{t}(e)} \quad \text{where} \quad \bar{\mathbf{K}}_{uu}^{(e)} = \mathbf{R}_{\alpha u}^{(e)\mathrm{T}}(\mathbf{F}_{\alpha\alpha}^{(e)})^{-1}\mathbf{R}_{\alpha u}^{(e)}.$$

IV) In Fig. 10 a diagram for the DE-FE model is shown.

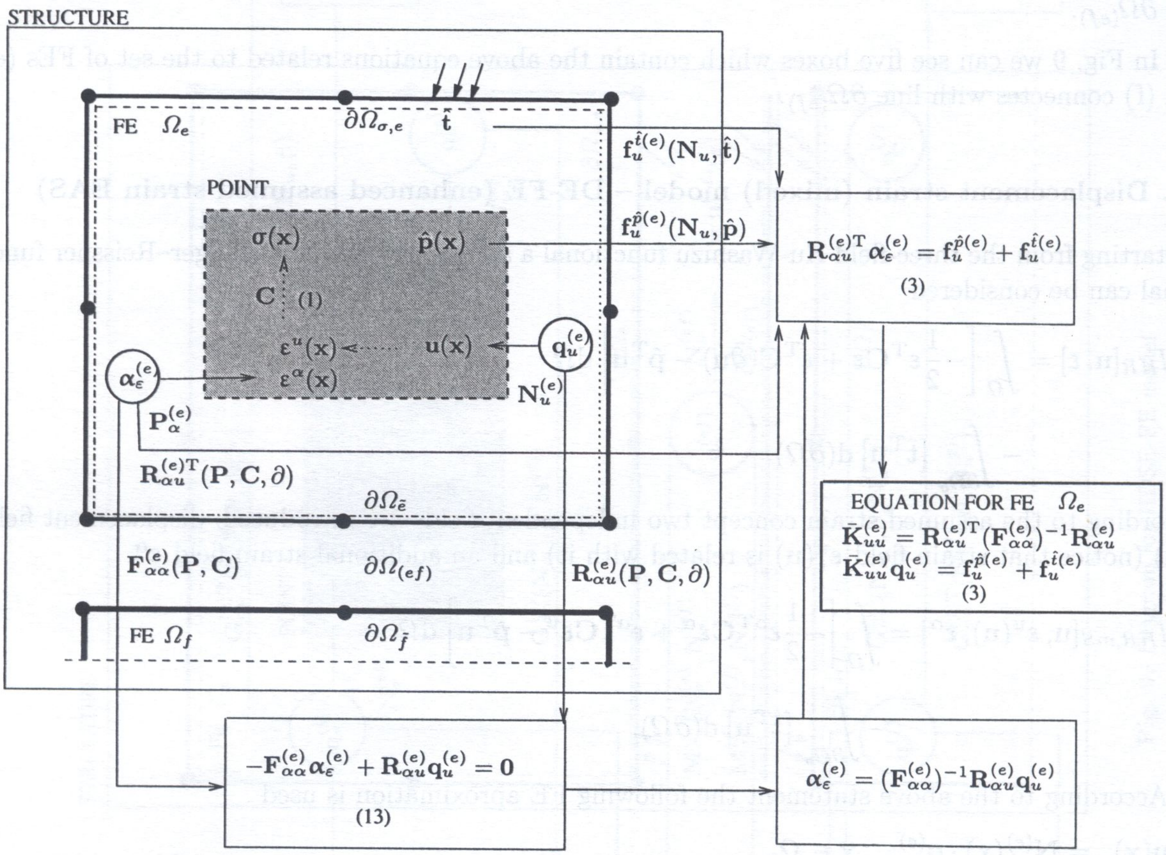

Fig. 10. Diagram for displacement-strain FE model – DE-FE

2.8. Displacement-stress-strain (mixed three-field) FE model – DSE-FE

I) In the general Hu-Washizu principle all variables $\mathbf{u}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$ may be varied free from constraints

$$I_{HW}[\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}] = \int_\Omega \left[+\frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{C}\boldsymbol{\varepsilon} - \boldsymbol{\sigma}^{\mathrm{T}}(\boldsymbol{\varepsilon} - \partial\mathbf{u}) - \hat{\mathbf{p}}^{\mathrm{T}}\mathbf{u} \right] \mathrm{d}\Omega - \int_{\partial\Omega_\sigma} [\hat{\mathbf{t}}^{\mathrm{T}}\mathbf{u}] \, \mathrm{d}(\partial\Omega).$$

The above functional is used not only for the derivation of DSE-FE model but it also gives the theoretical background for many various concepts (described in previous subsections).

II) The formulation of DSE-FE is based on the approximation of three fields

$$\mathbf{u}(\mathbf{x}) = \mathbf{N}_u^{(e)}(\mathbf{x}) \cdot \mathbf{q}_u^{(e)}, \quad \mathbf{x} \in \Omega_e,$$

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{N}_\varepsilon^{(e)}(\mathbf{x}) \cdot \mathbf{q}_\varepsilon^{(e)}, \quad \mathbf{x} \in \Omega_e,$$

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{N}_\sigma^{(e)}(\mathbf{x}) \cdot \mathbf{q}_\sigma^{(e)}, \quad \mathbf{x} \in \Omega_e.$$

According to the standard procedure

$$I_{HW}[\mathbf{q}_u, \mathbf{q}_\varepsilon, \mathbf{q}_\sigma] = \sum_{e=1}^{E} I_{HW}^{(e)},$$

$$I_{HW}^{(e)} = \frac{1}{2}\mathbf{q}_\varepsilon^{(e)\mathrm{T}}\mathbf{F}_{\varepsilon\varepsilon}^{(e)}\mathbf{q}_\varepsilon^{(e)} + \mathbf{q}_\sigma^{(e)\mathrm{T}}\mathbf{E}_{\sigma\varepsilon}^{(e)}\mathbf{q}_\varepsilon^{(e)} + \mathbf{q}_\sigma^{(e)\mathrm{T}}\mathbf{G}_{\sigma u}^{(e)}\mathbf{q}_u - \mathbf{q}_u^{(e)\mathrm{T}}\left(\mathbf{f}_u^{\hat{p}(e)} + \mathbf{f}_u^{\hat{t}(e)}\right),$$

$$\delta I_{HW}^{(e)} = \frac{\partial I_{HW}^{(e)}}{\partial \mathbf{q}_\varepsilon^{(e)}} \cdot \delta\mathbf{q}_\varepsilon^{(e)} + \frac{\partial I_{HW}^{(e)}}{\partial \mathbf{q}_\sigma^{(e)}} \cdot \delta\mathbf{q}_\sigma^{(e)} + \frac{\partial I_{HW}^{(e)}}{\partial \mathbf{q}_u^{(e)}} \cdot \delta\mathbf{q}_u^{(e)} = 0 \quad \rightarrow$$

$$\frac{\partial I_{HW}^{(e)}}{\partial \mathbf{q}_\varepsilon^{(e)}} = 0 \quad \rightarrow \quad \mathbf{F}_{\varepsilon\varepsilon}^{(e)}\mathbf{q}_\varepsilon^{(e)} + \mathbf{E}_{\sigma\varepsilon}^{(e)\mathrm{T}}\mathbf{q}_\sigma^{(e)} = \mathbf{0},$$

$$\frac{\partial I_{HW}^{(e)}}{\partial \mathbf{q}_\sigma^{(e)}} = 0 \quad \rightarrow \quad \mathbf{E}_{\sigma\varepsilon}^{(e)}\mathbf{q}_\varepsilon^{(e)} + \mathbf{G}_{\sigma u}^{(e)}\mathbf{q}_u^{(e)} = \mathbf{0},$$

$$\frac{\partial I_{HW}^{(e)}}{\partial \mathbf{q}_u^{(e)}} = 0 \quad \rightarrow \quad \mathbf{G}_{\sigma u}^{(e)\mathrm{T}}\mathbf{q}_\sigma^{(e)} = \mathbf{f}_u^{\hat{p}(e)} + \mathbf{f}_u^{\hat{t}(e)},$$

$$\begin{bmatrix} \mathbf{F}_{\varepsilon\varepsilon}^{(e)} & \mathbf{E}_{\sigma\varepsilon}^{(e)\mathrm{T}} & \mathbf{0} \\ \mathbf{E}_{\sigma\varepsilon}^{(e)} & \mathbf{0} & \mathbf{G}_{\sigma u}^{(e)} \\ \mathbf{0} & \mathbf{G}_{\sigma u}^{(e)\mathrm{T}} & \mathbf{0} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{q}_\varepsilon^{(e)} \\ \mathbf{q}_\sigma^{(e)} \\ \mathbf{q}_u^{(e)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{f}_u^{\hat{p}(e)} + \mathbf{f}_u^{\hat{t}(e)} \end{bmatrix}$$

we obtain a set of three equations describing the last model

$$\mathbf{F}_{\varepsilon\varepsilon}^{(e)}\mathbf{q}_\varepsilon^{(e)} + \mathbf{E}_{\sigma\varepsilon}^{(e)\mathrm{T}}\mathbf{q}_\sigma^{(e)} = \mathbf{0},$$

$$\mathbf{E}_{\sigma\varepsilon}^{(e)}\mathbf{q}_\varepsilon^{(e)} + \mathbf{G}_{\sigma u}^{(e)}\mathbf{q}_u^{(e)} = \mathbf{0},$$

$$\mathbf{G}_{\sigma u}^{(e)\mathrm{T}}\mathbf{q}_\sigma^{(e)} = \mathbf{f}_u^{\hat{p}(e)} + \mathbf{f}_u^{\hat{t}(e)}.$$

III) In the beginning we assume that fields $\mathbf{u}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$ are independent. The appriopriate Euler equations are responsible for: i) the physical equation relating $\mathbf{q}_\varepsilon$ and $\mathbf{q}_\sigma$, ii) the kinematic equation - compatibility condition for strain and displacement fields represented by $\mathbf{q}_\varepsilon$, $\mathbf{q}_u$, iii) the equlibrium equation described by $\mathbf{q}_\sigma$, $\mathbf{f}_u^{\hat{p}}$, $\mathbf{f}_u^{\hat{t}}$.

IV) In Fig. 11 we can see that no connections are present in the box related to the inner point P. These constraints are transferred to the FE level writing the relations in three boxes, which are enforced on displacement, strain, stress DOFs.

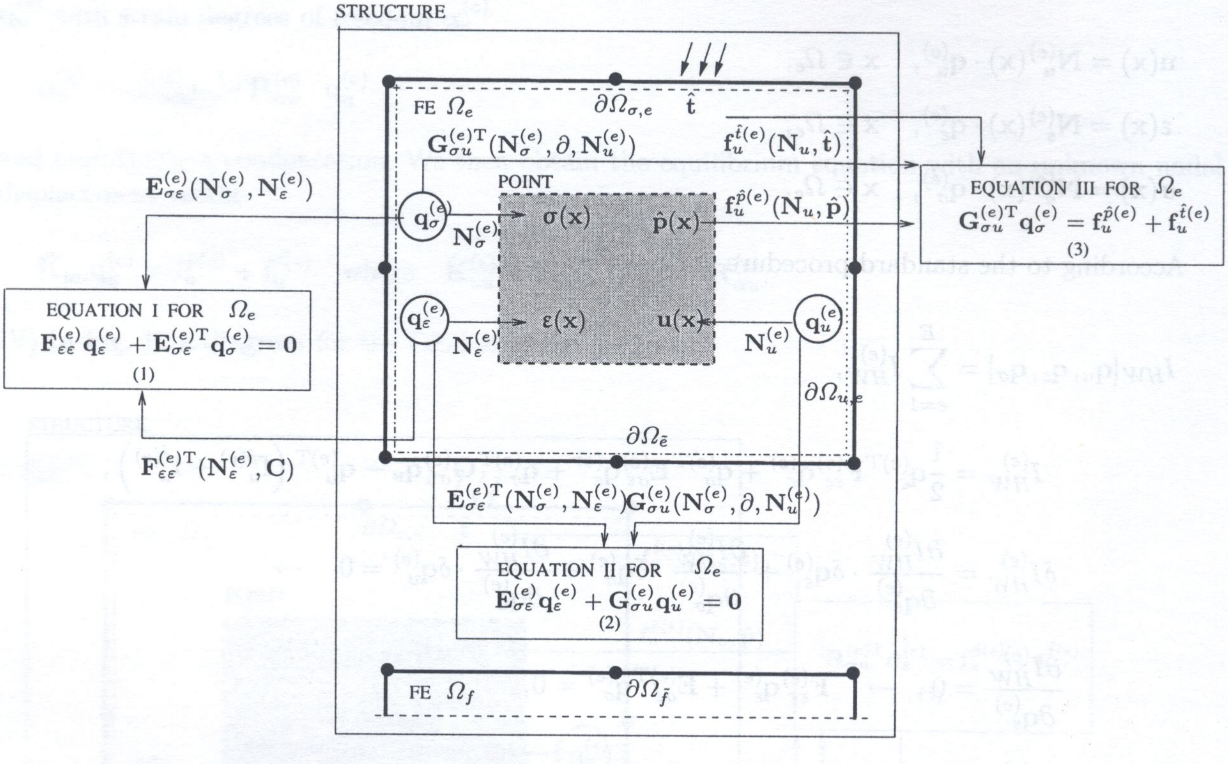

Fig. 11. Diagram for displacement-stress-strain (mixed) model – DSE-FE

3. Final remarks

The paper contains a presentation of nine FE models in a uniform manner. The description includes the statement of basic relations and their graphic vizualization in diagrams initiated by N-E. Wiberg in paper [1]. The diagrams give a possibility of confrontation of similarities and differences in the formulations of various FE models.

In two-field, three-field and hybrid approximation we have to deal with larger size algebraic problems. The equations resulting from these formulations have zero diagonal terms. In fact, the structure of algebraic problem is connected also with the number of degrees of freedom for the approximated fields, mesh density and boundary conditions. The mathematical criteria of solvability are given among others in [5]. To avoid singularity certain necessary and sufficient conditions must be complied with [5]. In some formulations the continuity requirements imposed on selected shape functions are difficult to satisfy and the problem is solved in many ways. However, in mixed models at the stage of functional formulation continuity requirements are relaxed and transferred from one to another field, using Green's theorem, i.e. integrating by parts [4, 5, 6].

The problem of convergence to the true solution (which is interesting for mathematicians and engineers) is related to the type of variational principle (saddle-point or extremum of a functional).

In the paper the presentation of a set of formulations in diagram form has been proposed with the aim of a better understanding which equations are incorporated in the functional to eliminate certain fields, which are satisfied *a priori*, which by shape function, and which at the end – in a variational manner.

In the description of some FE types the method of taking into account nonhomogeneous static and kinematic boundary conditions is presented.

Last of all, it should be noted that the investigations of optimal finite-element approximation for plates and shells still stimulate the development of mixed and hybrid/mixes FEs [6].

Acknowledgments

The author is indebted to doctor Jerzy Pamin for helpful remarks and discussions during the preparation of this paper.

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