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Optimality of grids based on a combined r-h adaptive strategy

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A critical review of earlier works on optimality of finite element grids has been made. A material force method of r-adaption to obtain optimal initial grids has been described. The study focuses on determining the configurational driving force and its convergence rate across an interior patch node for one-dimensional linear, quadratic element and two-dimensional bilinear quadrilateral elements. Numerical implementation is made on one and two-dimensional problems. Various aspects considered to define optimality in earlier works along with their predefined guidelines have also been worked out with some modifications for the material force method and it is shown that this method of adaption provides good optimal grids. The method is advantageous owing to its physical basis and mathematical vigor than earlier works. Based on the numerical studies conducted a combined adaptive strategy incorporating node disposition and mesh enrichment has been evolved to obtain an optimal mesh for a specified accuracy.

Notations

$e(x)$ $\|e(x)\|$	error function, norm of error function,
$u(x)$	exact solution of displacement,
$u_h(x)$	finite element solution of displacements,
N	interpolation function used for FE approximation,
$\hat{u}_h(x)$	smoothed or recovered finite element solution,
h_i	element size,
c	relaxation and correction factors,
f, q	body forces and traction terms,
Ω, Ω_i	domain and elemental volumes,
$\Gamma = \Gamma_D \cup \Gamma_N$	domain boundary union of Dirichlet and von Neumann boundaries,
$\Psi(u_i, X_A)$	deformation mapping function,
X_A, x_A	referential and present coordinates,
F_{iA}	deformation gradient,
$W(u_{i,j}, x_k)$	strain energy density,
σ_{ij}, C_{ij}	Cauchy stress tensor and energy momentum tensor,
g_k	configurational body forces,
G_e^I, G^k	elemental and assembled nodal configurational force,

G_x1, G_x2	configurational force at successive present configurations,
$\| \triangle G_i \|$	norm of change in configurational force,
η	relative error norm percentage.

1. Introduction

The theory of adaptive finite element grid design procedures, which can assure that the approximated finite element solution to a given problem is accurate enough together with minimum computational cost, is of importance. Such adaptive grid design procedures aim at defining optimal grids with respect to an initial and adapted mesh along with an appropriate choice of mesh adaptive techniques that can optimize a spatial discretization. The obligation is thus to obtain an optimal mesh in which the number of degrees of freedom is minimal for a specified accuracy, that are distributed in such a manner that estimated error in the solution is uniformly distributed over the mesh and potential energy is a minimum. The necessity of mixing numerical grid generation and adaptive methods is well established. The quality of finite element solution may thus be improved by optimizing the disposition of nodes. It is also possible to improve the quality of existing meshes iteratively using predefined guidelines for redistribution of nodes. To link optimality of an adapted mesh with accuracy the criterion for node disposition needs to be defined with respect to a given norm or measure of error. One such criterion could be that the estimated error from the current finite element solution can be transformed into a spatial distribution of the mesh parameters such that the error is equally distributed, resulting in an adapted optimal mesh. The mathematical formulation of optimum grid design problem defined to minimize the maximum value of error of a finite element approximation lacks uniqueness in procedures, hence it is essential to study various kinds of error measures of finite element approximations which reflect the nature of solution together with the extent of accuracy required in design.

Of the available adaptive strategies the r-version of refinement strategy, henceforth called r-adaption technique, is of interest to predict the characteristics of the optimal mesh. Here the nodes of the discretized domain are relocated iteratively in order to minimize the discretization error while preserving the number of unknowns and order of approximation of the field variable. Typically, the mesh density increases near the regions of steeper gradients of the field variable. In the focus of the present study the adaptation criteria for r-adaption may be classified according to the procedure used for node relocation. The classification includes firstly a direct approach, which seeks to equally distribute the error metric based on norm associated with the Hessian of the displacement field while the other considers the amount of departure from the material force equilibrium as a measure of error. A performance analysis on these methods has been made in an earlier work by authors [11] and it has been shown that the material force (configurational force) method is more robust. This study focuses on possibility and efficiency of using such a methodology for obtaining optimal meshes. In the r-adaption based on material force the imbalance in material equilibrium is considered as a measure of error. This departure from material equilibrium, which is reduced by minimization of the potential with respect to nodal coordinates is looked upon as an optimality criterion and is accomplished by relocation of nodes in finite element mesh. The material force equilibrium results in defining energy momentum tensor in material space [29, 24] whose components represent the change of total potential energy of a deformed body produced by unit material translation. For a homogeneous body, in the continuous case, force balance implies material force balance. However, in the discrete case nodal force balance does not imply nodal material force balance due to the presence of nodes and hence element interface. Thus in a discretized form considering the material force equilibrium the non-vanishing of the divergence of energy momentum tensor at the inter element boundaries is taken as an error indicator [11, 21, 22]. In the process of relocation the errors due to discretization or approximation occurring at the nodes are typically equally distributed for better solution over the entire domain. Although only an improvement of existing solution is possible through r-adaption the ultimate aim of achieving a specified accuracy can be realized by successive mesh enrichment. Further an adaption based on Material forces tends to result in bad shape elements and approximation. To get better Finite element solution a change in the topology of the domain is quite eminent after completion of mesh adaption iterations. Thus the aim of adaptive post processing technique to obtain softer discretization, along with stationary value of potential to get better displacement or stress solution across element boundaries with a good mesh is achieved through a combined refinement strategy. The present study as an extension to the earlier work thus focuses on obtaining a mesh with minimal number of degrees of freedom for a specified accuracy. In other words the focus is on exploring the possibility and efficiency of defining optimal final adapted mesh using a combined r-h adaptive strategy, with r-adaption based on configurational force method together with h-refinement based on discretization error.

In this paper firstly the focus is on a critical review of various aspects of optimality of finite element grids. The second section describes the use of the Configurational force method as a case for producing optimal initial meshes. The subsequent section discusses the use of a combined r-h strategy for obtaining optimal adapted final meshes for a specified level of accuracy. It has been demonstrated that some of the guidelines suggested from earlier works on mesh optimality are found to perform well when applied to configurational force method. In the last section numerical results are presented and discussed. The present study largely owes in its critical review to previous extensive and thorough studies on numerical grid generation methods reported in literature right from the earliest pioneering works of Carroll and Barker [3, 4] on grid optimality in one dimension to the present state of art in designing optimal finite element grids.

2. Critical review on optimality of finite element grids

Most rules for grid optimization are related to the minimization of a specific quantity or objective function, selected as the measure of quality of grid. Optimization of finite element solution as an algorithmic procedure for the generation of a finite element discretization yields the required accuracy for the minimum amount of effort [33]. There are two fundamental approaches for the optimization of finite element solution namely analytical and topological. The analytical methodology includes the nodal coordinates as unknowns in the equations of potential energy and therefore posses two difficulties namely high nonlinearity of equations and nonlinear constraints of the nodal coordinates thus making the solution tedious and time consuming. The topological investigation based on optimal mesh configurations has resulted in guidelines which enable the analyst to lay out a mesh configuration that provides a mesh topology with characteristics similar to that of optimum mesh for his specific problem.

In the earliest work on grid optimality [3, 4] it is shown that in utilizing the displacement formulation, the true minimum of potential energy Π must consider idealization of geometry as a primary parameter. As a consequence apart from equilibrium equation another residual equation (1) involving the gradients of the stiffness matrix and load vector resulting from changes in the idealization is considered. Thus the emphasis is to produce a true minimum on $\Pi = \Pi(u_i, l_k)$ necessitates not only considering the equilibrium equation $\partial\Pi/\partial u_i$ but also has the additional equation in geometry l_k.

$$\frac{\partial \Pi}{\partial l_k} = \frac{1}{2} u_i \frac{\partial K_{ij}}{\partial l_k} u_j - \frac{\partial P_i}{\partial l_k} u_i = r_k = 0, \tag{1}$$

where l_k is the length, K_{ij} is the stiffness coefficient, r_k is the residual vector, P_i is the applied load. The theorem that a necessary consequence of the following refinement scheme is that $\Pi_n \geq \Pi_{n+1} \geq \Pi_{n+2} \ldots \geq \Pi_{n+m} \geq \Pi_{\text{exact}}$, where m represents successive refinements of initial finite element mesh, n is the existence of an optimum sub-division such that $\Pi_{n+m}(l_i^*) \leq \Pi_{n+m}(l_i)$, where l_i^* designate the optimum mesh configuration. Although some guidelines are suggested based on numerical experiments on one dimension, these rules lack physical basis and aim of obtaining an optimal adapted mesh for a specified accuracy has not been addressed.

Similar work on introduction of nodal coordinates as independent unknowns in addition to displacements leading to a set of nonlinear equations in obtaining optimal grids have been reported [36].

An analytical treatment becomes intractable with regards to obtaining explicit expressions for derivatives of the functional and the solution of the system of nonlinear equations. Consequently this has lead to the use of numerical optimization procedures. The criterion for stationarity here is that all principle minors of the Hessian be positive i.e. $\partial^2 \Pi_{p2} > 0$ where Hessian is defined as

$$H = \begin{vmatrix} \dfrac{\partial^2 \Pi_{p2}}{\partial x_L \partial x_M} & \dfrac{\partial^2 \Pi_{p2}}{\partial x_L \partial \triangle_M} \\ \dfrac{\partial^2 \Pi_{p2}}{\partial \triangle_L \, \partial x_M} & \dfrac{\partial^2 \Pi_{p2}}{\partial \triangle_L \, \partial \triangle_M} \end{vmatrix} \tag{2}$$

The improvements in stress are observed through isobars, isostatics and isochromatics which are in turn used for arriving at predefined guidelines to obtain optimal meshes. The use of isoenergetics to indicate the proper contour shape along which an edge of element is to be aligned is quite justified [39] especially in case of a more general and complex problem where the computational effort in using the optimization algorithm would become untenable.

The emphasis is on minimizing the total potential energy and hence the consistent response function with respect to potential energy formulation would be the strain energy density function. This further leads to question of examining the optimal nodal strains. An averaging of the adjacent element strains would result in different solution especially when the attributes associated with elements are different. A first note on optimal girds taking in to account the element attributes has been reported in one of the earlier works [27]. In numerical treatment to consider the geometry of the element a tapered axially loaded rod with area at any section (for k-th element) A_k (where $A(\xi) = A_0(1 - c\xi)$) is considered and two conditions on potential namely $\partial \Pi / \partial u_k$ and $\partial \Pi / \partial A_k$ are imposed to arrive at the equilibrium and optimality conditions as

$$\begin{aligned} &A_k \varepsilon_k - A_{k+1} \varepsilon_{k+1} = 0, \\ &A(\xi_k) = \{A(\xi_{k-1}) A(\xi_{k+1})\}^{0.5}, \end{aligned} \tag{3}$$

where ε_k and ξ_k, are the strain and nondimensionalized coordinates. It has been shown that the elongation of the k-th element for $k = 1$ becomes $\lambda_k = \dfrac{2P\xi_1}{EA_0(2 - c\xi_1)}$ which is independent of k, and thus each element has a constant strain energy, namely $P\lambda/2$, with the displacement forming an arithmetic series $\left(u(\xi_k) = -\left(\dfrac{kPl}{EA_0 n} \right) \ln(1 - c) \right)$. The important fact pointed out is that the optimal nodal displacements relative to the exact solution have the same error at all nodes. Even though this forms the first contribution to extrapolation of characteristics of optimal girds but it is quite evident that it is because of the linear variation of taper of rod that is considered. Further based on earlier works [27, 36] it has been pointed out that optimized nodal strains have a larger percentage error relative to the exact solution in the low strain gradient region than that for unoptimized solution and this is attributed to the fact that the variable nodes were being drawn in to high strain gradient areas during the process of minimizing the potential energy [35]. The concept of equidistribution of strain energy that holds good for a one dimensional linear taper bar as given in earlier works [27] need not be true for other bars with different taper factors.

Other earlier works on grid optimality include the works by Carlos Felippa [8, 9] where in the grid optimization problem is studied in the case of grids of similar topology having a fixed number of degrees of freedom per node. A general formulation based on weighted residual error measures is specialized to field problems associated with a positive definite energy functional the minimization of which with respect to variable node locations is adopted as a grid optimality criterion.

It is observed that an optimized mesh is twice as efficient as an evenly divided mesh in terms of number of degrees of freedom required to produce same accuracy [36]. This increase in efficiency was closely studied by considering a modified problem by allowing a limited number of degrees of freedom to be introduced during the optimization of mesh [23]. The optimization procedure consisted of measuring the effect of successively introducing more number of degrees of freedom (and elements) in the region occupied by one element. This may be looked as a first step to look at optimality of an enriched mesh with a specified degree of accuracy but no criterion for introduction of degrees of freedom and number of degrees of freedom to be introduced are given. Further it has been shown that the element energy difference or the *element energy differential* which is a measure of introduction of degrees of freedom provides a quantitative measure of the efficiency of a given mesh and a qualitative measure for selecting further mesh refinements.

$$\frac{\Delta \Pi}{\Delta \bar{u}} = \frac{1}{2} < u_c > [K]\{u_c\} - \frac{1}{2} < u_r > [K]\{u_r\}, \tag{4}$$

where $\Delta \Pi / \Delta \bar{u}$ is the energy difference, u_c is the vector of displacements for the current mesh, and u_r is the vector of displacements for reduced number of degrees of freedom. An indirect measure of efficiency and possible accuracy of finite element mesh is presented based on Specific Energy Difference (SED). This provides a measure of strain energy contributed by the higher modes of displacement of element. From a theoretical view point this measure is useful in indicating the quality of nodal distribution with respect to discretization error for all higher order continuum elements irrespective of element type. The SED is determined as the difference between the specific strain energy at any point and specific strain energy at centroid of element. With refinement SED approaches zero and when total SED does not change with refinement in a particular region of structure with two solutions with different mesh refinements in that region then exact solution is available. Some grid optimization criterions are based on variations in the SED [23, 27]. The subsequent advances made [33, 34] include the use of results of previous analysis to improve idealization. Unlike other procedures a new mesh is generated using computer graphics. Since the criterion methods operate exclusively on solution based information the majority of the computational effort per iteration is in the reanalysis. Here it is argued that the best finite element solution is one that best approximates the strain energy. The solution is associated with best possible approximation to the integrand of the strain energy, the SED. In order to reduce the error involved in approximation; the authors propose that nodes should be distributed in such a way that the difference of SEDs between selected neighboring nodes is made constant.

The most productive method of grid enrichment is one which is not constrained to the previous mesh. There is a substantial amount of effort required in defining the initial mesh and for procedures defining a new mesh in each cycle this substantial effort must be repeated in each cycle. In particular to minimize the difference between total strain energy of finite element solution and the exact solution, integral approximation tells us that we must obtain the best possible approximation to the integrand of the strain energy integral which is SED. Since the definition of a near optimum finite element mesh depends on the variation in SED, and not the SED values themselves, it would be appropriate to employ any procedure that can identify areas of high SED variations. One semi intuitive procedure for doing this was attempted but it was found that such procedures would not work for general problems. Therefore it was concluded that some form of initial analysis was required. The basic requirements of this approach are the definition of an initial finite element mesh. The performance of the analysis and the extraction of the data required to properly place the *key nodes* on the boundary curves. In the works discussed no study is made on the procedure adopted for node relocation and convergence rates.

Grid optimization techniques are related to the minimization of a specific quantity or objective function, selected as the measure of quality of grid. The error associated with the interpolation of the true solution using functions from the finite element space forms the objective function [6]. This is justified because interpolation error bounds the finite element error. Element lengths and area are selected as design variables respectively in one and two dimensional problems. The necessary conditions for a minimum of this problem appear in a form that are unsuitable for computations but they can be successfully replaced by more manageable equations that become the desired guidelines. In case of one dimensional grid it is required to find the vector of element lengths $h = (h_1, h_2, h_3, ...h_M)$.

$$\text{minimize } B^2(u,h) = \sum_{k=1}^{M} h_k^{\beta} |u|^2_{k+1,k} \quad \text{subject to} \quad \sum_{k=1}^{M} h_k = 1 \quad \text{and} \quad h_k \geq 0. \tag{5}$$

Using grading functions $p(t)$ the disposition of the nodes on the domain can be defined and $B^2(u,h)$ can be expressed as a functional with $p(t)$ as dependent variable. Considering the optimality of p we get $f_K = h_K^{\beta} |u|^2_{k+1,K} = \lambda$ a constant, $K = 1, 2, ...M$, which helps us to define the true optimum h^* and thus above equation can be written as, $B^2(u,h) = B^2(u,h^*)(1 + O(h))$ where $h = \max_K\{h_K\}$. For node relocation an iterative algorithm is developed to modify existing grids to satisfy the condition $f_K = A_K |u|^2_{k+1,K} =$ constant. This method is not computationally efficient and takes considerable number of iterations. In two dimensions the motion of a node is determined by the magnitude of the f_K in the elements surrounding the node which is developed from the idea of assigning a mass like quantity to geometric centroid of each element with the relocation of mesh nodes in a direction towards the center of mass.

$$x_n^{\nu+1} = \frac{\sum_{K \in \{N\}} \overline{X}_k^{\nu} (F_K/A_K)^{\nu}}{\sum_{K \in \{N\}} (F_K/A_K)^{\nu}}, \tag{6}$$

where $x_n^{\nu+1}$ is the new location of the n-th node, $\{N\}$ is the set of all elements that contain x_n, $\overline{X}_K^{\nu}$ is the location of geometric center of K in the ν-th iteration. The authors have shown no proof for convergence of solution. Improvement per iteration depends upon the behavior of solution u and on the initial grid. Also the condition of functional being constant has not been satisfied exactly. Remembering that the solution to which the mesh is being adapted is inertial, relocation of nodes new, non integer positions will effectively move the associated function $(F_K/A_K)^{\nu}$ away from its correct inertial location. An effort to converge this iterative process or to relocate the mesh to resolve rapidly moving features will usually require redistribution of F_K as the grid node position changes, in the same manner that the solution itself is redistributed. Also the overall computational work expended will depend directly on the number of iterations.

Another form of looking at the optimality condition is that of the variations in the stain energy density (SED) and Degree of freedom density (DOFD). Along any selected path in a mesh these curves are plotted together. The condition for the optimum finite element solution is satisfied when the SED curve coincides with the DOFD curve [13]. This is based on an iterative procedure by examining the variation of total strain energy with change in length ratio along any selected path. Selection of computation path and methodology for computation of DOFD along the path are heuristic.

Some gird optimization techniques are based on minimization of the trace of stiffness matrix [18, 19]. A significant decrease in the potential energy from that of a uniform mesh is found for structural problems. In deriving the condition for optimality for the tapered bar in terms of length ratios it has been shown that for a uniform thickness bar all the meshes are optimal and the trace would be minimum which may not be true for an axially loaded tapered bar (arithmetic series). Similar works on finding optimal grids for structural problems such as transversely loaded beam have been reported [15]. Although the nonlinear equations are arrived at in this work, the exact numerical procedure and algorithm for solving the nonlinear equations have not been indicated. The above critical review on optimality of initial mesh is limited, there are other notable works discussing various grid adaption algorithms and issues [32].

The optimality of an adapted final mesh with a specified accuracy is important. The introduction of a concept of adaptive grid design for finite element analysis by combining numerical grid generation methods and adaptive finite element methods where in the development of a finite element model is considered as a design problem similar to structural optimization is notable [17]. There is report on use of a geometry based approach for grid optimization [14]. The advantage and disadvantages of grid optimization through repositioning finite element nodal points has been dealt and some limitations such as distortion, geometric complexity, and convergence problem are examined and overcome through mesh enrichment by h-refinement. A uniform mesh is considered for implementing the r-adaption for a one dimensional problem. It is expected that an r-adaption procedure should not result in change in topology of the domain. It is observed that the adaption procedure based on energy norm of error results in topology change. A progressive halving procedure has been adopted for mesh enrichment which may not necessarily lead to optimal adapted meshes [14].

Though not from a grid optimization point of view there has been some earlier works on r-adaptive strategy [1] which are based on sub domain division for either r- or h on the divided domain with no refinement on the interface of these domains. Although the sub domain division is based on desired element size, these are not based on error estimators and there seems to be less mathematical vigor for refinement strategies adopted. The interface between domains poses topological constraints on h-adaptive remeshing as well as on the r-adaptive remeshing. There has been report on use of r-h-p refinement strategies [18, 19] but the adaption procedure lacks robustness. A combined r-h adaptive refinement strategy has been proposed and formulated in an earlier work by authors [11]. The r-adaption is based on configurational force method. The h-refinement is based on Zienkiewicz–Zhu error estimator. The refinement strategy takes care that the initial topology is maintained to an extent and is based on an efficient refinement criterion. A combined h- and r- refinement would yield useful results only if the adaption is performed in cycles in succession for every new mesh. The combined r-h strategy resolves the unhealthy mesh distortions and provides better convergence of the solution. The following section describes the r-adaption based on configurational force approach with appropriate algorithms for node relocation as a case for obtaining optimal initial mesh. The h-refinement strategy based on best guess stress type estimator has also been discussed, along with a brief description of refinement strategy. The next section deals with the numerical implementation of combined r-h strategy for linear one and two dimensional problems in structural mechanics as a case to obtain optimal adapted meshes. The last section describes the effectiveness in use of various predefined guidelines along with configurational force method to obtain optimal meshes.

3. Optimal grids based on material force method

The general procedure to obtain optimal girds based on material forces and error estimates is indicated in Fig 1a. The r-adaption is based on a method of achieving material force equilibrium. The departure from material force equilibrium is considered as a measure of error and is reduced by minimization of the potential with respect to nodal coordinates. This is accomplished by relocating nodes in a finite element mesh. Thus, considering the material force equilibrium defines the energy momentum tensor in material space [2, 24]. The components of the energy momentum tensor represent the change of total potential energy of a deformed body produced by unit material translation. For a homogeneous body, in the continuous case, force balance implies material force balance. However, in the discrete case nodal force balance does not imply nodal material force balance due to the presence of nodes and hence element interface. Thus in a discretized form considering the material force equilibrium the non-vanishing of the divergence of energy momentum tensor at the inter element boundaries is taken as an error indicator. In this section the error indicator is derived by considering the material force equilibrium in a similar manner as is done in physical equilibrium. The displacement vector for a solid in the region $\Omega_0 \in \Re^3$ in referential description is given as $u_i = x_i - X_I$. The deformation mapping for a lagrangian description is defined as

$$x_i = x_i(X_1, X_2, X_3, t) \Rightarrow x_i = x_i(X_I) \quad \text{or} \quad \Rightarrow \Psi(u_i, X_A) \tag{7}$$

with the displacement gradient in Ω_0 given by $F_{iA} = \delta_{iA} + u_{i,A}$. The deformation mapping takes on prescribed values $\bar{\Psi}$ over the displacement part of Γ ($\Gamma = \Gamma_D \cup \Gamma_N$) of the undeformed boundary.

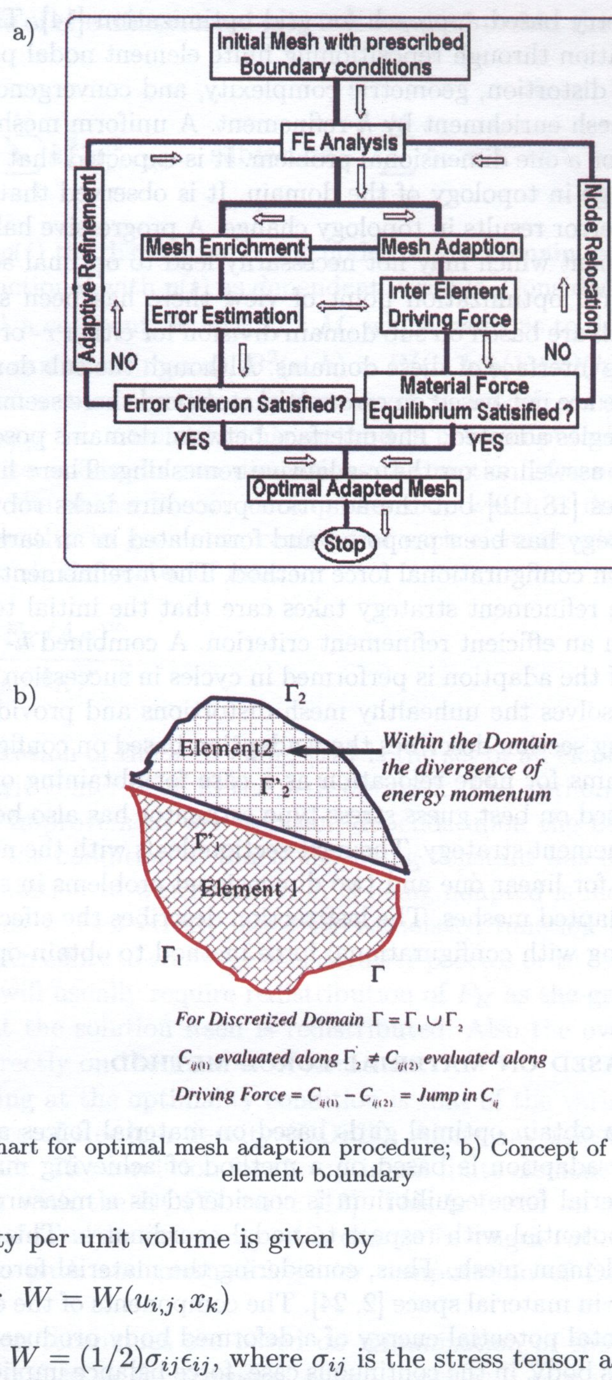


Fig. 1. a) Flow chart for optimal mesh adaption procedure; b) Concept of driving force at inter element boundary

The strain energy density per unit volume is given by

$$W = W(x^i_{,I}, X^K) \quad \text{or} \quad W = W(u_{i,j}, x_k) \tag{8}$$

for an isotropic material $W = (1/2)\sigma_{ij}\epsilon_{ij}$, where σ_{ij} is the stress tensor and ϵ_{ij} is the strain tensor. The physical equilibrium equation is given by $\sigma_{ij,j} + f_i = 0$, where f_i is the body force. The material gradient of strain energy results in the configurational force equilibrium and is given by

$$C_{kj,j} + g_k = 0, \tag{9}$$

where the configurational stress tensor is given by $C_{kj} = W\delta_{kj} - \sigma_{ij}u_{i,k}$ and the configurational force arising due to body forces are given by $g_k = -f_i u_{i,k}$. It is required to compute the discrete configurational forces arising out of discretization. In the absence of body forces the weighted residual form of the balance law equation using a vectorial test function h and integrating over the domain Ω_0 is given by

$$\int_{\Omega_0} C_{ij,j}\eta_i \, d\Omega_0 = 0. \tag{10}$$

The weak form in the absence of body forces obtained by integrating by parts can be written as

$$-\int_{\Omega_0} C_{ij}\eta_{i,j}\,\mathrm{d}\Omega_0+\int_{\Gamma} C_{ij}n_j\eta_i\,\mathrm{d}\Gamma+\int_{\Gamma_e\notin\Gamma} C_{ij}n_j\eta_i\,\mathrm{d}\Gamma=0. \tag{11}$$

As a consequence of considering stationary boundaries the test function η vanishes on the boundaries of the domain Γ and hence the second term becomes zero. The divergence of the energy momentum tensor is zero for a homogeneous body without body forces. This is used to check the discrete solution obtained through finite element analysis. As finite element results are approximate solutions the non-vanishing divergence of the energy momentum tensor provides an error indicator. The discrete jump in the energy momentum tensor (see Fig. 1b) occurring at the element boundaries Γ_e (third term of the weak form equation) is the driving force used as an error measure in the node relocation process. The balance law in its weak form is analogous to the bilinear form of the governing differential equation with jump in the energy momentum tensor being similar to the traction jump occurring at the inter element boundaries. The discretized form of the above weak form can be written by inserting an element wise interpolation of the test function η and its gradient. Thus we can write

$$\eta_i=\sum_I N^I\eta_i^I \quad\text{and}\quad \eta_{i,\,j}=\sum_I N^I_{,\,j}\eta_i^I. \tag{12}$$

Thus Eq. (12) reduces to the form

$$\sum_I\left[-\int_{\Omega_0} C_{ij}N^I_{i,j}\mathrm{d}\Omega_0+\int_{\Gamma_e\notin\Gamma} C_{ij}n_jN_i^I\,\mathrm{d}\Gamma\right]\eta_i^I=0. \tag{13}$$

The second term which is a traction related to the discontinuity is the configuration force of the element that needs to be numerically evaluated. Since η_I are arbitrary, each of the above in the summation over I should go to zero. Thus, the first term is equal to the negative of the second that is evaluated as the discrete configuration force, G_e^I given by

$$\int_{\Gamma_e\notin\Gamma} C_{ij}n_jN_i^I\,\mathrm{d}\Gamma=\int_{\Omega_0} C_{ij}N^I_{i,j}\mathrm{d}\Omega_0=\begin{Bmatrix} G^I_{e\,1}\\ G^I_{e\,2}\end{Bmatrix}=G_e^I. \tag{14}$$

These configurational forces on assemblage should go to zero in the domain. This also means that the configurational forces of the elements are equidistributed for an optimal mesh. The assembled total configurational force is given by G^k. The assemblage spans over the number of elements connected to a particular node. The driving force terms across an interior patch node have been derived for various types of element and are presented in the results and discussion section. The general procedure is indicated in Fig. 2b.

3.1. Node relocation procedure

The interior nodes are updated by an iterative rule such as $X^K = X^K - c\ G^K$. The constant c is chosen sufficiently small to achieve convergence (to avoid unhealthy mesh distortions). For better convergence a nonlinear conjugate gradient method, known as Polak–Reberie method [37] for minimization of energy function has been incorporated [11]. The algorithm has been explained in Fig. 2a. The method has two levels of iteration. The outer loop is the undeformed coordinate iterative update or nodal coordinate update. Nodal coordinate iterative loop contains solution for equilibrium solution for deformed coordinate for a fixed mesh. This ensures that the configurational forces for undeformed coordinate update correspond to equilibrium solution. A linear projection method, which is similar to the successive over relaxation iterative technique, shows better convergence [11].

a)

Steps in Polak Reberie Algorithm

Step 1: For the assumed initial mesh, determine the response, perform equilibration and determine the material force. Let us presume the initial system is X_0. *Thus for the known value of* X_0, *evaluate the function* $G_0 = G(X_0)$. *Let* $d_0 = -G_0$.

Step 2: Choose a Small Value of arbitrary Correction factor (σ) *that is smaller than element size. Find* $G(X_k + \sigma d_k) \rightarrow G_\sigma$ *(say)*

Step 3: Compute $c_k = -\sigma * (G_k^T . d_k) / (G_\sigma^T . d_k - G_k^T . d_k)$ *Update* $X_{k+1} = X_k + c_k\, d_k$

Step 4: Compute G_{k+1} *using the updated* X_{k+} *Set* $\beta_k = G_{k+1}^T (G_{k+1} - G_k) / (G_k^T G_k)$ *Update* $d_{k+1} = -G_{k+1} + \beta_k\, d_k$

Step 5: Check: Is convergence or maximum Iterations reached? If true then End the iterations. Else, *Loop over the index k =0,1,2….N (i.e. till convergence is reached) Go to Step 2.*

b)

Algorithm for Determining the Driving force

1) For the element under consideration assume the Displacement variation and variation of the test function

Displacement function $\hat{u} = u \cong u_h = \sum_{e=1}^{N}\sum_{i=1}^{n} u_i^e \psi_i^e = \sum_{I=1}^{M} U_I \Phi_I$

ψ_i^e = *Basis function and* u_i^e = *nodal values*

Test function $\eta_i = \sum_I N^I \eta_i^I$ *and* $\eta_{i,j} = \sum_I N^I{}_{,j}\, \eta_i^I$

N^I = *Basis function and* η_i^I = *Nodal values*

2) Evaluate the Configurational force tensor

$C_{kj} = W\delta_{kj} - \sigma_{ij} u_{i,k}$ *Where W = Strain Energy*

σ_{ij} = *Stress Tensor and* $u_{i,k}$ = *Displacement Gradient*

3) Evaluate the Driving force at an k^{th} *interior patch node*

$$\int_{\Gamma_e \not\subset \Gamma} C_{ij}\, n_j\, N_i^I d\Gamma = \int_{\Omega e} C_{ij}\, N^I_{,j} d\Omega_0 = \begin{Bmatrix} G_{e\,1}^{\,I} \\ G_{e\,2}^{\,I} \end{Bmatrix} = G_e^{\,I} \text{ and } G^K = \bigcup_{e=1}^{ne} G_e^I$$

Fig. 2. a) Steps in Polak Reberie algorithm; b) Algorithm for finding the driving force

4. Combined refinement strategy

It is observed that there is no change in the topology of the domain when corrections are made for configurational forces. There is only an increase or decrease in the element size h_i. The aim of adaptive post processing technique is to obtain softer discretization, along with stationary value of potential and to get better displacement or stress solution across element boundaries with a good mesh. The criteria for goodness of mesh are based upon strain energy, displacement and stress values at selected critical points of a structure. An adaption based on material forces tends to result in bad shape elements and approximation. This is from the understanding that the displacement polynomial approximation made within the element assumes extreme values at the nodes. To get better finite element solution we need to change the topology of the domain once the stationary value of the potential is reached after completion of mesh adaption iterations. Furthermore an optimal mesh is one in which the number of degrees of freedom is minimal for a specified accuracy. This can be achieved only through mesh enrichment. The process of adaption and enrichment may follow one another as one single cycle or may be repeated in cycles. The h-adaptive strategy is based on best guess type or Zienkiewicz–Zhu (1987) error estimator [40]. The general form the estimator is as given below

$$\|e\| = \sqrt{\left\{\int_{\Omega} \{e_{\sigma}\}^{T} | C|^{-1}\{e_{\sigma}\} \mathrm{d}\Omega\right\}}, \tag{15}$$

where $\{e_{\sigma}\} = \{\sigma^{*}\} - \{\sigma_{h}\}$ and, $\{\sigma\}^{*}$ $\{\sigma_{h}\}$ are the best guess stress and finite element stress respectively. The best guess stress is obtained through a simple projection technique as given by Zienkiewicz–Zhu (1990). The absolute value of the error over domain is calculated. The global percentage error η is given by

$$\eta = \frac{\|e\|}{\|u\|} * 100, \tag{16}$$

where, $\|e\|$ is a suitable error norm and $\|u\|$ is the displacement norm. Since the exact displacement norm is not known, we use an approximate norm $\|u^{*}\|$. Specifying the global error in the energy norm $\bar{\eta}$ in the form of a percentage of total energy norm one can compute the permissible error in the energy norm. Thus for a uniform distribution of error one can compute the permissible error and refinement index for each element. The mesh density can thus be computed based on present size of the element, order of approximation and refinement index.

5. Optimality of grids

One of the possible approaches to the reduction of the computational costs in finite element analysis is the selection of *optimal grids* which produce *best* answers in the sense of minimizing a discretization error measure, for a fixed level of computational effort. The grid optimization problem is studied and presented in two stages. In the first stage the consideration is of grids of similar topology having a fixed number of degrees of freedom so as to arrive at an initial optimal mesh. The configurational force method of r-adaption as a case to obtain initial adapted meshes has been explained in this paper. Here the minimization of the potential energy forms the optimality criterion. In the second stage the consideration is of obtaining nearly optimal meshes through adaptive modification strategies based on h-refinement i.e to obtain grids with optimal degrees of freedom for a specified accuracy. The optimality criterion is based on uniform distribution of error in the new mesh. Combined r-h adaptive refinement strategy arising as a result of stage one and two is thus presented as a case to obtain optimal meshes.

5.1. Optimality of initial adapted mesh

In the r-adaption based on a method of achieving material force equilibrium the imbalance in material equilibrium is considered as a measure of error. This departure from material equilibrium is reduced by minimization of the potential with respect to nodal coordinates and is looked upon as an optimality criterion. This is accomplished by relocating nodes in a finite element mesh. The material force equilibrium results in defining energy momentum tensor in material space whose components represent the change of total potential energy of a deformed body produced by unit material translation. In the process of relocation the errors due to discretization or approximation occurring at the nodes are typically equally distributed for better solution over the entire domain. The configurational force method agrees well with predefined guidelines suggested based on earlier works. The adaption process results in alignment of element edges along the isoenergetics and isochromatics.

5.2. Optimality of final adapted mesh

In the process of obtaining the optimal initial mesh it is observed that there is no change in the topology of the domain when corrections are made for configurational forces. There is only an increase or decrease in the element size h_i. The aim of optimal adaptive post processing technique is to obtain softer discretization, along with stationary value of potential and to get better displacement or stress solution across element boundaries with a good mesh. An adaption based on material forces to obtain optimal initial mesh with minimization of potential energy as optimality criterion tends to result in bad shape elements and approximation. To get better finite element solution we need to change the topology of the domain once the stationary value of the potential is reached after completion of mesh adaption iterations. This results in the need for second stage of optimality in which an optimal mesh is defined for a specified accuracy. This can be achieved only through mesh enrichment. The process of adaption and enrichment may follow one another as one single cycle or may be repeated in cycles. The optimality criterion for an adapted mesh is based on an idea that on a nearly optimal mesh the estimated absolute error must be equidistributed on each element. This criterion has as its objective the definition of new mesh sizes in such a way that we have the same absolute error in each of the sub domains that define each element of the previous mesh. The global absolute error can thus be defined as

$$\|e_{es}\|_n^2 \;=\; \sum_{e=1}^{ne_p} \|e_{es}^{(e)_p}\|_n^2 \;=\; ne_p\,\|e_{es}^{(e)_p}\|_n^2, \tag{17}$$

where ne_p is the number of elements in the previous mesh. We then obtain $\|e_{es}^{(e)_p}\|_n$ using the following expression $\|e_{es}^{(e)_p}\|_n = \left[\dfrac{1}{ne_p}\right]^{1/2} \|e_{es}\|_n$ and to obtain a previously specified absolute error in all the new elements contained in each element of the previous mesh we use the equation

$$h_n^{(e)_p} \approx h_p^{(e)} \left[\frac{\|e_{es}^{(e)_p}\|_n}{\|e_{es}^{(e)}\|_p}\right]^{1/c}, \tag{18}$$

where $c = p$ which is the order of the approximation polynomial. Since the optimal adapted mesh is obtained through a combined strategy of initial and adapted meshes, it is very much required to maintain the initial topology of the domain to an extent.

6. Results and discussions

In this section we report the results of numerical tests that establish effectiveness of using the material force method for obtaining initial optimal mesh. The method has been tested for meshes of one-dimensional linear and quadratic element and in two dimensions for meshes of linear quadrilateral elements. The driving force at an interior patch node for above elements has been derived which may be directly used for node relocation. An optimal adapted mesh is obtained through a combined r-h adaptive strategy. Numerical studies have been made on implementing in the first case node relocation followed by mesh enrichment in succession Various predefined guidelines used by earlier authors for describing the characteristics of the optimal mesh have also been worked out for the configurational force method.

6.1. One-dimensional example

A linear elastic axial rod fixed at one end and free at other end which is under a uniform body force b as shown in Fig. 4a was considered. The elastic axial rod was discretized using one dimensional linear and quadratic element to obtain initial meshes as shown in Fig. 4b. A set of five nodes were considered with one node at free end to define geometry and positions of other nodes are as indicated in Fig. 4b. We choose $E = 1N/mm^2$, and $b(x) = 1 * xN/m$, which is a linearly varying load acting along the length of the bar. Mesh adaption based on configurational forces was performed with appropriate choice of correction factor. Figure 4c shows the final adapted optimal meshes obtained for quadratic and linear elements. Figure 4d shows a plot of the normalized driving force versus number of iterations. It is seen that the conjugate gradient algorithm together with a linear step projection considerably improves the convergence rates. It is observed that the convergence rates are faster in case of a quadratic element than in case of linear element although initially its is slower. This is evident from the driving force terms given in the Eq. (19) and Eq. (20). It is seen that there is not much contribution from the midside node, also since the node relocation process considers only the patch node that is connected to the adjacent element, and movement of the midside node is based on end nodes which may affect the convergence process. Figure 5e shows the plot of potential energy of the system, for the same number of node relocation iterations it is observed that the quadratic element results in a flexible discretization with a higher value of the potential than linear elements. Contrary to the near uniform meshes obtained as in case of a linear element for the same number of node relocation iterations the quadratic element doesn't result in a uniform mesh.

6.2. Evaluation of the driving force terms

Following Eq. (14) and considering that the end nodes are prevented from relocation to preserve geometry the total driving force at the mid side patch node are given below. In case of quadratic elements for evaluation of integral of Eq. (14) we resort to two point Gaussian quadrature with sampling points and weights taken as $\xi_i = \pm 0.57735$ and $W_i = 1.0$.

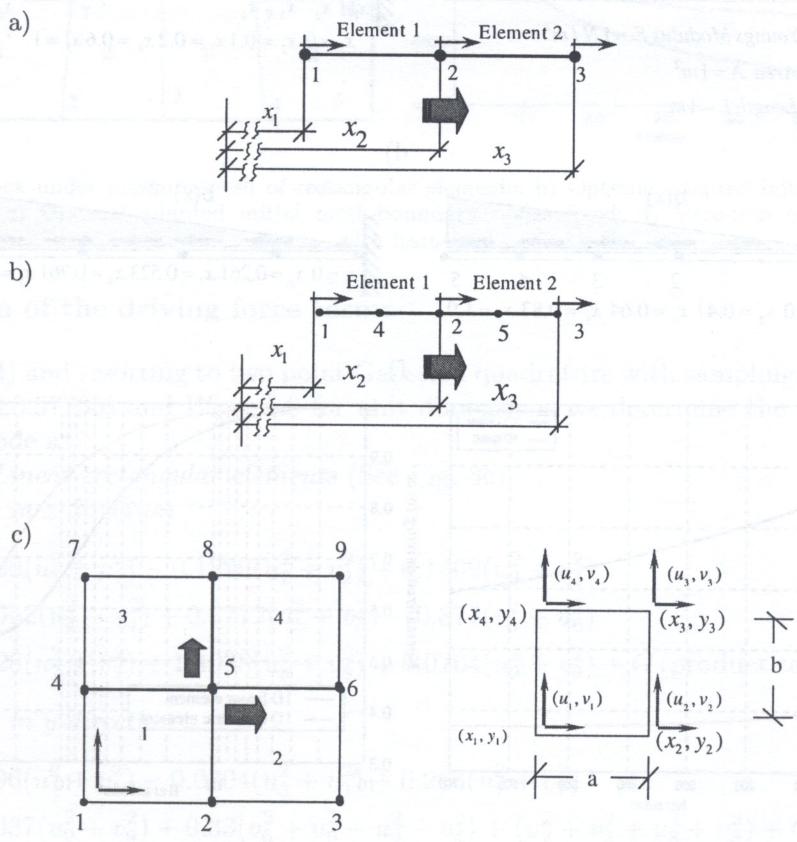


Fig. 3. a) Patch of linear one dimensional elements; b) Patch of quadratic one dimensional elements; c) Patch of linear rectangular elements

Patch of Linear One-dimensional Elements (see Fig. 3a)

$$\textit{Driving force} \quad G_i = \frac{-AE}{l^2}\left[0.5*(u_1^2 - u_3^2) + (u_3\,u_2 \;-\; u_1\,u_2)\right]; \tag{19}$$

Patch of Quadratic One-dimensional Elements (See Fig. 3b)

$$\textit{Driving force} \quad G_i = \frac{-AE}{l^2}\left[C_1(u_1^2 - u_3^2) + C_2(u_4^2 - u_5^2) + C_3(u_1u_2 - u_2u_3) + C_4(u_2u_4 - u_2u_5)\right], \tag{20}$$

where $C_1 = 1/6$, $C_2 = 16 * C_1$, $C_3 = 2 * C_1$, $C_4 = 2 * C_2$.

6.3. Two-dimensional example- block under pressure

A homogeneous square block of linear elastic isotropic material with nondimensionalized length of four units with a symmetric loading is considered. The vertical displacements on the bottom edge are fixed and a plane strain state is assumed. The block is discretized using four noded bilinear elements. The initial mesh is shown in Fig. 5a. For the given loading and boundary conditions mesh adaption is performed by Polak–Reberie conjugate gradient node relocation algorithm to get adapted mesh as shown in Fig. 5b. The boundary nodes are generally fixed during the adaption process Fig. 5b. Some times the boundary nodes can also be made to move in one direction Fig. 5c. This process causes a further reduction in the potential of the system as shown in Fig. 5d.

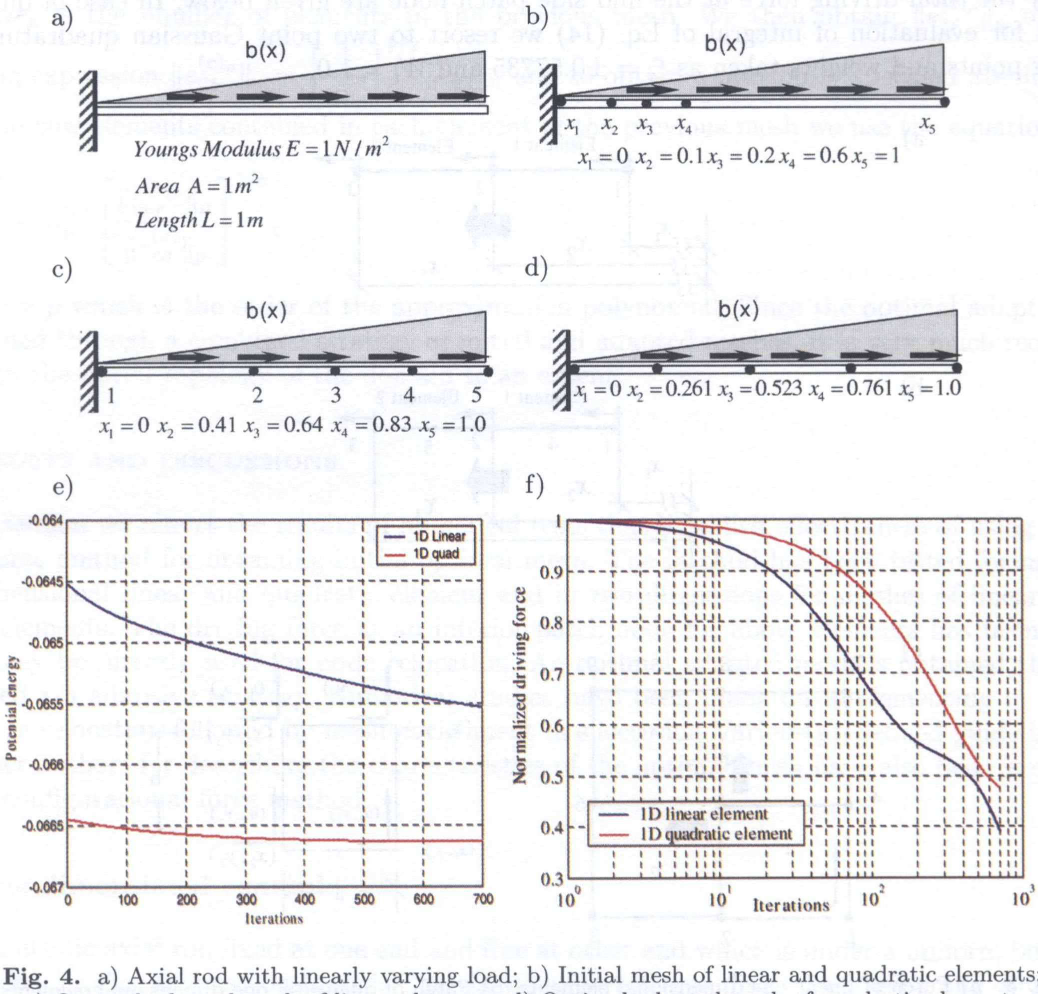


Fig. 4. a) Axial rod with linearly varying load; b) Initial mesh of linear and quadratic elements; c) Optimal adapted mesh of linear elements; d) Optimal adapted mesh of quadratic elements

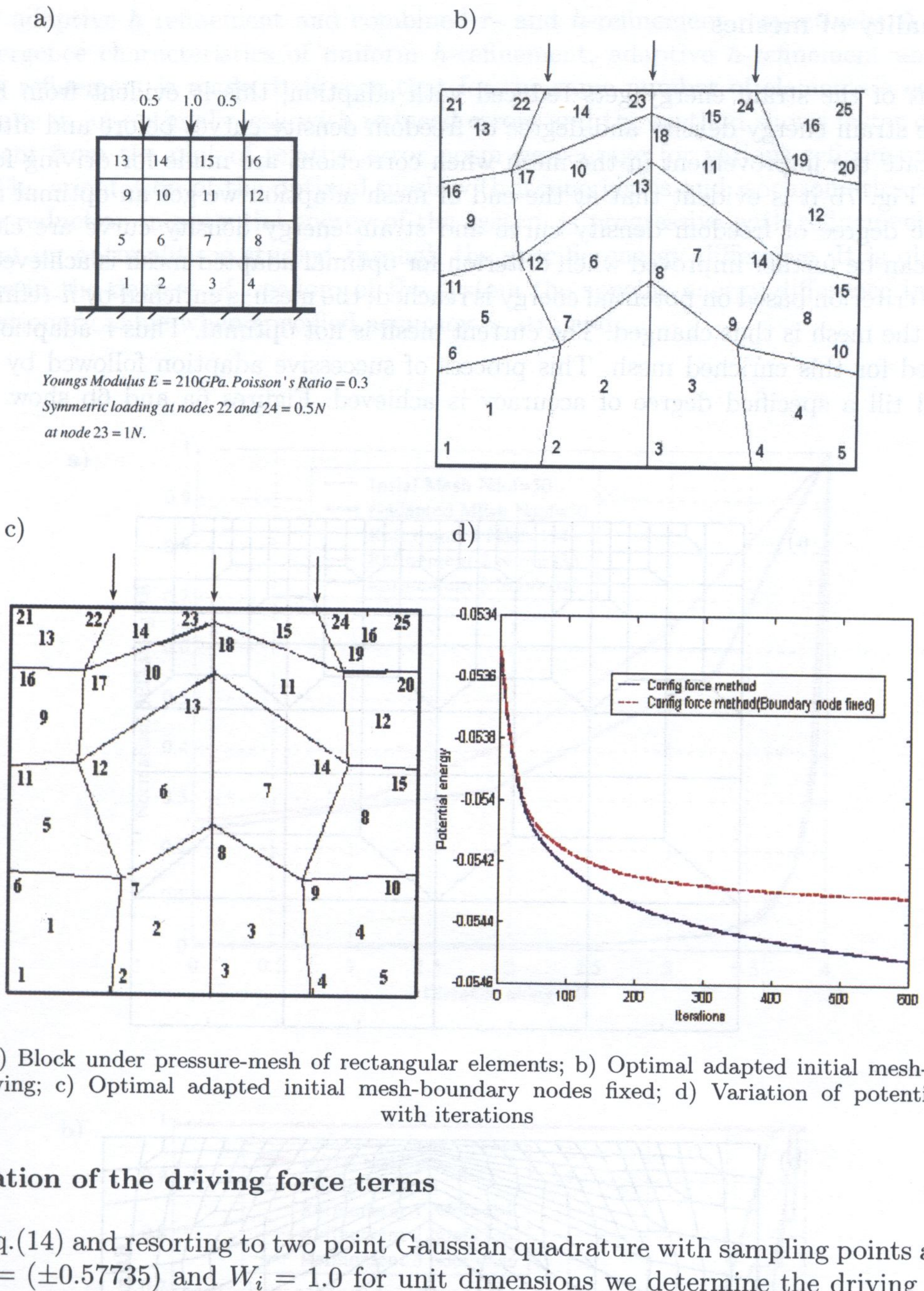


Fig. 5. a) Block under pressure-mesh of rectangular elements; b) Optimal adapted initial mesh-boundary nodes moving; c) Optimal adapted initial mesh-boundary nodes fixed; d) Variation of potential energy with iterations

6.4. Evaluation of the driving force terms

Following Eq.(14) and resorting to two point Gaussian quadrature with sampling points and weights taken as $\xi_i = (\pm 0.57735)$ and $W_i = 1.0$ for unit dimensions we determine the driving force at an interior patch node as

For a Patch of Linear rectangular elements (See Fig. 3c)

Driving force in x-direction

$$\begin{aligned} GF_x = & -0.269(u_1^2 + v_1^2) - 0.1229(u_2^2 + v_2^2) + 0.1409(u_3^2 + v_3^2) \\ & +1.4552(u_4^2 + v_4^2) + 0.5772(u_5^2 + v_5^2) - 0.877(v_6^2 + u_6^2) \\ & -0.526(u_7^2 + v_7^2) + 1.4038(u_8^2 + v_8^2) + 0.0704(u_9^2 + v_9^2) + G\,(\text{product terms}); \end{aligned}$$

Driving force in y-direction

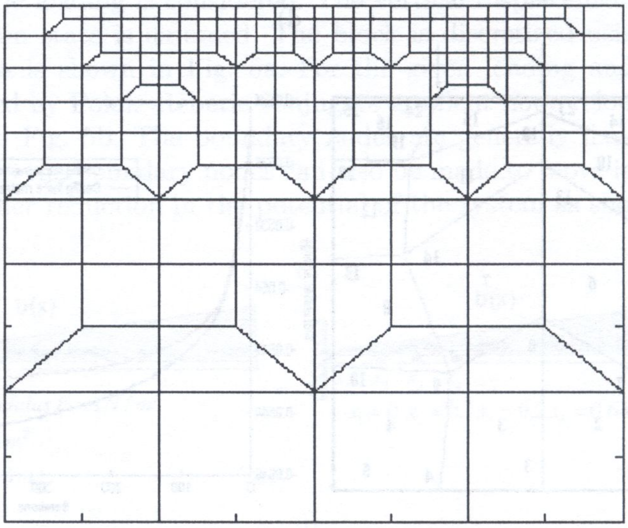
$$\begin{aligned} GF_y = & -0.596(u_1^2 + v_1^2) - 0.0704(u_3^2 + v_3^2) + 0.263(u_7^2 + v_7^2) \\ & +0.4037(u_9^2 + v_9^2) + 0.33(v_6^2 + u_6^2 - u_4^2 - v_4^2) + (u_2^2 + v_2^2 + u_8^2 + v_8^2) + G\,(\text{product terms}). \end{aligned}$$

The expressions of driving force terms for a quadrilateral element are given in Appendix.

6.5. Optimality of meshes

The gradient of the strain energy gets reduced with adaption, this is evident from Fig. 7a and Fig. 7b. The strain energy density and degree of freedom density curves before and after adaption clearly indicate the improvement in the mesh when corrections are made for driving forces. From Fig. 7a and Fig. 7b it is evident that at the end of mesh adaption we get an optimal initial mesh in which the degree of freedom density curve and strain energy density curve are close to each other. This can be further improved when criterion for optimal adapted mesh is achieved. Once the convergence criterion based on potential energy is reached, the mesh is enriched by h-refinement. The topology of the mesh is thus changed. The current mesh is not optimal. Thus r-adaption iterations are continued for this enriched mesh. This process of successive adaption followed by enrichment is continued till a specified degree of accuracy is achieved. Figures 6a and 6b show the meshes

a)



b)

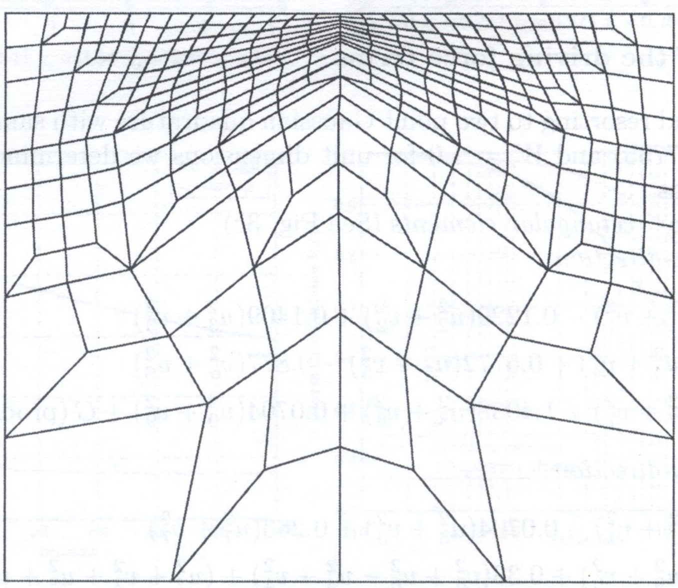


Fig. 6. a) Final mesh obtained from adaptive h-refinement; b) Optimal mesh obtained from adaptive r-h refinement

obtained by adaptive h refinement and combined r- and h-refinement respectively. A comparison of the convergence characteristics of uniform h-refinement, adaptive h-refinement and combined r-h adaptive refinement is made. It is seen that for the same number of elements a combined r-h strategy results in an optimal mesh with reduced errors and the method shows faster convergence. This is evident from the plot of relative error norm percentage for various refinements as shown in Fig. 8a. The orientation of the optimal mesh with isoenergetics and isochromatics is clear from Fig. 8b. The reduction in potential energy of the system is progressive with refinement. The effect of enrichment on optimality is studied through the specific energy difference. It is observed that with increase in the degrees of freedom of the system the specific energy difference increases and reaches a stationary value when specified accuracy is attained.

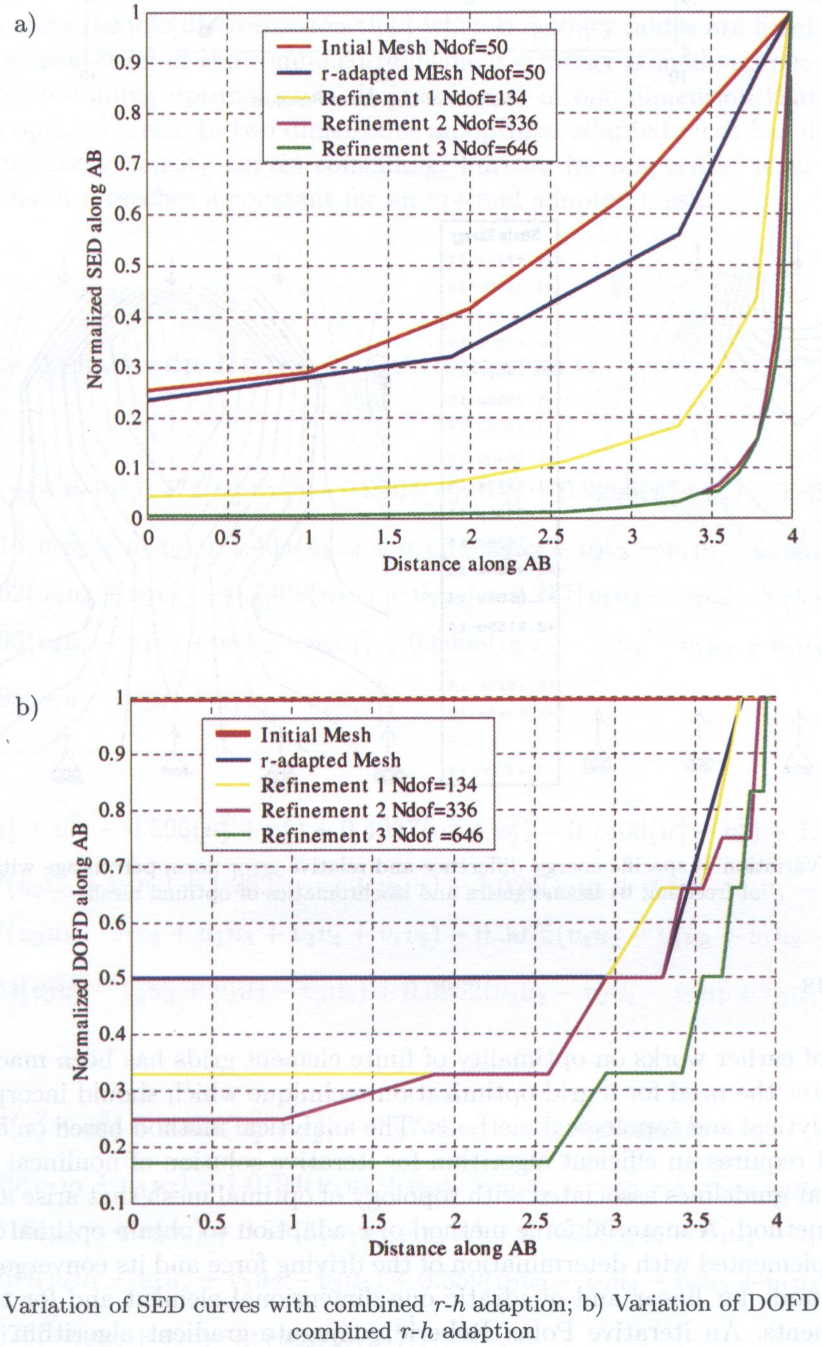


Fig. 7. a) Variation of SED curves with combined r-h adaption; b) Variation of DOFD curves with combined r-h adaption

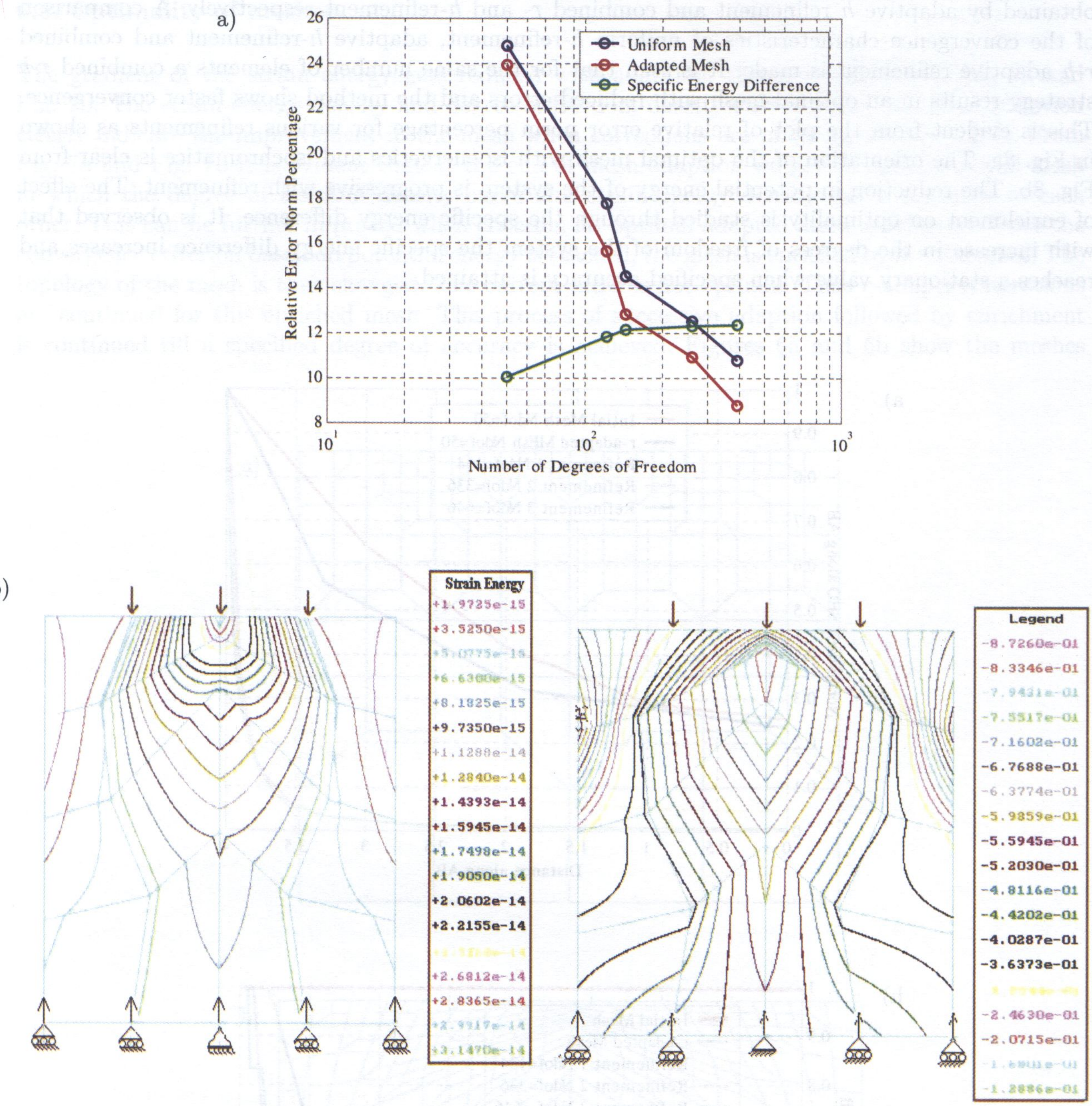


Fig. 8. a) Variation of specific energy difference and relative error norm percentage with degrees of freedom; b) Isoenergetics and isochromatics of optimal mesh

7. Conclusions

A critical review of earlier works on optimality of finite element grids has been made in this study. The review indicates the need for a grid optimization technique which should incorporate the basis from both the analytical and topological methods. The analytical method based on energy has more physical basis and requires an efficient algorithm for iterative solution of nonlinear equations. It is required to arrive at guidelines associated with topology of optimal mesh that arise as a consequence of the analytical method. A material force method of r-adaption to obtain optimal initial grids has therefore been implemented with determination of the driving force and its convergence rates across an interior patch node for linear and quadratic one dimensional element and for two dimensional quadrilateral elements. An iterative Polak Reberie conjugate gradient algorithm [37] is used for node relocation. Various aspects considered to define optimality with respect to topology in earlier works along with their predefined guidelines have been worked out with some modifications for the configurational force method and it is shown that this method of adaption provides good optimal grids. The method is advantageous owing to its physical basis and mathematical vigor than earlier works. The combined adaptive strategy incorporating node disposition and mesh enrichment to obtain an optimal mesh for a specified accuracy performs well.

From the numerical results on implementation of r-adaption based on configurational force method it is observed that in one dimensional situation there is not much improvement in convergence or the driving force terms in case of one dimensional problem as we move from linear to quadratic elements. Although in using a quadratic element there is a considerable increase in the potential energy of the system indicating that it is more flexible discretization. In fact the limitation in case of quadratic elements is that the relocation of the mid side node depends upon the end nodes; this greatly affects the convergence rates. In two dimensions the case of moving boundary nodes produces a more flexible discretization than when boundary nodes are fixed.

Numerical implementation of the combined refinement strategy is made on one and two dimensional problems for obtaining optimal grids. It is observed in one dimension that linear elements tend to a uniform optimal mesh. In two dimensions an optimal adapted mesh has degree of freedom density and strain energy density curves coinciding. Further for a specified level of accuracy the specific energy difference reaches a constant for an optimal adapted mesh.

Appendix

Driving Force Terms for Bilinear Quadrilateral

NODE1

$$\begin{aligned} GF_X^1 = \Big[&0.526(u_1^2+v_1^2) - 0.474(u_2^2+v_2^2) + 0.0704(u_3^2+v_3^2) + 0.9296(u_4^2+v_4^2) - 1.0516(v_1v_2+u_1u_2) \\ &- 1.0516(v_1v_2+u_1u_2) + 1.404(u_1u_4+v_1v_4+u_2u_3+v_2v_3-v_1v_3-u_1u_3) \\ &+ 0.5962(u_2u_4+v_2v_4) - 0.1409(u_3u_4+v_3v_4) + 0.237(v_4u_4-v_3u_4-v_4u_3+v_3u_3) \\ &+ 0.4296(v_4u_1-v_4u_2+v_3u_2-v_3u_1) + 0.6408(v_2u_3-v_2u_4-v_1u_3+v_1u_4) \\ &+ 0.6295(v_2u_2-v_2u_1+v_1u_1-v_1u_2)\Big]; \end{aligned}$$

$$\begin{aligned} GF_Y^1 = \Big[&0.86(u_1^2+v_1^2) - 0.596(u_2^2+v_2^2) + 0.4037(u_3^2+v_3^2) - 0.1408(u_4^2+v_4^2) - 1.71(u_1u_4+v_3v_4) \\ &- 0.807(u_2u_3+v_2v_3) + 1.262(u_2u_4+v_2v_4) + 1.026(v_4u_4+v_1u_1-v_4u_1-v_1u_4) \\ &+ 0.737(u_3u_4-v_1v_3+u_1u_3+v_3v_4+v_1v_2) - 0.3075(v_4u_2-v_4u_3+v_1u_3-v_1u_2) \\ &- 0.5704(v_3u_2-v_2u_2+v_2u_3-v_3u_3) + 0.0962(v_2u_1-v_2u_4-v_3u_1+v_3u_4)\Big]. \end{aligned}$$

NODE2

$$\begin{aligned} GF_X^1 = \Big[&0.596(u_1^2+v_1^2) - 0.404(u_2^2+v_2^2) + 0.4742(u_3^2+v_3^2) - 0.526(u_4^2+v_4^2) \\ &- 0.929(v_1v_3+u_1u_3) - 1.0704(u_1u_4+v_1v_4+u_2u_3+v_2v_3-v_2v_4-u_2u_4) \\ &+ 1.0516(u_3u_4+v_3v_4) + 0.8075(u_1u_2+v_1v_2) + 0.7629(v_4u_4-v_3u_4-v_4u_3+v_3u_3) \\ &+ 0.5704(v_4u_1-v_4u_2+v_3u_2-v_3u_1) + 0.359(v_2u_3-v_2u_4-v_1u_3+v_1u_4) \\ &- 0.3076(-v_2u_2+v_2u_1-v_1u_1+v_1u_2)\Big]; \end{aligned}$$

$$GF_Y^2 = \Big[-0.737(u_1^2+v_1^2) + 0.1408(u_2^2+v_2^2) - 0.859(u_3^2+v_3^2) + 0.263(u_4^2+v_4^2) - 0.526(u_1u_4+v_1v_4)$$
$$-0.282(u_2u_3+v_2v_3) + 0.4037(u_1u_3+v_1v_3) - 0.6925(v_4u_4+v_1u_1-v_4u_1-v_1u_4)$$
$$+1.596(u_3u_4-v_2v_4+u_1u_2-u_2u_4+v_3v_4+v_1v_2) + 0.6408(v_4u_2-v_4u_3+v_1u_3-v_1u_2)$$
$$+0.237(v_3u_2-v_2u_2+v_2u_3-v_3u_3) - 0.4296(v_2u_1-v_2u_4-v_3u_1+v_3u_4)\Big]$$

NODE3

$$GF_X^3 = \Big[0.269(u_1^2+v_1^2) + 0.737(u_2^2+v_2^2) - 0.1408(u_3^2+v_3^2) + 0.859(u_4^2+v_4^2)$$
$$+0.287(v_3v_4+u_3u_4) - 1.596(u_1u_4+v_1v_4+u_2u_3+v_2v_3-v_1v_3-u_1u_3)$$
$$-0.4037(u_2u_4+v_2v_4) + 0.5257(u_1u_2+v_1v_2) - 0.3075(v_4u_4-v_3u_4-v_4u_3+v_3u_3)$$
$$-1.0258(v_4u_1-v_4u_2+v_3u_2-v_3u_1) - 0.237(v_2u_3-v_2u_4-v_1u_3+v_1u_4)$$
$$+0.4296(-v_2u_2+v_2u_1-v_1u_1+v_1u_2)\Big];$$

$$GF_Y^3 = \Big[-0.596(u_1^2+v_1^2) + 0.5257(u_2^2+v_2^2) - 0.474(u_3^2+v_3^2) + 0.4037(u_4^2+v_4^2)$$
$$+1.1924(u_1u_4+v_1v_4) + 0.948(u_2u_3+v_2v_3) - 1.07(u_2u_4+v_2v_4)$$
$$-0.763(v_4u_4+v_1u_1-v_4u_1-v_1u_4) - 0.929(u_3u_4-v_1v_3+u_1u_2-u_1u_3+v_3v_4+v_1v_2)$$
$$-0.0962(v_4u_2-v_4u_3+v_1u_3-v_1u_2) + 0.641(v_3u_2-v_2u_2+v_2u_3-v_3u_3)$$
$$-0.692(v_2u_1-v_2u_4-v_3u_1+v_3u_4)\Big].$$

NODE4

$$GF_X^4 = \Big[-0.86(u_1^2+v_1^2) + 0.1409(u_2^2+v_2^2) - 0.403(u_3^2+v_3^2) + 0.596(u_4^2+v_4^2)$$
$$+0.737(v_1v_3+u_1u_3) + 1.263(u_1u_4+v_1v_4+u_2u_3+v_2v_3-v_2v_4-u_2u_4)$$
$$-1.19(u_3u_4+v_3v_4) - 0.282(u_1u_2+v_1v_2) - 0.692(v_4u_4-v_3u_4-v_4u_3+v_3u_3)$$
$$+0.258(v_4u_1-v_4u_2+v_3u_2-v_3u_1) - 0.763(v_2u_3-v_2u_4-v-1u_3+v_1u_4)$$
$$+0.5704(-v_2u_2+v_2u_1-v_1u_1+v_1u_2)\Big];$$

$$GF_Y^4 = \Big[0.474(u_1^2+v_1^2) - 0.0704(u_2^2+v_2^2) + 0.929(u_3^2+v_3^2) - 0.5257(u_4^2+v_4^2)$$
$$+1.0515(u_1u_4+v_1v_4) + 0.1408(u_2u_3+v_2v_3) - 0.596(u_1u_3+v_1v_3)$$
$$+0.43(v_4u_4+v_1u_1-v_4u_1-v_1u_4) - 1.4037(u_3u_4+v_1v_2-u_2u_4+u_1u_2-v_2v_4+v_3v_4)$$
$$-0.237(v_4u_2-v_4u_3+v_1u_3-v_1u_2) - 0.3075(v_3u_2-v_2u_2+v_2u_3-v_3u_3)$$
$$+1.026(v_2u_1-v_2u_4-V-3u_1+v_3u_4)\Big].$$

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