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Wave polynomials for solving different types of two-dimensional wave equations

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The paper demonstrates a specific power series expansion technique used to obtain the approximate solution of the two-dimensional wave equation in some unusual cases. The solution for inhomogeneous wave equation, for more complicated shape geometry of the body, discrete boundary conditions and a membrane whose thickness is not constant is shown. As solving functions (Trefftz functions), so-called wave polynomials are used. Recurrent formulas for the particular solution are obtained. Some examples are included.

Keywords: wave equation, wave polynomials, Trefftz method, membrane vibrations

1. Introduction

There are three groups of the methods applied to solve partial differential equations. The first group consists of so called analytical methods which means, that the solution fully satisfies both field equation and all given conditions. The examples include: integral transformation method, separation of the variables method, Green function method and so on. Analytical solutions are not always useful for numerical calculations. For example, when the solution is given as Bessel function series, some problems with numerical convergence arise. Moreover, analytical methods are applicable only for simple shape of the body. The second group contains such methods as Finite Element Method or Boundary Element Method. In this case the solution approximately satisfies both field equation and all given initial and boundary conditions and the shape does not have to be simple. In the third group the solution exactly satisfies the equation and approximately all given conditions. Trefftz method is an example here. So is the method presented in this paper. Its key idea is to determine functions (polynomials) satisfying the given differential equation and fitted to the governing initial and boundary conditions. In this sense it is a variant of the Trefftz method [1, 2].

The method was first described in the paper [3] where it was applied to solve one-dimensional heat conduction problems. The heat polynomials in a similar form were also used in the paper [4] to solve unsteady heat conduction problems. Both papers describe one-dimensional direct problem for heat equation in the Cartesian coordinate system. The method is further discussed in the Cartesian coordinate system in papers [5, 6], describing heat polynomials for the two- and three-dimensional case. The application of the heat polynomials in polar and cylindrical coordinates is shown in the papers [7–9]. A slightly different approach for one-dimensional heat polynomials is presented in the paper [10].

In scientific literature inverse problems are often discussed. We deal with the direct problem, if all boundary and initial conditions for the given equation are known. In the case of inverse problems we usually have to find an extrapolation of the function. The method described here is applicable both to direct and inverse problems. In the papers [5–9, 11, 12] the method is used to solve inverse heat conduction problems. The paper [13] presents interesting concept of using heat polynomials as a new type of finite element base functions.

All papers mentioned above refer to the heat conduction equation. The paper [14] deals with numerous applications of the method in various differential equations, for example, in Laplace, Poisson and Helmholtz equations. The one-dimensional wave equation is solved there too. The two-dimensional wave polynomials and their derivatives extensively described in the paper [16] are used here. Basically, the wave equation can be solved by means of various methods. Some of them are better for infinite bodies while others for finite ones but of simple shape geometry. The method presented here is useful for finite bodies of relatively general shape geometry.

The properties of Taylor series are important for the application of the method

$$f(x+\Delta x, y+\Delta y, t+\Delta t) = f\,(x,y,t) + \frac{\mathrm{d}f}{1!} + \frac{\mathrm{d}^2 f}{2!} + \cdots + \frac{\mathrm{d}^N f}{N!} + R_{N+1} \tag{1}$$

where

$$\mathrm{d}^n f = \left(\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial t}\Delta t\right)^n,$$

and R_{N+1} is the remainder term. In this paper some unusual cases of the vibrating membrane described by wave equation are considered. In Sec. 2 the wave polynomials' method, discussed in the paper [16], is presented. The inhomogeneous two-dimensional wave equation is solved in Sec. 3. Section 4 describes a solution for more complicated shape geometry of the body. Here we consider a membrane in the shape of right and curvilinear triangle. A membrane with variable thickness is examined in Sec. 5. Section 6 deals with discrete boundary conditions for a square membrane. Approximation of the δ-Dirac for the stationary case of a circular membrane is described in Sec. 7. Section 8 contains concluding remarks.

2. Wave polynomials' method

The wave polynomials' method for the two-dimensional wave equation is accurately described in the paper [16]. Basing on it let us consider the dimensionless wave equation

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}. \tag{2}$$

The wave polynomial method discussed below is a variant of the Trefftz method. As a solution of wave equation (2) we take a linear combination of wave polynomials V_n:

$$w \approx u = \sum_{n=1}^{N} c_n V_n. \tag{3}$$

For example, the wave polynomials satisfying (2) are (see [16])

$$V_1 = 1, \qquad V_2 = x, \qquad V_3 = y,$$
$$V_4 = t, \qquad V_5 = -\frac{x^2}{2} - \frac{t^2}{2}, \qquad V_6 = -xy,$$
$$V_7 = -xt, \qquad V_8 = -yt, \qquad V_9 = -\frac{y^2}{2} - \frac{t^2}{2}, \ \ldots$$

Because all polynomials V_n satisfy Eq. (2), the linear combination u satisfies Eq. (2). The coefficients c_n are chosen so as to minimize the error for fulfilling a given boundary and initial condition corresponding to Eq. (2) (see examples).

3. Inhomogeneous wave equation

Let us consider the inhomogeneous wave equation

$$L(w) = Q(x, y, t), \tag{4}$$

where $L = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$. We assume that the function $Q(x, y, t)$ is analytical or $Q(x, y, t) \in C^{N+1}$. As an approximation of the solution we get:

$$w \approx u = \sum_{n=1}^{N} c_n V_n + w_p. \tag{5}$$

Because all polynomials V_n satisfy the wave equation (2), their linear combination also satisfies it. Additionally, w_p denotes the particular solution for the inhomogeneous wave equation (4). Boundary and initial conditions determine the coefficients c_n.

3.1. Particular solution

Just like for other equations [14] the solution w_p by means of power series for Q, can be calculated as follows:

$$\begin{aligned} w_p &= L^{-1}(Q) \\ &= L^{-1}\left(\sum_{n=0}^{\infty} \sum_{k+l+m=n} \frac{\partial^{(k+l+m)} Q(x_0, y_0, t_0)}{\partial x^k \partial y^l \partial t^m} \frac{\hat{x}^k \hat{y}^l \hat{t}^m}{k!l!m!} \right) \\ &= \sum_{n=0}^{\infty} \sum_{k+l+m=n} a_{klm} L^{-1}(\hat{x}^k \hat{y}^l \hat{t}^m), \end{aligned} \tag{6}$$

where $\hat{x} = x - x_0, \hat{y} = y - y_0, \hat{t} = t - t_0$ and $a_{klm} = \frac{1}{k!l!m!} \frac{\partial^{(k+l+m)} Q(x_0, y_0, t_0)}{\partial x^k \partial y^l \partial t^m}$.

If $Q(x, y, t) \in C^{N+1}$ the sum is finite and we can calculate the particular solution only approximately. In practice we use the finite sum in both cases. Therefore, we obtain an approximate solution. It is easy to prove that for the particular solution recurrent formulas can be given:

$$\begin{aligned} w_{p1}(x^k y^l t^m) &= L^{-1}(x^k y^l t^m) \\ &= \frac{1}{(k+2)(k+1)} (-x^{k+2} y^l t^m + m(m-1) L^{-1}(x^{k+2} y^l t^{m-2}) \\ &\quad - l(l-1) L^{-1}(x^{k+2} y^{l-2} t^m)) \end{aligned} \tag{7}$$

or

$$w_{p2}(x^k y^l t^m) = L^{-1}(x^k y^l t^m)$$
$$= \frac{1}{(l+2)(l+1)}(-x^k y^{l+2} t^m + m(m-1)L^{-1}(x^k y^{l+2} t^{m-2})$$
$$- k(k-1)L^{-1}(x^{k-2} y^{l+2} t^m)) \tag{8}$$

or

$$w_{p3}(x^k y^l t^m) = L^{-1}(x^k y^l t^m)$$
$$= \frac{1}{(m+2)(m+1)}(-x^k y^l t^(m+2) + k(k-1)L^{-1}(x^{k-2} y^l t^{m+2})$$
$$+ l(l-1)L^{-1}(x^k y^{l-2} t^{m+2})). \tag{9}$$

In these formulas it is assumed that a term on the right-hand side equals zero if the corresponding subscript takes a negative value.

Proof. As an example we prove relation (7). To prove it, it is sufficient to calculate $L(w_{p1})$ as follows:

$$L(w_{p1}) = \frac{1}{(k+2)(k+1)}\Big(- m(m-1)x^{k+2} y^l t^{m-2}$$
$$+ (k+2)(k+1)x^k y^l t^m + l(l-1)x^{k+2} y^{l-2} t^m$$
$$+ m(m-1)x^{k+2} y^l t^{m-2} - l(l-1)x^{k+2} y^{l-2} t^m\Big) = x^k y^l t^m.$$

This proves the relation (7). The proof for relations (8) and (9) is similar.

3.2. Example: vibration of a square membrane

The first example shows the vibration of a square membrane with constant and non-constant source. The equation of motion reads

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + Q(x, y, t), \quad (x, y) \in (0, 1) \times (0, 1), \quad t > 0$$

and is completed by initial and boundary conditions, respectively:

$$w(x, y, 0) = 0, \qquad \frac{\partial w(x, y, 0)}{\partial t} = 0, \tag{10}$$

$$w(0, y, t) = w(1, y, t) = w(x, 0, t) = w(x, 1, t) = 0. \tag{11}$$

We consider two kinds of the source. The first is $Q = \text{const} = 0.1$ and the second is $Q = \frac{1}{10}\sin(2\pi t)$. The exact solutions for these cases can be found in [17].

In the case $Q = \text{const} = 0.1$ the exact solution is given as

$$w(x, y, t) = \sum_{k,l} \frac{\sin(k\pi x)\sin(l\pi y)}{\sqrt{(k\pi)^2 + (l\pi)^2}}$$
$$\cdot \int_0^1 \int_0^1 \int_0^t \sin(k\pi x)\sin(l\pi y)\sin\Big(\sqrt{(k\pi)^2 + (l\pi)^2}(t - \tau)\Big) d\tau dy dx.$$

The solution $w(x,y,t)$ is approximated in accordance with Eq. (5). Here we have

$$w_p = \frac{\frac{1}{10}w_{p1}(1) + \frac{1}{10}w_{p2}(1) + \frac{1}{10}w_{p3}(1)}{3} = \frac{-x^2 - y^2 + t^2}{60}.$$

Because the function u satisfies the wave equation (linear combination of wave polynomials), we minimize $\| w - u \|$ recording to boundary and initial conditions. We look for an approximate solution u in the time interval $(0, \Delta t)$. Therefore, the coefficients c_n have to be chosen appropriately so as to minimize the functional

$$\begin{aligned}
I = & \underbrace{\int_0^1\int_0^1 \Big[u(x,y,0) - \underbrace{w(x,y,0)}_{=0}\Big]^2 \mathrm{d}y\mathrm{d}x}_{\text{cond.(10)}} \\
& + \underbrace{\int_0^1\int_0^1 \left[\frac{\partial u(x,y,0)}{\partial t} - \underbrace{\frac{\partial w(x,y,0)}{\partial t}}_{=0}\right]^2 \mathrm{d}y\mathrm{d}x}_{\text{cond.(10)}} \\
& + \underbrace{\int_0^{\Delta t}\int_0^1 [u(0,y,t) - \underbrace{w(0,y,t)}_{=0}]^2 \mathrm{d}y\mathrm{d}t}_{\text{cond.(11)}} \\
& + \underbrace{\int_0^{\Delta t}\int_0^1 [u(1,y,t) - \underbrace{w(1,y,t)}_{=0}]^2 \mathrm{d}y\mathrm{d}t}_{\text{cond.(11)}} \\
& + \underbrace{\int_0^{\Delta t}\int_0^1 [u(x,0,t) - \underbrace{w(x,0,t)}_{=0}]^2 \mathrm{d}x\mathrm{d}t}_{\text{cond.(11)}} \\
& + \underbrace{\int_0^{\Delta t}\int_0^1 [u(x,1,t) - \underbrace{w(x,1,t)}_{=0}]^2 \mathrm{d}x\mathrm{d}t}_{\text{cond.(11)}}.
\end{aligned} \tag{12}$$

The necessary condition to minimize functional I is

$$\frac{\partial I}{\partial c_1} = \cdots = \frac{\partial I}{\partial c_N} = 0. \tag{13}$$

In the case $Q = \frac{1}{10}\sin(2\pi t)$, the exact solution is given as

$$w(x,y,t) = \sum_{k,l} \frac{\sin(k\pi x)\sin(l\pi y)}{\sqrt{(k\pi)^2 + (l\pi)^2}} \cdot \int_0^1\int_0^1\int_0^t \sin(k\pi x)\sin(l\pi y)\sin(2\pi h)\sin(\sqrt{(k\pi)^2 + (l\pi)^2}(t-\tau))\mathrm{d}\tau\mathrm{d}y\mathrm{d}x.$$

The solution $w(x,y,t)$ is approximated in accordance with Eq. (5) again. Here we put

$$w_p = \frac{\frac{1}{10}\sum_{k=1}^{K} \left.\frac{d^k \sin(2\pi t)}{dt^k}\right|_{t=0} \frac{w_{p1}(t^k) + w_{p2}(t^k) w_{p3}(t^k)}{k!}}{3}.$$

The coefficients c_n have to be chosen as in the previous case. All the results below are obtained for $\Delta t = 1$.

Figure 1 shows the exact and the approximate solutions (by polynomials from order 0 to 9) for point $x = y = 0.5$: a) $Q = \text{const} = 0.1$, b) $Q = \frac{1}{10}\sin(2\pi t)$. Note, that the approximate solution in the Fig. 1b) is very good in time interval $(0; 1.5)$ although it was calculated in time interval $(0; 1)$.

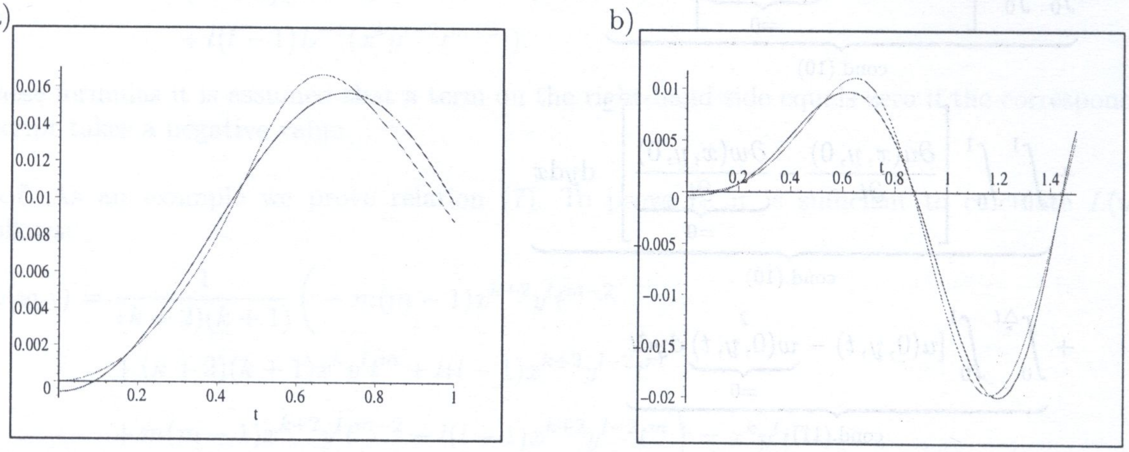

Fig. 1. Exact and approximate solutions for point $x = y = 0.5$: a) $Q = \text{const} = 0.1$, b) $Q = \frac{1}{10}\sin(2\pi t)$

4. More complicated shape of the body

The wave polynomials' method is most useful for bodies of more complicated shape. Unfortunately, the exact solution is frequently unknown.

Two-dimensional wave equation describes vibrations of the membrane. In this section we consider two cases: membrane in the shape of a right triangle and a curvilinear triangle.

4.1. Membrane in the shape of right triangle

Let us consider the wave equation for a right triangle, describing the vibrations of the membrane:

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}, \qquad x \in (0,1), \qquad 0 < y < -x+1, \qquad t > 0$$

to be completed by initial and boundary conditions, respectively:

$$w(x,y,0) = w_0(x,y) = -xy(x+y-1), \qquad \frac{\partial w(x,y,0)}{\partial t} = 0, \tag{14}$$

$$w(0,y,t) = w(x,0,t) = w(x,y,t)|_{x+y=1} = 0. \tag{15}$$

The solution $w(x,y,t)$ is approximated in accordance with Eq. (3). Because the function u satisfies the wave equation (linear combination of wave polynomials), we minimize $\| w - u \|$ recording to boundary and initial conditions. We look for an approximate solution u in the time interval $(0, \Delta t)$. Therefore, the coefficients c_n have to be chosen appropriately to minimize the functional

$$
\begin{aligned}
I = & \underbrace{\int_0^1 \int_0^1 [u(x,y,0) - w_0(x,y)]^2 \mathrm{d}y \mathrm{d}x}_{\text{cond.(14)}} \\
& \underbrace{+ \int_0^1 \int_0^1 \left[\frac{\partial u(x,y,0)}{\partial t} - \underbrace{\frac{\partial w(x,y,0)}{\partial t}}_{=0} \right]^2 \mathrm{d}y \mathrm{d}x}_{\text{cond.(14)}} \\
& \underbrace{+ \int_0^{\Delta t} \int_0^1 [u(0,y,t) - \underbrace{w(0,y,t)}_{=0}]^2 \mathrm{d}y \mathrm{d}t}_{\text{cond.(15)}} \\
& \underbrace{+ \int_0^{\Delta t} \int_0^1 [u(x,0,t) - \underbrace{w(x,0,t)}_{=0}]^2 \mathrm{d}x \mathrm{d}t}_{\text{cond.(15)}} \\
& \underbrace{+ \sqrt{2} \int_0^{\Delta t} \int_0^1 [u(x,-x+1,t) - \underbrace{w(x,-x+1,t)}_{=0}]^2 \mathrm{d}x \mathrm{d}t}_{\text{cond.(15)}} .
\end{aligned} \tag{16}
$$

The necessary condition to minimize the functional I is described by conditions (13). As a result linear system of equations can be written as

$$
AC = B \tag{17}
$$

where $C = [c_1, \ldots, c_N]^{\mathrm{T}}$ and the elements of the matrices A and B are

$$
\begin{aligned}
a_{ij} = & \int_0^1 \int_0^{-x+1} V_i(x,y,0) V_j(x,y,0) \mathrm{d}y \mathrm{d}x \\
& + \int_0^1 \int_0^{-x+1} \frac{\partial V_i(x,y,0)}{\partial t} \frac{\partial V_j(x,y,0)}{\partial t} \mathrm{d}y \mathrm{d}x \\
& + \int_0^{\Delta t} \int_0^1 V_i(0,y,t) V_j(0,y,t) \mathrm{d}y \mathrm{d}t \\
& + \int_0^{\Delta t} \int_0^1 V_i(x,0,t) V_j(x,0,t) \mathrm{d}x \mathrm{d}t \\
& + \sqrt{2} \int_0^{\Delta t} \int_0^1 V_i(x,-x+1,t) V_j(x,-x+1,t) \mathrm{d}x \mathrm{d}t, \\
b_i = & \int_0^1 \int_0^{-x+1} V_i(x,y,0) w_0(x,y) \mathrm{d}y \mathrm{d}x.
\end{aligned}
$$

Note that $a_{ij} = a_{ji}$ (matrix A is symmetrical) which facilitates calculations. From Eq. (17) we obtain coefficients c_n:

$$
C = A^{-1} B.
$$

In time intervals $(\Delta t, 2\Delta t)$, $(2\Delta t, 3\Delta t)$,... we proceed analogously. Here, the initial condition for time interval $((m-1)\Delta t, m\Delta t)$ is the value of function u at the end of interval $((m-2)\Delta t, (m-1)\Delta t)$. All the results below are obtained for $\Delta t = 1$.

We obtain the approximation in the whole time interval $(0, \Delta t)$. For example, in Fig. 2, for time $t = 0$: a) the exact solution, b) an approximation by polynomials from order 0 to 9, c) the difference between a) and b) are shown. Figure 3 shows an approximation by polynomials from order 0 to 9 for time: a) $t = 0.15$, b) $t = 0.5$, c) $t = 0.8$. Figures 2 and 3 show that the boundary conditions are adequately fulfilled and the physical character of the vibrations is preserved.

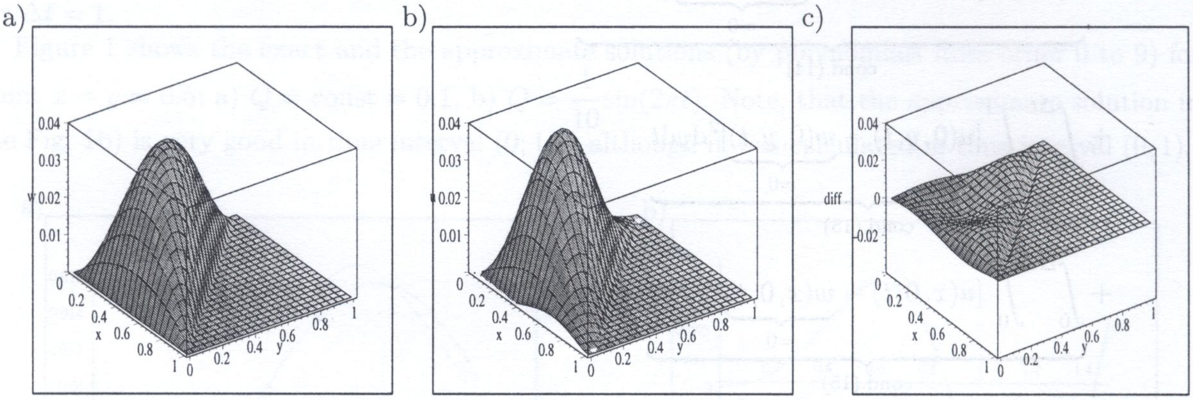

Fig. 2. Solution for time $t = 0$: a) exact, b) approximation, c) difference

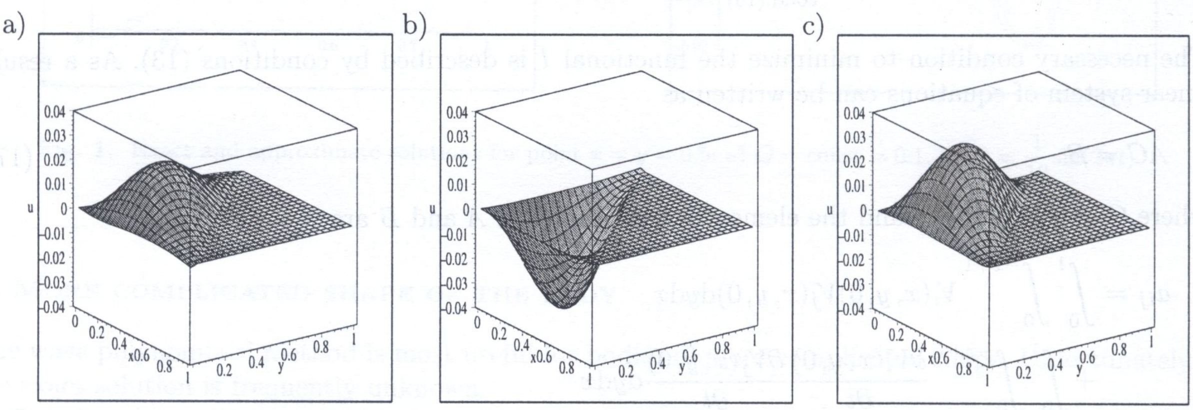

Fig. 3. Solution for time: a) $t = 0.15$, b) $t = 0.5$, c) $t = 0.8$

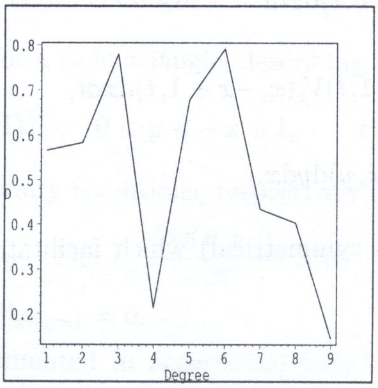

Fig. 4. Difference $D(K)$

Let u_K denote the approximation (3) by polynomials from order 0 to K. To verify the quality of the solution we define an average, relative difference between solutions u_K and u_{K-1}:

$$D(K) = \sqrt{\frac{\int_0^{\Delta t}\int_0^{-x+1}\int_0^1 (u_K(x,y,t) - u_{K-1}(x,y,t))^2 \mathrm{d}x\mathrm{d}y\mathrm{d}t}{\int_0^{\Delta t}\int_0^{-x+1}\int_0^1 (u_{K-1}(x,y,t))^2 \mathrm{d}x\mathrm{d}y\mathrm{d}t}}. \tag{18}$$

Figure 4 shows the difference $D(K)$ which depends on the order K for $\Delta t = 1$. Figure 4 suggests that in this case the method is convergent. Obviously, this is not a proof of convergence. The detailed discussion on convergence of this method can be found in the papers [14] and [16].

4.2. Membrane in the shape of a curvilinear triangle

Let us consider the wave equation in a curvilinear triangle describing the vibrations of the membrane:

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \qquad x \in (0,1), \qquad 0 < y < \sqrt{1-x^2}, \quad t > 0$$

to be completed by initial and boundary conditions, respectively:

$$w(x,y,0) = w_0(x,y) = -xy(x^2+y^2-1), \qquad \frac{\partial w(x,y,0)}{\partial t} = 0, \tag{19}$$

$$w(0,y,t) = w(x,0,t) = w(x,y,t)|_{x^2+y^2=1} = 0. \tag{20}$$

Just as in the case of the right triangle we can obtain the approximation in the whole time interval $(0,\Delta t)$. In Fig. 5, for time $t = 0$ a) the exact solution, b) an approximation by polynomials from order 0 to 9, c) the difference between a) and b) are shown. All the results below are obtained for $\Delta t = 1$. Figure 6 shows an approximation by polynomials from order 0 to 9 for time a) $t = 0.15$, b) $t = 0.45$, c) $t = 0.8$. Figures 5 and 6 show that the boundary conditions are well fulfilled and the physical character of the vibrations is preserved. By analogy with Sec. (4.1) we define an average, relative difference between solutions u_K and u_{K-1}:

$$D(K) = \sqrt{\frac{\int_0^{\Delta t}\int_0^{\sqrt{1-x^2}}\int_0^1 (u_K(x,y,t) - u_{K-1}(x,y,t))^2 \mathrm{d}x\mathrm{d}y\mathrm{d}t}{\int_0^{\Delta t}\int_0^{\sqrt{1-x^2}}\int_0^1 (u_{K-1}(x,y,t))^2 \mathrm{d}x\mathrm{d}y\mathrm{d}t}}. \tag{21}$$

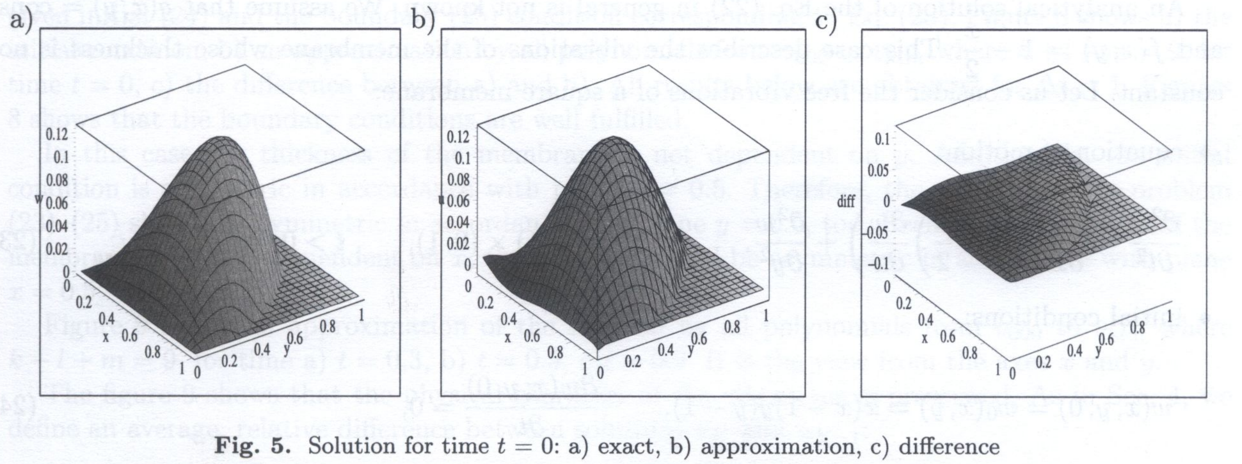

Fig. 5. Solution for time $t = 0$: a) exact, b) approximation, c) difference

Figure 7 shows the difference $D(K)$ which depends on the order K for $\Delta t = 1$. Figure 7 implies that in this case the method is convergent.

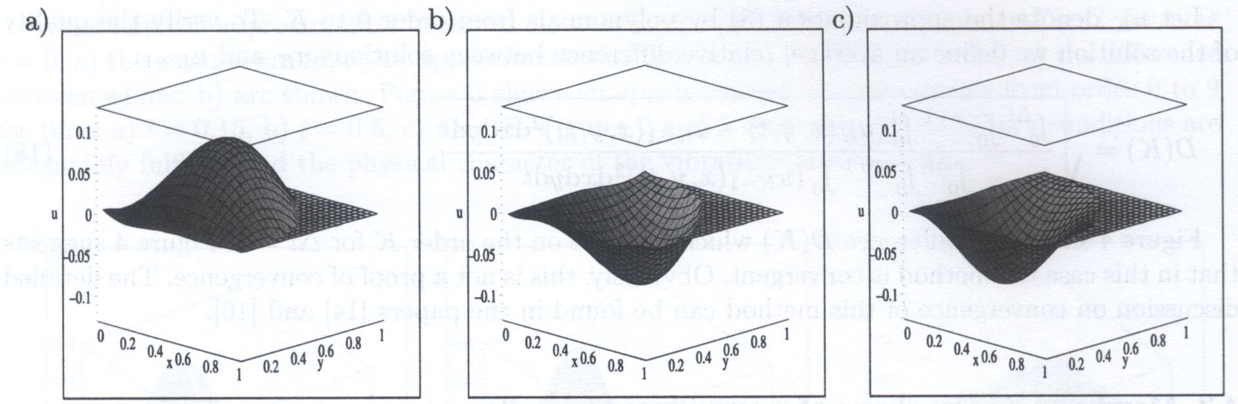

Fig. 6. Solution for time: a) $t = 0.15$, b) $t = 0.45$, c) $t = 0.8$

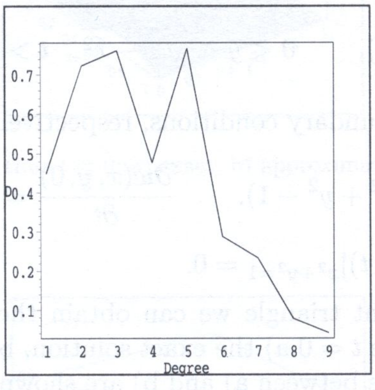

Fig. 7. Difference $D(K)$

5. Membrane with variable thickness

The vibrations of a membrane with variable thickness are described by the equation

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(f(x,y)\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(g(x,y)\frac{\partial w}{\partial y}\right). \tag{22}$$

An analytical solution of the Eq. (22) in general is not known. We assume that $g(x,y) = \text{const}$ and $f(x,y) = 1 - \dfrac{x}{2}$. This case describes the vibrations of the membrane whose thickness is not constant. Let us consider the free vibrations of a square membrane:

- equation of motion:

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(\left(1 - \frac{x}{2}\right)\frac{\partial w}{\partial x}\right) + \frac{\partial^2 w}{\partial y^2}, \qquad (x,y) \in (0,1) \times (0,1), \qquad t > 0, \tag{23}$$

- initial conditions:

$$w(x,y,0) = w_0(x,y) = x(x-1)y(y-1), \qquad \frac{\partial w(x,y,0)}{\partial t} = 0, \tag{24}$$

- boundary conditions:

$$w(0,y,t) = w(1,y,t) = w(x,0,t) = w(x,1,t) = 0. \tag{25}$$

As a solution of the wave equation (23) we take a linear combination of wave polynomials V_n in accordance with Eq. (3). Here the wave polynomials (Trefftz functions) should satisfy Eq. (23). Similarly as for other equations (see [14] and [16]), the wave polynomials for Eq. (23) can be obtained using Taylor series (1) for function w. Let the function $w(x,y,t)$ satisfy the wave equation (23). We assume that $w \in C^{N+1}$ in the neighbourhood of $(x_0, y_0, t_0) = (0,0,0)$. Then, the Taylor series for function w is

$$w(x,y,t) = w(0,0,0) + \frac{\partial w}{\partial x}x + \frac{\partial w}{\partial y}y + \frac{\partial w}{\partial t}t + \frac{\partial^2 w}{\partial x^2}\frac{x^2}{2} + \frac{\partial^2 w}{\partial y^2}\frac{y^2}{2} + \frac{\partial^2 w}{\partial t^2}\frac{t^2}{2} + \frac{\partial^2 w}{\partial x \partial y}xy + \frac{\partial^2 w}{\partial x \partial t}xt + \frac{\partial^2 w}{\partial y \partial t}yt + \cdots + R_{N+1}.$$

Eliminating the derivative $\dfrac{\partial^2 w}{\partial t^2}$ by Eq. (23) yields

$$\begin{aligned} w(x,y,t) = {} & w\ (0,0,0) + \frac{\partial w}{\partial x}\left(x - \frac{t^2}{4}\right) + \frac{\partial w}{\partial y}y + \frac{\partial w}{\partial t}t \\ & + \frac{\partial^2 w}{\partial x^2}\left(\frac{x^2}{2} + \frac{t^2}{2} - \frac{xt^2}{2} + \frac{t^4}{48}\right) + \frac{\partial^2 w}{\partial y^2}\left(\frac{y^2}{2} + \frac{t^2}{2}\right) \\ & + \frac{\partial^2 w}{\partial x \partial y}\left(xy - \frac{yt^2}{4}\right) + \frac{\partial^2 w}{\partial x \partial t}\left(xt - \frac{t^3}{12}\right) + \frac{\partial^2 w}{\partial y \partial t}yt + \ldots . \end{aligned} \tag{26}$$

The coefficients following the derivation terms on the right-hand side are wave polynomials satisfying (23). Let v_{klm} denote the coefficient at the derivative $\dfrac{\partial w^{k+l+m}}{\partial x^k \partial y^l \partial t^m}$. Note that there is no v_{002}. Further $V_1 = v_{000}, V_2 = v_{100}, V_3 = v_{010}, V_4 = v_{001}, V_5 = v_{200}, V_6 = v_{020}, V_7 = v_{110}, V_8 = v_{101}, V_9 = v_{011}, \ldots$. From (26) we have

$$\begin{aligned} & V_1 = 1, \qquad V_2 = x - \frac{t^2}{4}, \qquad V_3 = y, \qquad V_4 = t \\ & V_5 = \frac{x^2}{2} + \frac{t^2}{2} - \frac{xt^2}{2} + \frac{t^4}{48}, \qquad V_6 = \frac{y^2}{2} + \frac{t^2}{2} \\ & V_7 = xy - \frac{yt^2}{4}, \qquad V_8 = xt - \frac{t^3}{12}, \qquad V_9 = yt, \ldots \end{aligned} \tag{27}$$

The polynomials (27) satisfy Eq. (23). Hence their linear combination satisfies this equation, too. The coefficients c_n in (3) are chosen (as in Secs. 3 and 4) so as to minimize the error for fulfilling the given initial (24) and the boundary (25) condition corresponding to Eq. (23). Figure 8 shows a) the initial condition, b) an approximation by all polynomials from v_{000} to v_{klm} where $k+l+m=9$, for time $t=0$, c) the difference between a) and b). All results below are obtained for $\Delta t = 1$. Figures 8 shows that the boundary conditions are well fulfilled.

In this case the thickness of the membrane is not dependent on y. Additionally, the initial condition is symmetric in accordance with plane $y = 0.5$. Therefore, the solution of the problem (23)–(25) should be symmetric in accordance with plane $y = 0.5$, too. Because the thickness of the membrane is linearly dependent on x, the solution should be asymmetric in accordance with plane $x = 0.5$.

Figure 9 shows an approximation of the solution by all polynomials from v_{000} to v_{klm} where $k+l+m=9$, for time a) $t=0.3$, b) $t=0.5$, c)$t=0.9$. It is the view from the axes x and y.

The figure 9 shows that the physical character of the vibrations is preserwed. As in Sec. 4, we define an average, relative difference between solutions u_K and u_{K-1}:

$$D(K) = \sqrt{\frac{\int_0^{\Delta t}\int_0^1\int_0^1 (u_K(x,y,t) - u_{K-1}(x,y,t))^2 \mathrm{d}x\mathrm{d}y\mathrm{d}t}{\int_0^{\Delta t}\int_0^1\int_0^1 (u_{K-1}(x,y,t))^2 \mathrm{d}x\mathrm{d}y\mathrm{d}t}}.$$

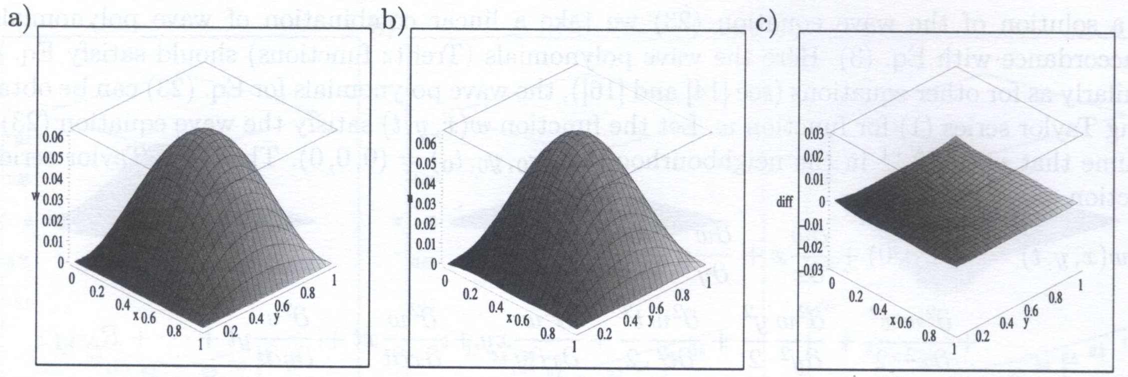

Fig. 8. Solution for time $t = 0$: a) exact, b) approximation, c) difference

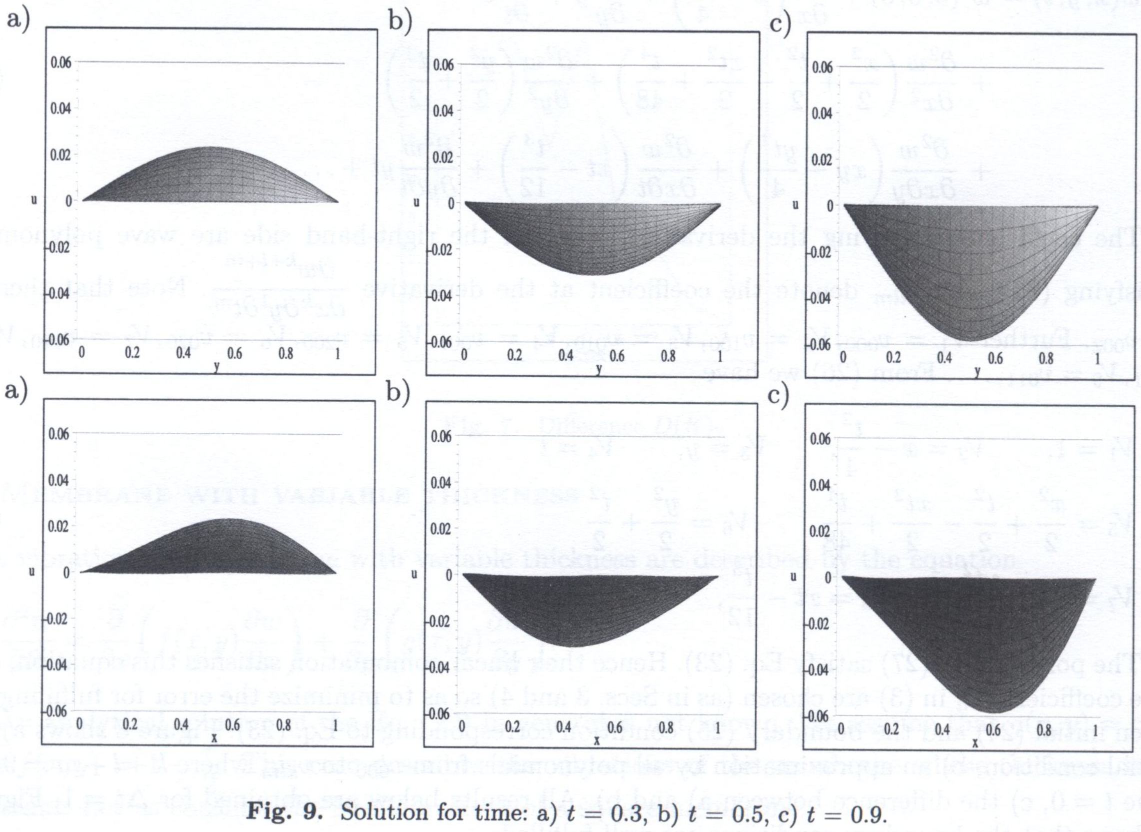

Fig. 9. Solution for time: a) $t = 0.3$, b) $t = 0.5$, c) $t = 0.9$.

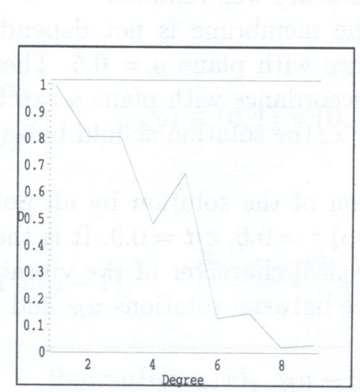

Fig. 10. Difference $D(K)$

Here u_K means the approximate solution by all polynomials from v_{000} to v_{klm} where $k+l+m=K$.

Figure 10 shows the difference $D(K)$ which depends on the order K for $\Delta t = 1$. Figure 10 suggests that in this case the method is convergent.

6. Discrete boundary conditions

The considered wave polynomials' method is relatively flexible when applied to continuous or discrete initial and boundary conditions. Moreover, it is used to solve inverse problems, for which one or more conditions are not known. Initial and boundary conditions are used to calculate the coefficients c_n in the approximate solution (see (3) and (5)). The adequate functional is dependent upon the kind of initial and boundary conditions. The integrals appear in the functional in the case of continuous conditions. When discrete conditions are given the integrals are substituted by sums.

Let us consider the vibrations of the membrane suspended in discrete points, described by the following mathematical model:

- equation of motion:

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}, \qquad (x,y) \in (0,1)\times(0,1), \qquad t>0, \tag{28}$$

- initial conditions:

$$w(x,y,0) = w_0(x,y) = x(x-1)y(y-1), \qquad \frac{\partial w(x,y,0)}{\partial t} = 0, \tag{29}$$

- boundary conditions:

$$w(x_i,y_j,t) = 0, \qquad \text{for } (x_i,y_j) \in B, \tag{30}$$

where

$$\begin{aligned} B = \{&(0;0),(0;0,25)(0;0,5),(0;0,75),(0;1),(0,25;0),(0,25;1),(0,5;0),\\ &(0,5;1),(0,75;0),(0,75;1),(1;0),(1;0,25)(1;0,5),(1;0,75),(1;1)\}. \end{aligned}$$

The solution $w(x,y,t)$ is approximated in accordance with equation (3). Because the function u satisfies the wave equation (linear combination of wave polynomials), we minimize $\| w-u \|$ recording to the boundary and the initial conditions. We look for an approximate solution u in the time interval $(0,\Delta t)$. Therefore, the coefficients c_n have to be chosen appropriately to minimize the functional

$$\begin{aligned} I = &\underbrace{\int_0^1\int_0^1 [u(x,y,0)-w_0(x,y)]^2 \mathrm{d}y\mathrm{d}x}_{\text{cond.(29)}} \\ &+ \underbrace{\int_0^1\int_0^1 \Bigg[\frac{\partial u(x,y,0)}{\partial t} - \underbrace{\frac{\partial w(x,y,0)}{\partial t}}_{=0}\Bigg]^2 \mathrm{d}y\mathrm{d}x}_{\text{cond.(29)}} \\ &+ \underbrace{\sum_{(x_i,y_j)\in B}\left(\int_0^{\Delta t}\Big[u(x_i,y_j,t) - \underbrace{w(x_i,y_j,t)}_{=0}\Big]^2 \mathrm{d}t\right)}_{\text{cond.(30)}}. \end{aligned}$$

The coefficients c_n are calculated in Secs. 3 and 4. All the results below are obtained for ($\Delta t = 1$.)

We obtain the approximation in the whole time interval $(0, \Delta t)$. For example, in the Fig. 11 for $t = 0$, a) the exact solution, b) an approximation by polynomials from order 0 to 9, c) the difference between a) and b) are shown. Figure 12 presents an approximation by polynomials from order 0 to 9 for time a) $t = 0.2$, b) $t = 0.6$, c) $t = 0.9$. Figures 11 and 12 show that the boundary conditions are well fulfilled and the physical character of the vibrations is preserved. The coefficients c_n are calculated as in Secs 3 and 4. All the results below are obtained for $\Delta t = 1$.

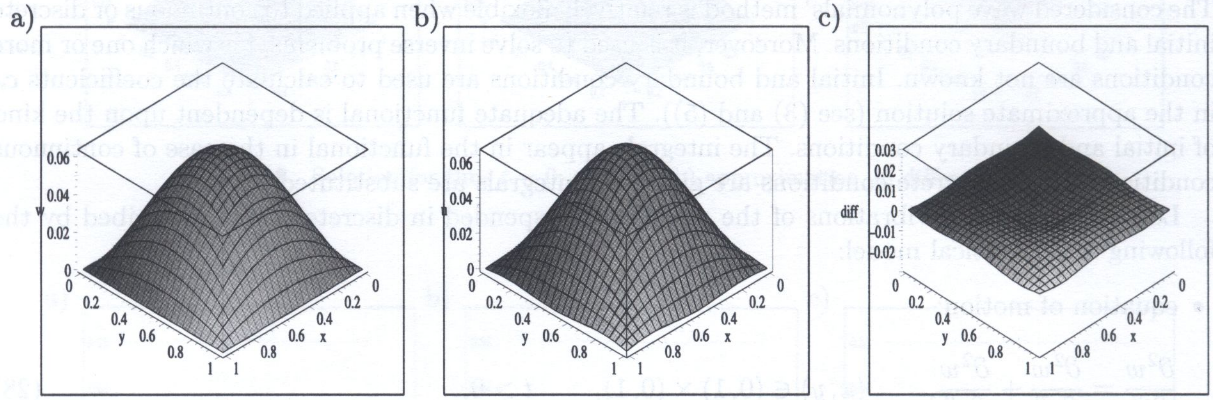

Fig. 11. Solution for time $t = 0$: a) exact, b) approximation, c) difference

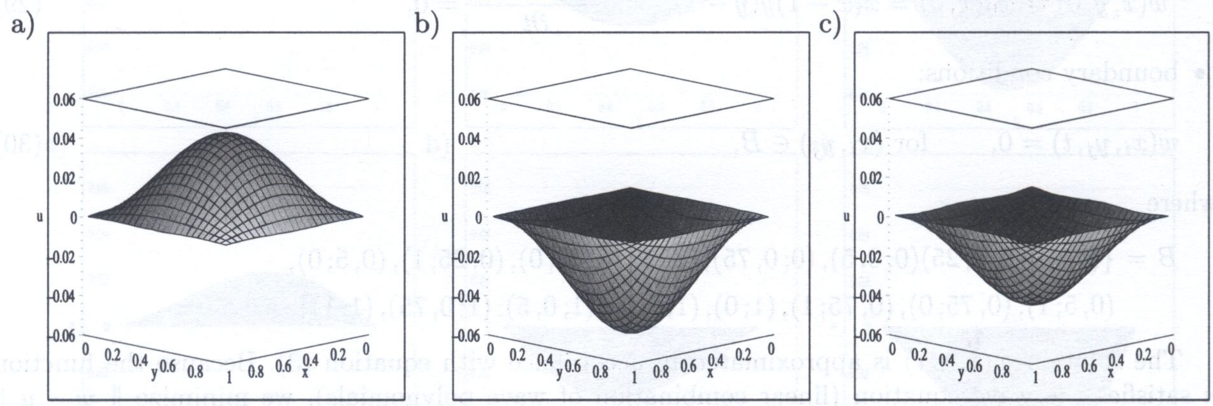

Fig. 12. Solution for time: a) $t = 0.2$, b) $t = 0.6$, c) $t = 0.9$

7. Approximation of the δ-Dirac for stationary case of circular membrane

A wave polynomial method can be also used for an approximate solution of Poisson's equation in a polar coordinate system. Let us consider axial symmetric stationary case of the membrane, described by the following mathematical model:

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} = -\frac{\delta(r)}{10}, \qquad r \in (0, R), \tag{31}$$

$$w(R) = 0. \tag{32}$$

The exact solution of the problem (31)–(32) and for $R = 0.2$ is

$$w(r) = \frac{1}{20\pi}\ln\left(\frac{1}{5r}\right).$$

To calculate the approximate solution of the problem (31)–(32) we use the definition of the δ-Dirac and we assume, that [18]

$$\delta(r) \approx \frac{k^2 e^{-k^2 r^2}}{\pi}. \tag{33}$$

Then, an approximation of the Eq. (31) is

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} = -\frac{k^2 e^{-k^2 r^2}}{10\pi}. \tag{34}$$

As an approximation of the solution we follow Eq. (5). Here the polynomials V_n are well known harmonic polynomials in a polar coordinate system and w_p is a particular solution. To solve Eq. (34) we follow steps described in Sec. 3. We expand the right side of the Eq. (34) into Taylor series. As in formula (6) we use the invert operator for Eq. (34). It is easy to prove that for the operator $L = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$ we have [14]

$$L^{-1}(r^n) = \frac{r^{n+2}}{(n+2)^2}.$$

Figure 13 shows the exact solution of the problem (31)–(32) and the approximate solutions for $N = 7$ in formula (5) and for $k = 5, 10, 15, 20, 25, 30$ in formula (33). Figure 13 shows, that if $k \to \infty$, the approximate solution is convergent to the exact solution.

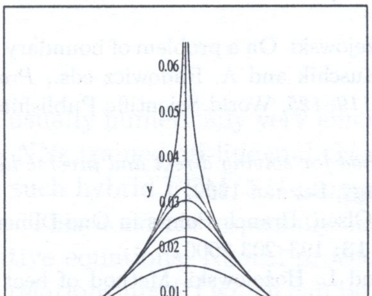

Fig. 13. Exact and approximate solution of Poisson equation

8. Concluding remarks

A new technique for solving the two-dimensional homogeneous and inhomogeneous wave equation has been developed. The wave polynomials' method presented in this paper is a straightforward method for solving wave equations in finite bodies. It is also useful when the shape of the body is more complicated. The coefficients c_n are determined by calculating integrals - for most shapes it does not cause any problems. It is very interesting that this method can be used for various modifications of wave equation (for example, a membrane with variable thickness). In many cases the method leads to determining Trefftz functions for the considered equation. Thanks to it we obtain the solution wholly satisfying the given equation. Therefore, the wave polynomials can be used as new base functions of finite elements. The method presented here is relatively flexible for initial and boundary conditions which can be continuous or discrete. Moreover, it is useful in the case of inverse problems where one or more conditions are not known. The examples show that the approximations of the exact solutions are very good.

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References

[1] E. Trefftz. Ein Gegenstük zum Ritzschen Verfahren. In: *Proceedings 2nd International Congres of Applied Mechanics,* 131–137, Zurich 1926.

[2] A.P. Zieliński and I. Herrera. Trefftz method: fitting boundary conditions. *Int. J. Num. Meth. Eng.,* **24:** 871–891, 1987.

[3] P.C. Rosenbloom and D.V. Widder. Expansion in terms of heat polynomials and associated functions. *Trans. Am. Math. Soc.,* **92:** 220–266, 1956.

[4] H. Yano, S. Fukutani and A. Kieda. A boundary residual method with heat polynomials for solving unsteady heat conduction problems. *Journal of the Franklin Institute,* **316** (4): 291–298, 1983.

[5] S. Futakiewicz and L. Hożejowski. Heat polynomials method in the n-dimensional direct and inverse heat conduction problems. In: A.J. Nowak, C.A. Brebbia, R. Bielecki and M. Zerroukat, eds., *Advanced Computational Method in Heat Transfer V, 103–112, Southampton UK and Boston USA: Computational Mechanics Publications* 1998.

[6] L. Hożejowski. *Heat polynomials and their application for solving direct and inverse heat condutions problems (PhD-Thesis),* (in Polish), 115 Kielce: University of Technology, 1999.

[7] S. Futakiewicz and L. Hożejowski. Heat polynomials in solving the direct and inverse heat conduction problems in a cylindrical system of coordinates. In: A.J. Nowak, C.A. Brebbia, R. Bielecki and M. Zerroukat. eds., *Advanced Computational Method in Heat Transfer V, 71–80* and Boston USA Computational Mechanics Publications, Southampton UK, 1998.

[8] S. Futakiewicz, K. Grysa and L. Hożejowski. On a problem of boundary temperature identification in a cylindrical layer. In: B.T. Maruszewski, W. Muschik and A. Radowicz eds., *Proceedings of the International Symposium on Trends in Continuum Physics, 119–125,* World Scientific Publishing, Singapore, New Jersey, London, Hong Kong , 1999.

[9] S. Futakiewicz. *Heat functions method for solving direkt and inverse heat condutions problems (PhD-Thesis)* (in Polish), 120, University of Technology, Poznań 1999.

[10] P. Johansen, M. Nielsen and O.F. Olsen. Branch Points in One-Dimensional Gaussian Scale Space. *Journal of Mathematical Imaging and Vision,* **13:** 193–203 2000.

[11] M.J. Ciałkowski, S. Futakiewicz and L. Hożejowski. Method of heat polynomials in solving the inverse heat conduction problems. *ZAMM* **79:** 709–710, 1999.

[12] M.J. Ciałkowski, S. Futakiewicz and L. Hożejowski. Heat polynomials applied to direct and inverse heat conduction problems. In: B.T. Maruszewski, W. Muschik and A. Radowicz eds., *Proceedings of the International Symposium on Trends in Continuum Physics, 79–88,* World Scientific Publishing Singapore, New Jersey, London, Hong Kong, 1999.

[13] M.J. Ciałkowski. Solution of inverse heat conduction problem with use new type of finite element base functions. In: B.T. Maruszewski, W. Muschik and A. Radowicz eds., *Proceedings of the International Symposium on Trends in Continuum Physics, 64–78,* World Scientific Publishing Singapore, New Jersey, London, Hong Kong 1999.

[14] M.J. Ciałkowski and A. Frąckowiak. *Heat functions and their application for solving heat transfer and mechanical problems.* (in Polish), 360 University of Technology Publishers Poznań 2000.

[15] I.N. Sneddon. *Elements of Partial Differential Equations,* 423. PWN, Warsaw 1962.

[16] A. Maciąg and J. Wauer. Solution of the two-dimensional wave equation by using wave polynomials, *Journal of Engrg. Math.* **51:** (4) 339–350, 2005.

[17] E.B. Magrab. *Vibrations of elastic structural members, 320.* Maryland USA: Sijthoff & Noordhoff, 1979.

[18] E. Kącki. *Partial differential equations in phisics and engineering problems* (in Polish), 506. WNT, Warsaw 1989